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BLOCK SPACES ON THE UNIT SPHERE IN R^n

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ABSTRACT. Let $B_q^{\mu,\nu}$ be the block space on the unit sphere introduced by S. Lu. We discuss the relation between $B_q^{\mu,\nu}$ and the L^p-space on the unit sphere. Then we give the characterization of $B_q^{\mu,\nu}$ and a simple proof of Theorem (12.11)(iii) of *Spaces generated by blocks* (Beijing Normal University Math. Ser., 1989).

Let Σ_{n-1} be the unit sphere in R^n $(n \geq 2)$, $1 < q \leq \infty$, $\mu \geq 0$, and $\nu > -1$. In [2] Lu introduced the spaces $B_q^{\mu,\nu}(\Sigma_{n-1})$ with respect to the study of singular integral operators.

In this note, we investigate the spaces $B_q^{\mu,\nu}(\Sigma_{n-1})$. Then the relation between $B_q^{\mu,\nu}(\Sigma_{n-1})$ and $L^q(\Sigma_{n-1})$ is clear; thus, a simple proof of [2, (12.11) Theorem (iii)] is derived.

Definition 1. (1) For $x_0' \in \Sigma_{n-1}$ and $0 < \theta_0 \leq 2$, $B(x_0', \theta_0) = \{x' \in \Sigma_{n-1} : |x_0' - x'| < \theta_0\}$ is called a cap. σ is the normalized rotation invariant measure on Σ_{n-1} with $\sigma(\Sigma_{n-1}) = e^{-1}$.

(2) For $1 < q \leq \infty$, a measurable function b is called a q-block on Σ_{n-1} if and only if b is a function on some cap $I = B(x_0', \theta_0)$ and $\|b\|_{L^q(\Sigma_{n-1})} \leq \sigma(I)^{-1/q'}$, where $1/q + 1/q' = 1$. We sometimes write $\sigma(I) = |I|$.

(3) $B_q^{\mu,\nu} = B_q^{\mu,\nu}(\Sigma_{n-1}) = \{\Omega \text{ on } \Sigma_{n-1} : \Omega = \sum_{n=1}^{\infty} c_k b_k$, where $c_k \in \mathbb{C}$; each b_k is a q-block with the support on a cap I_k; and $M_q^{\mu,\nu}(\{c_k\}) = \sum_{n=1}^{\infty} |c_k|(1 + \phi_{\mu,\nu}(|I_k|)) < \infty\}$, where $\phi_{\mu,\nu}(t) = \int_t^1 u^{-1-\mu} \log^\nu u^{-1}\, du$ $(0 < t < 1)$ and $= 0$ $(t \geq 1)$.

We remark that $\phi_{\mu,\nu}(t) \sim t^{-\mu} \log^\nu t^{-1}$ as $t \to 0$ $(\mu > 0)$ and $\phi_{\mu,\nu}(t) \sim \log^{\nu+1} t^{-1}$ as $t \to 0$ $(\mu = 0)$.

(4) $M_q^{\mu,\nu}(\Omega) = \inf\{M_q^{\mu,\nu}(\{c_k\}) : \Omega = \sum c_k b_k$ and each b_k is a q-block with the support on a cap $I_k\}$.

Then $M_q^{\mu,\nu}$ is the norm of $B_q^{\mu,\nu}$ and $(B_q^{\mu,\nu}, M_q^{\mu,\nu})$ is a Banach space.

(5) Using the notation $\phi_{\mu,\nu}$ of (3), we define, for $S \subset \Sigma_{n-1}$,

$$|S|_{\phi_{\mu,\nu}} = \inf\left\{\sum_{k=1}^{\infty} |I_k|\phi_{\mu,\nu}(|I_k|) : S \subset \bigcup_{k=1}^{\infty} I_k,\ \text{where } \{I_k\} \text{ are caps}\right\}$$

and $J_{\phi_{\mu,\nu}}(f) = \int_0^\infty |\{x : |f(x)| > \lambda\}|_{\phi_{\mu,\nu}}\, d\lambda$ for any measurable function f.

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Definition 2. Suppose $\Omega \in B_q^{\mu,\nu}$ is a homogeneous function of degree 0 which satisfies $\int_{\Sigma_{n-1}} \Omega(x')\,d\sigma(x') = 0$. Let h be a bounded radial function. We define an operator T^* on $L^p(R^n)$ $(1 < p < \infty)$ such that

$$T^*f(x) = \sup_{0<\varepsilon<\infty} |(\Omega(t)h(t)|t|^{-n}\chi_{\{|t|>\varepsilon\}} * f)(x)|.$$

By Definition 1, it is easy to see

Proposition 3. (1) $B_q^{\mu,\nu_2} \subset B_q^{\mu,\nu_1}$ $(\nu_2 > \nu_1 > -1,\ \mu \geq 0)$.
(2) $B_q^{\mu_2,\nu_2} \subset B_q^{\mu_1,\nu_1}$ $(0 \leq \mu_1 < \mu_2,\ \nu_i > -1,\ i = 1, 2)$.
(3) $L^q(\Sigma_{n-1}) \subset B_q^{\mu,\nu}$ $(1 < q \leq \infty,\ \mu \geq 0,\ \nu > -1)$.
(4) *If* $J_{\phi_{\mu,\nu}}(f) < \infty$, *then* $f \in B_\infty^{\mu,\nu}$.

