

ON THE ABSTRACT CHARACTERIZATION OF OPERATOR SPACES

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ABSTRACT. A direct proof is given for the matricial norm characterization of operator spaces.

1. INTRODUCTION

The abstract characterization given for linear spaces of bounded Hilbert space operators (“operator spaces”) in terms of “matricially normed spaces” [9] implies that quotients, mapping spaces, and various tensor products of operator spaces may again be regarded as operator spaces. Owing in part to this result, the theory of operator spaces is having an increasingly significant effect on operator algebra theory (see, e.g., [3, 4, 7]).

The proof given in [9] appealed to the theory of ordered operator spaces [1]. In this note we will show that one can give a purely metric proof of this important theorem by using a technique of Pisier [8] and Haagerup [5] (as modified in [2]).

2. THE MATRICIAL CHARACTERIZATION THEOREM

Given Hilbert spaces H and K , the linear space $B(H, K)$ of bounded operators $b: H \rightarrow K$ comes equipped with the usual norm. It follows that any operator space $V \subseteq B(H)$ is a normed vector space. However, due to the fact that a matrix of operators is again an operator, operator spaces inherit a richer “matricial norm structure”, which distinguishes them from normed vector spaces. To be specific, we may regard an $m \times n$ matrix $b = [b_{ij}]$ of operators $b_{ij} \in B(H)$ as an operator from H^n to H^m , where

$$H^n = \overbrace{H \oplus \cdots \oplus H}^n.$$

As a result, we may identify the space $\mathbf{M}_{m,n}(V)$ of $m \times n$ matrices over V with a subspace of $B(H^n, H^m)$. This provides a distinguished norm on the matrix space $\mathbf{M}_{m,n}(V)$. The appropriate morphisms in the category of operator spaces

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are those that are well behaved with respect to all of these matrix norms (see below).

Given a vector space V , we let $\mathbf{M}_{m,n}(V)$ denote the vector space of $m \times n$ matrices $\mathbf{v} = [v_{ij}]$ ($v_{ij} \in V$), and we let $\mathbf{M}_n(V) = \mathbf{M}_{n,n}(V)$, $\mathbf{M}_{m,n} = \mathbf{M}_{m,n}(\mathbb{C})$, and $\mathbf{M}_n = \mathbf{M}_{n,n}$. If we systematically keep track of the indices, we may use the usual identifications $\mathbf{M}_{m,n}(\mathbf{M}_{p,q}) = \mathbf{M}_{mp,nq}$, i.e., we may “delete internal brackets”. Given $\mathbf{v} \in \mathbf{M}_{m,n}(V)$, $\mathbf{w} \in \mathbf{M}_{p,q}(V)$, and scalar matrices $\alpha \in \mathbf{M}_{p,m}$, $\beta \in \mathbf{M}_{n,q}$, we define matrices $\mathbf{v} \oplus \mathbf{w} \in \mathbf{M}_{m+p,n+q}(V)$ and $\alpha \mathbf{v} \beta \in \mathbf{M}_{p,q}(V)$ by

$$\mathbf{v} \oplus \mathbf{w} = \begin{bmatrix} \mathbf{v} & 0 \\ 0 & \mathbf{w} \end{bmatrix} \quad \text{and} \quad \alpha \mathbf{v} \beta = \left[\sum_{ij} \alpha_{ki} v_{ij} \beta_{jl} \right].$$

Letting \mathbb{C}^n have the usual Hilbert space norm, we have that $\mathbf{M}_{m,n} = B(\mathbb{C}^n, \mathbb{C}^m)$ is an operator space. From elementary operator considerations, the matricial norms on an operator space V are connected by the relations

(M1) $\|\mathbf{v} \oplus \mathbf{w}\| = \max\{\|\mathbf{v}\|, \|\mathbf{w}\|\}$ ($\mathbf{v} \in \mathbf{M}_{m,n}(V)$, $\mathbf{w} \in \mathbf{M}_{p,q}(V)$),

(M2) $\|\alpha \mathbf{v} \beta\| \leq \|\alpha\| \|\mathbf{v}\| \|\beta\|$ ($\mathbf{v} \in \mathbf{M}_{m,n}(V)$, $\alpha \in \mathbf{M}_{p,m}$, $\beta \in \mathbf{M}_{n,q}$).

