

SHADOWING PROPERTY OF CONTINUOUS MAPS WITH ZERO TOPOLOGICAL ENTROPY

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ABSTRACT. The study of the shadowing property has a long history but for interval maps it is rather new. Recent research in this direction is mainly focused on the positive entropy maps and work for zero entropy is still seldom to be found in the literature. In this paper we give a characterization of zero topological entropy maps which have the shadowing property. Moreover, our condition is necessary for any continuous function to have the shadowing property.

1. INTRODUCTION

Let $C^0(I, I)$ denote the class of continuous maps $I \rightarrow I$, where I is a compact real interval. The orbit (or trajectory) of $x \in I$ with respect to f is the sequence $\text{orb}(x) = \{f^n(x)\}_{n=0}^\infty$ where f^n denotes the n th iterate of f . Denote the set of periodic points of f by $\text{Per}(f)$, the set of the fixed points of f by $\text{Fix}(f)$, and the topological entropy of f by $E(f)$. The interval $J \subset I$ is called a periodic interval with period $\text{per}(J) = k \in \mathbb{N}$ if $f^k(J) = J$ and $f^i(J) \cap f^j(J) = \emptyset$ for $0 \leq i \neq j < k$. If J is degenerate to a point then it may be called a periodic point. We will denote a closed interval with $x \leq y$ by $[x, y]$, and a closed interval where no information about order of x, y is provided by $[x, y]^*$.

Definition 1.1. If $f \in C^0(I, I)$ and $\delta > 0$ is given, then a sequence $\mathbf{X}_\delta = \{\mathbf{x}_i\}_{i=0}^\infty$ of points in I is called a δ -chain of f (or a δ -pseudo orbit of f) provided that

$$|f(\mathbf{x}_i) - \mathbf{x}_{i+1}| \leq \delta \quad \text{for every } i \geq 0.$$

Given $\varepsilon > 0$, a δ -chain \mathbf{X}_δ is said to be ε -shadowed by $y \in I$ if

$$|f^i(y) - \mathbf{x}_i| \leq \varepsilon \quad \text{for every } i \geq 0;$$

f is said to have the *shadowing property* if for any $\varepsilon > 0$ there is $\delta > 0$ such that every δ -chain of f can be ε -shadowed by a point in I .

Definition 1.2. Let $f \in C^0(I, I)$. We will call f a *shrink function* if and only if for every sequence $\{J_k\}_{k=0}^\infty$ of periodic intervals such that $J_{k+1} \subset J_k$ and $\text{per}(J_{k+1}) > \text{per}(J_k)$ we have that $\lim_{k \rightarrow \infty} |J_k| = 0$.

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Definition 1.3. We will call a one-sided neighborhood $[p, q]^*$ of the periodic point p an m - f -nontrapping neighborhood of p if $f^m(p) = p$ and, for every $x \in [p, q]^*$, $x \in f^m([p, x]^*)$. If $m = 1$ then we will say that $[p, q]^*$ is an f -nontrapping neighborhood of p .

Definition 1.4. We will call $f \in C^0(I, I)$ a *nondegenerate function* if the following condition holds:

If $x \in I$, $p \in \text{Per}(f)$, $[p, q]^*$ is an m - f -nontrapping neighborhood of p , and $\lim_{n \rightarrow \infty} f^{mn}(x) = p$, then for every neighborhood O_x of x and for all $z_1, z_2 \in (p, q)^*$ there is an $n_0 \in \mathbb{N}$ such that $[z_1, z_2]^* \subset f^{mn_0}(O_x)$.

Main Theorem. Let $f \in C^0(I, I)$ and $E(f) = 0$. Then f has the shadowing property if and only if f is a nondegenerate shrink function.

Remark 1.5 (cf. Preiss and Smítal [13]). Let $f \in C^0(I, I)$ and $E(f) = 0$. If f has the shadowing property then f is nonchaotic stable (this means that a perturbed map can be chaotic but the chaos must be small whenever the perturbation is small).

Remark 1.6. If we use the results from [11], we can easily obtain similar results for continuous maps of the circle.

2. NECESSARY CONDITIONS

Let I be a compact interval and A a subset of I . Denote by $|A|$ the diameter of A , $\text{int}(A)$ the interior of A , $\text{conv}(A)$ the convex hull of A , and \bar{A} the closure of A .

First we recall the following well-known fact (see [8]):

Lemma 2.1. Let $f \in C^0(I, I)$ and $n \in \mathbb{N}$. Then f has the shadowing property if and only if f^n does.

Proposition 2.2. Let $f \in C^0(I, I)$. If f is a nonshrink function then it does not have the shadowing property.

