PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 119, Number 2, October 1993

A REPRESENTATION LATTICE ISOMORPHISM FOR THE PERIPHERICAL SPECTRUM

JOSEP MARTÍNEZ

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. In this paper we construct a representation isometric lattice isomorphism for the peripherical spectrum of a positive operator on a Banach lattice. By a representation lattice homomorphism, we mean that the peripherical spectrum of the operator is identified with the spectrum of the induced isometric lattice homomorphism. A simple proof of a "zero-two" law follows easily from our representation technique.

1. THE REPRESENTATION LATTICE HOMOMORPHISM

We develop our technique in the context of Banach lattices with p-additive norm. Following Zaanen [Zaa],

1.1. **Definition.** Let $1 \le p < \infty$. A Banach lattice E for which $\|x+y\|^p = \|x\|^p + \|y\|^p$ whenever $x \wedge y = 0$ is called an *abstract* $\mathscr{L}^p$*-space* (or an AL_p-*space*).

In fact, the norm of an AL_p-space $(1 \le p < \infty)$ is p-superadditive for all positive elements. In other words, if E is an AL_p-space, then

$$\|x+y\|^p \ge \|x\|^p + \|y\|^p \quad \forall x, y \ge 0.$$

As a straightforward consequence of the p-superadditivity of the norm, we get the following basic property of AL_p-spaces:

(1) If T is a contraction on E that verifies $0 \le x \le Tx$ for some $x \in E$, then $Tx = x$.

Now let E be an AL_p-space $(1 \le p < \infty)$, and let $\mathscr{F}$ denote a free ultrafilter on $\mathbf{N}$. The $\mathscr{F}$-product $\widehat{E}_{\mathscr{F}}$ is actually an AL_p-space (see [S1, Chapter V, §1]). We denote by P the following isometric lattice isomorphism on $\widehat{E}_{\mathscr{F}}$:

$$P((x_1, x_2, \dots) + c_{\mathscr{F}}(E)) = (x_2, x_3, \dots) + c_{\mathscr{F}}(E),$$

whose inverse isometric lattice homomorphism is given by

$$Q((x_1, x_2, \dots) + c_{\mathscr{F}}(E)) = (0, x_1, x_2, \dots) + c_{\mathscr{F}}(E).$$

Received by the editors February 13, 1992.
1991 *Mathematics Subject Classification.* Primary 47B55.
The author was supported by an F.P.I. grant from the Spanish Ministerio de Educación y Ciencia.

1.2. **Definition.** Let T be a positive contraction on E, and denote by $\widehat{T}_{\mathscr{F}}$ its canonical extension to the $\mathscr{F}$-product $\widehat{E}_{\mathscr{F}}$. We define the *limit space of* T as the Banach space $E(T) = \operatorname{Ker}(P - \widehat{T}_{\mathscr{F}})$ and the *limit operator of* T by $\widetilde{T} = \widehat{T}_{\mathscr{F}}|_{E(T)} = P|_{E(T)}$. The fact that

$$\{T^n x\} + c_{\mathscr{F}}(E) \in E(T) \quad \forall x \in E$$

justifies our terminology. The Banach subspace

$$A(T) = \overline{\{\{T^n x\} + c_{\mathscr{F}}(E) : x \in E\}} \subseteq E(T)$$

will be called the *asymptotic space* of T.

As an immediate consequence of property (1), $E(T) = \operatorname{Ker}(I - Q\widehat{T}_{\mathscr{F}})$ is in fact a sublattice of $\widehat{E}_{\mathscr{F}}$ and so an AL_p-space. Moreover, as $\widetilde{T}$ is the restriction of an isometric lattice homomorphism, $\widetilde{T}$ is obviously an isometric lattice homomorphism.

We can now state our basic lemma, which translates the techniques of Allan-Ransford [A] and Phong-Lyubich [L] to the setting of Banach lattices.

1.3. **Theorem.** *Let T be a positive contraction on the AL_p-space E with $1 \le p < \infty$, and let $\widetilde{T}$ denote its limit operator. Then we have*

$$\Gamma \cap \sigma_p(T) \subseteq \sigma(\widetilde{T}) \subseteq \Gamma \cap \sigma(T), \quad \textit{where } \Gamma = \{\lambda \in \mathbf{C} : |\lambda| = 1\}.$$

Proof. Let $\lambda \in \rho(T)$. As

$$\widehat{R(\lambda, T)}(E(T)) \subseteq E(T),$$

we get $\lambda \in \rho(\widetilde{T})$, where the resolvent is given by

$$R(\lambda, \widetilde{T}) = \widehat{R(\lambda, T)}|_{E(T)}.$$

But $\widetilde{T}$ is an invertible isometry and so the inclusion $\sigma(\widetilde{T}) \subseteq \Gamma \cap \sigma(T)$ is already proved.

On the other hand, if $\lambda \in \Gamma \cap \sigma_p(T)$, there exists a nonzero vector x with $Tx = \lambda x$. Defining now

$$\tilde{x} = (x, \lambda x, \lambda^2 x, \dots) + c_{\mathscr{F}}(E) \in E(T),$$

we obtain $\widetilde{T}\tilde{x} = \lambda \tilde{x}$, $\|\tilde{x}\| = \|x\| \neq 0$; therefore, $\lambda \in \sigma(\widetilde{T})$.