An L^∞ -matricially normed space is a vector space V with norms given on each matrix space $\mathbf{M}_{m,n}(V)$ satisfying (M1) and (M2). Given two such spaces V and W , a linear map $\varphi: V \rightarrow W$ determines a linear map

$$\varphi^{(m,n)}: \mathbf{M}_{m,n}(V) \rightarrow \mathbf{M}_{m,n}(W): [v_{ij}] \mapsto [\varphi(v_{ij})],$$

and we let $\varphi^{(n)} = \varphi^{(n,n)}$. φ is said to be *completely bounded* (resp. *completely contractive*, *completely isometric*) if it satisfies

$$\|\varphi\|_{cb} = \sup\{\|\varphi^{(n)}\|: n \in \mathbb{N}\} = \sup\{\|\varphi^{(m,n)}\|: m, n \in \mathbb{N}\} < \infty$$

(resp. $\|\varphi\|_{cb} \leq 1$, $\varphi^{(n)}$ is isometric for each $n \in \mathbb{N}$).

Our object is to prove

Theorem A. *Any L^∞ -matricially normed space V is completely isometric to an operator space.*

Theorem A is a consequence of the following completely bounded analogue of the Hahn-Banach Theorem.

Theorem B. *Given an L^∞ -matricially normed space V and an element $\mathbf{v} \in \mathbf{M}_n(V)$, there exists a complete contraction $\varphi: V \rightarrow \mathbf{M}_n$ such that $\|\varphi^{(n)}(\mathbf{v})\| = \|\mathbf{v}\|$.*

To see how Theorem A follows from Theorem B, let \mathcal{S}_n be the collection of all complete contractions $\varphi: V \rightarrow \mathbf{M}_n$. The mapping

$$\Phi: V \rightarrow \bigoplus_{n \in \mathbb{N}} \bigoplus_{\varphi \in \mathcal{S}_n} \mathbf{M}_n: v \mapsto (\varphi(v))$$

is trivially completely contractive, and from Theorem B it is completely isometric.

Given $\mathbf{v} \in \mathbf{M}_n(V)$, the classical Hahn-Banach Theorem implies that there is an element $F \in \mathbf{M}_n(V)^*$ such that $|F(\mathbf{v})| = \|\mathbf{v}\|$. Thus Theorem B follows from

Theorem C. *Suppose that $F \in \mathbf{M}_n(V)^*$ is a contraction. Then there exists a completely contractive map $\varphi: V \rightarrow \mathbf{M}_n$ and contractive $1 \times n^2$ and $n^2 \times 1$ matrices γ and δ such that*

$$F(\mathbf{v}) = \gamma\varphi^{(n)}(\mathbf{v})\delta,$$

for all $\mathbf{v} \in \mathbf{M}_n(V)$.

Proof. We begin by proving that there exist states p_0 and q_0 on \mathbf{M}_n such that

$$(1) \quad |F(\alpha\mathbf{v}\beta)| \leq p_0(\alpha\alpha^*)^{1/2}\|\mathbf{v}\|q_0(\beta^*\beta)^{1/2}$$

for any matrices $\alpha \in \mathbf{M}_{np}$, $\mathbf{v} = [v_{ij}] \in \mathbf{M}_p(V)$, and $\beta \in \mathbf{M}_{pn}$. It suffices to find p_0 and q_0 such that

$$|F(\alpha\mathbf{v}\beta)| \leq \frac{1}{2}(p_0(\alpha\alpha^*) + q_0(\beta^*\beta))$$

for all such α and β and all \mathbf{v} with $\|\mathbf{v}\| = 1$, since then for all $t > 0$

$$|F(\alpha\mathbf{v}\beta)| = |F(t^{1/2}\alpha\mathbf{v}t^{-1/2}\beta)| \leq \frac{1}{2}(tp_0(\alpha\alpha^*) + t^{-1}q_0(\beta^*\beta)),$$

and the result follows by minimizing the right-hand side. In turn it suffices to find p_0 and q_0 such that

$$\operatorname{Re} F(\alpha\mathbf{v}\beta) \leq \frac{1}{2}(p_0(\alpha\alpha^*) + q_0(\beta^*\beta)),$$

since then choosing $\theta \in [0, 2\pi]$ with $e^{i\theta}F(\alpha\mathbf{v}\beta) = |F(\alpha\mathbf{v}\beta)|$, it follows that

$$|F(\alpha\mathbf{v}\beta)| = F(e^{i\theta}\alpha\mathbf{v}\beta) \leq \frac{1}{2}(p_0(\alpha\alpha^*) + q_0(\beta^*\beta)).$$

We will define p_0 and q_0 to be the barycenters of measures on the state space S of the C^* -algebra \mathbf{M}_n .