Proof. If f is a nonshrink function then there is a sequence $\{J_k\}_{k=0}^\infty$ of periodic intervals from Definition 1.2 such that $\lim_{k \rightarrow \infty} |J_k| > 0$. Let $J = \bigcap_{k=0}^\infty J_k$. We have that $J = [p, q]$ where $p \neq q$.

Set $\varepsilon = \frac{|p-q|}{2}$ and $x_0 = \frac{p+q}{2}$, and let $\delta > 0$. There are $j, k \in \mathbb{N}$ such that

$$(1) \quad |f^j(J_k)| < \delta,$$

but there is $x_j \in f^j(J_k)$ such that $f^{2^k-j}(x_j) = x_0$. Set

$$\begin{aligned} \mathbf{x}_i &= f^i(\mathbf{x}_0) \quad \text{for } 0 < i < j, \\ \mathbf{x}_{j+i} &= f^i(\mathbf{x}_j) \quad \text{for } 0 < i. \end{aligned}$$

We have that $\{\mathbf{x}_i\}_{i=0}^\infty$ is a δ -chain (see (1)) and $\mathbf{x}_0 = \mathbf{x}_{2^k}$.

Now if $y \notin J$ then $|y - x_0| > \varepsilon$ and if $y \in J$ then $|f^{2^k}(y) - x_{2^k}| > \varepsilon$ because $J \cap f^i(J) = \emptyset$ for $i \in \mathbb{N}$.

Hence we have found a δ -chain which is not ε -shadowed. \square

Proposition 2.3. Let $f \in C^0(I, I)$ be a degenerate function. Then f does not have the shadowing property.

Proof. Let $x \in I$, $\lim_{n \rightarrow \infty} f^{mn}(x) = p$, $[p, q]^*$ be an m - f -nontrapping neighborhood of p , and let there be $z_1, z_2 \in (p, q)^*$ and a neighborhood O_x of x such that

$$(2) \quad [z_1, z_2]^* \not\subset f^{mn}(O_x) \quad \text{for all } n \in \{0, 1, \dots\}.$$

Without loss of generality assume that $[p, q]^* = [p, q]$ and $z_1 < z_2$. Moreover, we can assume that $m = 1$ (see Lemma 2.1).

Let $\varepsilon > 0$ such that

$$(3) \quad |q - z_2| > \varepsilon \quad \text{and} \quad [x - \varepsilon, x + \varepsilon] \subset O_x.$$

Now we claim that for any $\delta > 0$ there exists X_δ which is not ε -shadowed.

There is an $n_\delta \in \mathbb{N}$ such that

$$(4) \quad f^n(x) < z_1 \quad \text{for all } n \geq n_\delta,$$

$$(5) \quad |p - f^{n_\delta}(x)| < \delta.$$

Let $k_\delta \in \mathbb{N}$ be such that $k_\delta > \frac{|q-p|}{\delta}$. Set $\mathbf{x}_{n_\delta+k_\delta} = q$, and for $i \in \{1, \dots, k_\delta\}$ define

$$\begin{aligned} \text{if } \mathbf{x}_{n_\delta+k_\delta-i+1} - \delta \leq p & \quad \text{then } \mathbf{x}_{n_\delta+k_\delta-i} = p, \\ \text{if } \mathbf{x}_{n_\delta+k_\delta-i+1} - \delta > p & \quad \text{then } \mathbf{x}_{n_\delta+k_\delta-i} \in [p, \mathbf{x}_{n_\delta+k_\delta-i+1} - \delta] \\ & \quad \text{and } f(\mathbf{x}_{n_\delta+k_\delta-i}) = \mathbf{x}_{n_\delta+k_\delta-i+1} - \delta \end{aligned}$$

(this definition is correct because $[p, q]$ is an f -nontrapping neighborhood of p), and let $\mathbf{x}_i = f^i(x)$ for $i \in \{0, \dots, n_\delta - 1\}$.

Hence $\{\mathbf{x}_i\}_{i=0}^{n_\delta+k_\delta}$ is a δ -chain (see (5)). If $y \notin O_x$ then $|y - \mathbf{x}_0| > \varepsilon$ (see (3)) and if $y \in O_x$ then $f^{n_\delta+k_\delta}(y) < z_2$ (see (2) and (4)) and so $|\mathbf{x}_{n_\delta+k_\delta} - f^{n_\delta+k_\delta}(y)| > \varepsilon$ (see (3)).

So our δ -chain cannot be ε -shadowed. \square

Remark 2.4. Note that we do not use the assumption that $E(f) = 0$, so Proposition 2.2 and Proposition 2.3 give us a necessary condition for any function from $C^0(I, I)$ to have the shadowing property.