Applications

Let T be a positive contraction on L^1. In 1970 Orstein and Sucheston [O] showed that

$$\sup_{\|f\|_1 \le 1} \lim_{n \to \infty} \|T^n f - T^{n+1} f\|_1 \tag{1}$$

is either 0 or 2. This surprising result opened a new direction of research. Wittmann [W] extended this "zero-two" law to AL_p-spaces. On the other hand, Zaharopol [Zah], Katznelson-Tzafriri [K], and Schaefer [S2] proved that, given a positive linear contraction in an arbitrary Banach lattice, the limit $\lim_n \||T^n - T^{n+1}\||$ is either 0 or 2.

We now deduce a simple proof of a uniform "zero-two" law from the above representation technique. We need the following modification of an Arendt-Schaefer-Wolff result (see [Ar, Lemma 3.3]):

2.1. **Lemma.** *Let T be a positive isometry on the Banach lattice E, and suppose that $r(I-T)<\sqrt{3}$. Then we have $T=I$.*

Proof. By the classical result of Gelfand (see [A]), we only need to show $\sigma(T)=1$. If this is not verified, as $\sigma(T)$ is cyclic (see [S1]), there must be an element $a\in\sigma(T)$ such that $\frac{2}{3}\pi\leq\arg a\leq\frac{4}{3}\pi$ holds; this implies $-1\leq\operatorname{Re}(a)\leq-\frac{1}{2}$. From this inequality we obtain

$$r(I-T)^2\geq|a-1|^2=|a|^2-2\operatorname{Re}(a)+1\geq 3.$$

2.2. **Theorem.** *Let E be an AL_p-space $(1\leq p<\infty)$, and let T be a positive contraction on E. Then the following statements are equivalent*:

(a) $\lim_{n\to\infty}\|T^n-T^{n+1}\|=0$.
(b) $\lim_{n\to\infty}\|T^n-T^{n+1}\|<\sqrt{3}$.

Proof. (b) $\to$ (a) Given a free ultrafilter $\mathscr{F}$ on E, if we denote by $S=\widehat{T}_{\mathscr{F}}$ the canonical extension of T to the $\mathscr{F}$-product AL_p-space $\widehat{E}_{\mathscr{F}}$, we have (see [S1, Chapter V, §1])

$$\sigma_{ap}(T)=\sigma_p(S),\qquad \|T^n-T^{n+1}\|=\|S^n-S^{n+1}\|.$$

By the Katznelson-Tzafriri theorem [K], it suffices to prove $\sigma_p(S)\cap\Gamma=\sigma(T)\cap\Gamma\subseteq\{1\}$. However, given $x\in E(T)$, denoting by $\widetilde{S}$ the limit operator of S we get

$$\|x-\widetilde{S}x\|=\|Q^n(\widehat{S}^n-\widehat{S}^{n+1})x\|\leq\|S^n-S^{n+1}\|\,\|x\|$$

and so we deduce $\|I-\widetilde{S}\|<\sqrt{3}$. Lemma 2.1 now shows that $\widetilde{S}=I$, and then from Theorem 1.3 we conclude

$$\Gamma\cap\sigma_p(S)\subseteq\sigma(\widetilde{S})=\{1\}.$$

Acknowledgment

The author wishes to thank Professors F. Andreu and J. Mazón for many stimulating conversations concerning the subject of this paper.

References

[A] G. R. Allan and T. J. Ransford, *Power-dominated elements in Banach algebras*, Studia Math. **94** (1989), 63–79.

[Ar] W. Arendt, H. H. Schaefer, and M. Wolff, *On lattice isomorphisms with positive real spectrum and groups of positive operators*, Math. Z. **164** (1978), 115–123.

[K] Y. Katznelson and C. Tzafriri, *On power bounded operators*, J. Funct. Anal. **68** (1986), 313–328.

[L] Y. I. Lyubich and Vu Quoc Phong, *Asymptotic stability of linear differential equations in Banach spaces*, Studia Math. **88** (1988), 37–42.

[O] D. Orstein and L. Sucheston, *An operator theorem on L^1 convergence to zero with applications to Markov kernels*, Ann. Math. Statist. **41** (1970), 1631–1639.

[S1] H. H. Schaefer, *Banach lattices and positive operators*, Springer, Berlin, Heidelberg, and New York, 1974.

[S2] ——, *The zero-two law for positive contractions is valid in all Banach lattices*, Israel J. Math. **59** (1987), 241–244.

[W] R. Wittmann, *Analogues of the zero-two law for positive linear contractions in L^p and $C(X)$*, Israel J. Math. **59** (1987), 8–28.

[Zaa] A. C. Zaanen, *Riesz spaces*. II, North-Holland, New York, 1983.

[Zah] R. Zaharopol, *The modulus of a regular linear operator and the "zero-two" law in L^p-spaces* $(1 < p < \infty,\ p \neq 2)$, J. Funct. Anal. **68** (1986), 300–312.

Departament d'Anàlisi Matemàtica, Universitat de València, Dr. Moliner 50, 46100 Burjassot, Spain
E-mail address: martinjo@vm.ci.uv.es