We let $C(S \times S)$ be the real continuous functions on $S \times S$, and let $\mathcal{E} \subseteq C(S \times S)$ denote the functions of the form

$$e_{\alpha, \mathbf{v}, \beta}(p, q) = p(\alpha\alpha^*) + q(\beta^*\beta) - 2 \operatorname{Re} F(\alpha\mathbf{v}\beta)$$

with $\|\mathbf{v}\| = 1$. This is a cone because it is obvious that $\alpha\varepsilon \subseteq \varepsilon$ for $\alpha \geq 0$ and

$$\begin{aligned} &(e_{\alpha, \mathbf{v}, \beta} + e_{\alpha', \mathbf{v}', \beta'})(p, q) \\ &= p([\alpha\alpha'] + [\alpha'\alpha']^*) + q\left(\begin{bmatrix} \beta \\ \beta' \end{bmatrix}^* \begin{bmatrix} \beta \\ \beta' \end{bmatrix}\right) - 2 \operatorname{Re} F\left([\alpha\alpha'](\mathbf{v} \oplus \mathbf{v}') \begin{bmatrix} \beta \\ \beta' \end{bmatrix}\right) \\ &= e_{\alpha'', \mathbf{v}'', \beta''}(p, q), \end{aligned}$$

where

$$\alpha'' = [\alpha\alpha'], \quad \mathbf{v}'' = \mathbf{v} \oplus \mathbf{v}', \quad \beta'' = \begin{bmatrix} \beta \\ \beta' \end{bmatrix},$$

and $\|\mathbf{v}''\| = 1$ by condition (M1). The function $e_{\alpha, \mathbf{v}, \beta}$ cannot be strictly negative since choosing $p_1, q_1 \in S$ with $p_1(\alpha\alpha^*) = \|\alpha\alpha^*\|$, $q_1(\beta^*\beta) = \|\beta^*\beta\|$, it follows that

$$\begin{aligned} e_{\alpha, \mathbf{v}, \beta}(p_1, q_1) &= p_1(\alpha\alpha^*) + q_1(\beta^*\beta) - 2 \operatorname{Re} F(\alpha\mathbf{v}\beta) \\ &\geq \|\alpha\alpha^*\| + \|\beta^*\beta\| - 2|F(\alpha\mathbf{v}\beta)| = 2(\frac{1}{2}(\|\alpha\|^2 + \|\beta\|^2) - |F(\alpha\mathbf{v}\beta)|) \\ &\geq 2(\|\alpha\| \|\mathbf{v}\| \|\beta\| - |F(\alpha\mathbf{v}\beta)|) \geq 0. \end{aligned}$$

Thus letting \mathcal{K} be the cone of strictly negative functions on $S \times S$, we have $\mathcal{E} \cap \mathcal{K} = \emptyset$. Since \mathcal{K} has interior, we have from the geometric form of the classical Hahn-Banach Theorem a measure $0 \neq \mu \in C(S \times S)^*$ such that $\mu|_{\mathcal{E}} \geq 0$ and $\mu|_{\mathcal{K}} \leq 0$. From the second inequality we conclude that μ is a positive measure, and we may assume that μ is a probability measure. Letting $p_0 = \iint p \, d\mu(p, q)$ and $q_0 = \iint q \, d\mu(p, q)$, i.e., taking the barycenters of the projections of the probability measures on the first and second coordinates, we have from the first inequality that

$$0 \leq \iint e_{\alpha, \nu, \beta}(p, q) \, d\mu(p, q) = p_0(\alpha\alpha^*) + q_0(\beta^*\beta) - 2 \operatorname{Re} F(\alpha\nu\beta)$$

for any α, ν, β as above, and thus we have (1).

Letting π be the standard representation of \mathbf{M}_n , i.e., the representation

$$\pi(a) = \overbrace{a \oplus \cdots \oplus a}^n$$

on $(\mathbb{C}^n)^n = \mathbb{C}^{n^2}$, there exist unit vectors ξ and η in \mathbb{C}^{n^2} with $p_0(a) = \pi(a)\xi \cdot \xi$ and $q_0(a) = \pi(a)\eta \cdot \eta$ for $a \in \mathbf{M}_n$ (this well-known fact follows easily from [6, 4.6.18(ii)]). We let K and L be the subspaces of all vectors of the form $\pi(\alpha)^*\xi$ ($\alpha \in \mathbf{M}_{n,1}$) and $\pi(\beta)\eta$ ($\beta \in \mathbf{M}_{1,n}$), respectively. Defining $\theta: \mathbb{C}^n \rightarrow (\mathbb{C}^n)^n = \mathbb{C}^{n^2}$ by

$$\theta(\alpha_1, \dots, \alpha_n) = ((\alpha_1, 0, \dots, 0), (\alpha_2, 0, \dots, 0), \dots, (\alpha_n, 0, \dots, 0)),$$

we have that K and L are subspaces of $\theta(\mathbb{C}^n)$. We define a sesquilinear form $\langle \cdot, \cdot \rangle_v$ on $L \times K$ by

$$\langle \pi(\beta)\eta, \pi(\alpha)^*\xi \rangle_v = F(\alpha\nu\beta).$$

This is bounded by $\|v\|$ since we have that

$$\begin{aligned} |\langle \pi(\beta)\eta, \pi(\alpha)^*\xi \rangle_v| &\leq p_0(\alpha\alpha^*)^{1/2} \|v\| q_0(\beta^*\beta)^{1/2} \\ &= \|v\| \|\pi(\alpha)^*\xi\| \|\pi(\beta)\eta\|. \end{aligned}$$