Then we obtain the following with relation to $B_q^{\mu,\nu}$ for $\mu > 0$.

Proposition 4. *Suppose* $1 < p \leq q \leq \infty$. *Then, for* $\mu > 1/p'$, *we have* $B_q^{\mu,\nu} \subset L^p(\Sigma_{n-1})$ *for any* $\nu > -1$.

Proof. By Proposition 3, we have $B_q^{\mu,\nu} \subset B_q^{\mu-\varepsilon,0}$ for some $\varepsilon > 0$ $(\mu - \varepsilon > 1/p')$. Then it is sufficient to prove $B_q^{\mu,0} \subset L^p(\Sigma_{n-1})$ for $\mu > 1/p'$. Also for $f \in B_q^{\mu,0}$, we may assume that $f = \sum a_k b_k$, where $a_k \in \mathbb{C}$ and b_k is on a cap I_k with $\sum |a_k||I_k|^{-\mu} < \infty$.

On the other hand, $\|f\|_{L^p(\Sigma_{n-1})} \leq \sum |a_k|\|b_k\|_{L^p(\Sigma_{n-1})}$. So by $\mu > 1/p'$, we obtain $\|f\|_{L^p(\Sigma_{n-1})} \leq \sum |a_k||I_k|^{-1/p'} \leq \sum |a_k||I_k|^{-\mu} < \infty$. Q.E.D.

By Proposition 4 and Chen's result [1], it is easy to prove

Corollary 5. *Let* $\mu > 0$, $1 < q \leq \infty$, $\Omega \in B_q^{\mu,\nu}$, *and suppose that* Ω *is homogeneous of degree zero and satisfies the cancellation property* (*i.e.*, $\int_{\Sigma_{n-1}} \Omega\,d\sigma = 0$). *Let* h *be any bounded radial function. Then* T^* *is a bounded linear operator on* $L^p(R^n)$ $(1 < p < \infty)$. *In particular,* [2, (12.11) *Theorem* (iii)] *is proved.*

Theorem 6. $B_q^{\mu,\nu} = L^q(\Sigma_{n-1})$ *if and only if* $\mu > 1/q'$ *or* $\mu = 1/q'$ *and* $\nu \geq 0$.

Proof. It is sufficient to prove the necessary condition. Let $\mu < 1/q'$ and $\varepsilon > 0$ with $1/q' - \varepsilon > \mu$. Then if we define $a_k = \exp(-(1/q' - \varepsilon/2)k)$, a cap I_k with $|I_k| = \exp(-k)$, and $f(x) = \sum_{k=1}^\infty a_k|I_k|^{-1}\chi_{I_k}(x)$, where χ_{I_k} is the characteristic function on I_k, it is easy to prove $f \in B_q^{\mu+\varepsilon/2,0}$ and $f \notin L^q(\Sigma_{n-1})$. Also suppose $\mu = 1/q'$ and $\nu < 0$. Then if we define that I_k is a cap with $|I_k| = \exp(-k^{3/|\nu|})$, $a_k = k|I_k|^{1/q'}$, and $f(x) = \sum_{k=1}^\infty a_k|I_k|^{-1}\chi_{I_k}(x)$, we obtain that $f \in B_q^{\mu,\nu}$ and $f \notin L^q(\Sigma_{n-1})$. Q.E.D.

Next we study $B_q^{\mu,\nu}$ for $\mu = 0$.

Proposition 7. *For any* $\nu > -1$, $B_\infty^{0,\nu}$ *is not contained in* $\bigcup_{p>1} L^p(\Sigma_{n-1})$.

Proof. Let $a_k = k^{-3}$, and assume a cap I_k with $|I_k| = \exp(-(1+\nu)^{-1}k)$. Then if we define $f = \sum a_k|I_k|^{-1}\chi_{I_k}$, we obtain $f \in B_\infty^{0,\nu}$ and $f \notin L^p(\Sigma_{n-1})$ for all $1 < p \leq \infty$. Q.E.D.

Remark. [2, (12.11) Theorem (i) and (ii), p. 140] are not contained in Chen's result.

Proposition 8. *For* $-1 < \nu < 0$, $B_{\infty}^{0,\nu}$ *is not contained in* $L \log L$ *class.*

Proof. Let α be a positive number with $\alpha|\nu| > 3$, $a_k = k^{-(\alpha-1)}$, and a cap I_k with $|I_k| = \exp(-k^{\alpha})$. If we define $f = \sum a_k |I_k|^{-1} \chi_{I_k}$, we obtain the desired result. Q.E.D.

References

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