Letting K_0 and L_0 be the subspaces of \mathbb{C}^n with $\theta(K_0) = K$ and $\theta(L_0) = L$, we have a corresponding sesquilinear function on $L_0 \times K_0$, and this determines a unique linear map $\rho(v): L_0 \rightarrow K_0$ of norm $\leq \|v\|$ such that

$$F(\alpha\nu\beta) = \rho(v)\theta^*\pi(\beta)\eta \cdot \theta^*\pi(\alpha)^*\xi.$$

Letting $\varphi(v) = \rho(v)E$ where E is the projection of \mathbb{C}^n on L_0 and using matrix notation, we may consider $\varphi(v)$ as an element of \mathbf{M}_n . Thus regarding ξ and η as $n^2 \times 1$ matrices,

$$(2) \quad F(\alpha\nu\beta) = \xi^*(\pi(\alpha)\theta)\varphi(v)(\theta^*\pi(\beta)\eta)$$

where

$$\theta^* \pi(\beta) = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & \cdots & \beta_1 & \beta_2 & \cdots \\ & & \cdots & & & \cdots \end{bmatrix},$$

$$\pi(\alpha)\theta = \begin{bmatrix} \alpha_1 & 0 & \cdots \\ \alpha_2 & 0 & \cdots \\ \vdots & \vdots & \vdots \\ 0 & \alpha_1 & \cdots \\ 0 & \alpha_2 & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

are $n \times n^2$ and $n^2 \times n$ matrices. It follows that if $\mathbf{v} = [v_{ij}] \in \mathbf{M}_n(V)$, $f_j = [0, \dots, 1_j, 0, \dots, 0] \in \mathbf{M}_{1,n}$, and $e_i = f_i^* \in \mathbf{M}_{n,1}$ then

$$F(\mathbf{v}) = \sum F(e_i v_{ij} f_j) = \sum \xi^* \pi(e_i) \theta \varphi(v_{ij}) \theta^* \pi(f_j) \eta = \gamma \varphi^{(n)}(\mathbf{v}) \delta$$

where $\gamma = \xi^* [\pi(e_1) \theta \cdots \pi(e_n) \theta]$ has norm ≤ 1 since

$$[\pi(e_1) \theta \cdots \pi(e_n) \theta] = \left[\begin{bmatrix} 1 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \vdots \\ 0 & 1 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots \\ 1 & 0 & \cdots \\ \vdots & \vdots & \vdots \\ 0 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix} \cdots \right]$$

has norm 1 (deleting the inner brackets, we obtain a permutation matrix, i.e., each row and each column contains precisely one “1”), and similarly

$$\delta = \begin{bmatrix} \theta^* \pi(f_1) \\ \vdots \\ \theta^* \pi(f_n) \end{bmatrix} \eta$$

has norm ≤ 1 .

Finally we have that φ is a complete contraction since given $\mathbf{v} \in \mathbf{M}_p(V)$ and vectors $\zeta = (\theta^* \pi(\alpha_k)^* \xi)$ and $\omega = (\theta^* \pi(\beta_l) \eta)$ in $(\mathbb{C}^n)^p$, and letting α and β be the row and column matrices with entries $\alpha_k \in \mathbf{M}_{n,1}$ and $\beta_l \in \mathbf{M}_{1,n}$, respectively, we have from (2) that

$$\begin{aligned} |\varphi^{(p)}(\mathbf{v}) \omega \cdot \zeta| &= \left| \sum \varphi(v_{kl}) \theta^* \pi(\beta_l) \eta \cdot \theta^* \pi(\alpha_k)^* \xi \right| \\ &= \left| \sum F(\alpha_k v_{kl} \beta_l) \right| = |F(\alpha \mathbf{v} \beta)| \leq p_0(\alpha \alpha^*)^{1/2} q_0(\beta^* \beta)^{1/2} \|\mathbf{v}\| \\ &= \left[\sum p_0(\alpha_k \alpha_k^*) \right]^{1/2} \left[\sum q_0(\beta_l^* \beta_l) \right]^{1/2} \|\mathbf{v}\| \\ &= \left[\sum \|\pi(\alpha_k)^* \xi\|^2 \right]^{1/2} \left[\sum \|\pi(\beta_l) \eta\|^2 \right]^{1/2} \|\mathbf{v}\| = \|\zeta\| \|\omega\| \|\mathbf{v}\|. \quad \square \end{aligned}$$

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