3. COVER SYSTEM

We begin with

Definition 3.1. Let $f \in C^0(I, I)$. We will call $\{M_i\}_{i=0}^\infty$ a *cover system* for f if and only if $M_i \subset I$ is a minimal closed set such that $M_i = \overline{\bigcup_{k \in K_i} I_k^i}$ where I_k^i is a closed periodic interval with period 2^i and, for all $p \in \text{Per}(f)$, $f^{2^i}(p) \neq p$ implies $p \in I_k^i$ for some $k \in K_i$.

Lemma 3.2. Let $f \in C^0(I, I)$, $E(f) = 0$, $p \in \text{Per}(f)$, and $c \in \text{Fix}(f)$. If $c \notin \text{conv}(\text{orb}(p))$, then $c \notin J$ where $J = \bigcup_{i=1}^\infty f^i(\text{conv}(\text{orb}(p)))$.

Proof. Assume that $c \notin \text{conv}(\text{orb}(p))$ and $c \in J$. Then $\text{per}(p) = m > 1$. Let $a = \min(\text{orb}(p))$ and $b = \max(\text{orb}(p))$. We can assume that $c > b$. Because $c \in J$, there is $d \in (a, b)$ and $k \in \mathbb{N}$ such that $f^{k+i}(d) > b$ for $i \in \{1, \dots, m\}$. But there are $i, j \in \{1, \dots, m\}$ such that $f^{k+i}(a) = a$ and $f^{k+j}(b) = a$. Hence $[a, b] \subset f^{k+i}([a, d]) \cap f^{k+j}([d, b])$, and we have that $E(f) > 0$ —a contradiction. \square

Lemma 3.3. *Let $f \in C^0(I, I)$ and $E(f) = 0$. Then for all $n \in \{0, 1, \dots\}$ and $p \in \text{Per}(f)$ such that $f^{2^n}(p) \neq p$ there is a closed periodic interval $J \subset I$ with period 2^n such that $p \in J$.*

Proof. If $n = 0$ then we can take $J = \bigcap_{i=1}^\infty f^i(I)$. Now we use induction. Let Lemma 3.3 hold for n . We will prove it for $n + 1$.

Let $p \in \text{Per}(f)$ and $f^{2^{n+1}}(p) \neq p$. From [2, Lemma 8] we have that $\text{orb}(p) = S_1 \cup S_2$ where $S_1 < c < S_2$, $c \in \text{Fix}(f)$, $f(S_1) = S_2$, $f(S_2) = S_1$, and S_1, S_2 are periodic orbits for $g = f^2$ with period $\frac{\text{per}(p)}{2} > 2^n$.

Let $J_1 = \overline{\bigcup_{i=1}^\infty f^{2i}(\text{conv}(S_1))}$ and $J_2 = \overline{\bigcup_{i=1}^\infty f^{2i}(\text{conv}(S_2))}$. We have that $f(J_1) = J_2$ and $f(J_2) = J_1$.

If, for example, $p \in S_1$, then $p \in \text{Per}(g|_{J_1})$ and $f^{2^n}(p) \neq p$. Hence there is closed periodic interval $J \subset J_1$ for function g such that $\text{per}(J) = 2^n$ and $p \in J$. But from Lemma 3.2 we have $J_1 \cap J_2 = \emptyset$ ($c \notin J_1 \cup J_2$), and from here it is easy to see that J is a periodic interval for function f with period 2^{n+1} . \square

Lemma 3.4. *Let $f \in C^0(I, I)$ and $E(f) = 0$. Let $\{J_k\}_{k \in K}$ be a set of periodic intervals with period 2^m , and let $J = \bigcup_{k \in K} J_k$ be an interval. Then J is a periodic interval with period 2^{m-1} or 2^m .*

Proof. We have that $f^{2^m}(J) = \overline{\bigcup_{k \in K} f^{2^m}(J_k)} = \overline{\bigcup_{k \in K} J_k} = J$, and hence it suffices to show that if $f^{2^{m-2}}(p) = p$ then $p \notin J$.

Assume that $p \in J$ and $f^{2^{m-2}}(p) = p$.

The set $P = \{x \in I; f^{2^{m-1}}(x) = x\}$ is closed and hence from the construction of J :

- there is an interval $J^* = [p, q]^* \subset J$; $p \neq q$ such that $P \cap (p, q)^* = \emptyset$ and we can take for J^* the maximal of such intervals, or
- there is an interval $J^* = [q_1, q_2] \subset J$; $q_1 \neq q_2$ such that $P \cap [q_1, q_2] = \{q_1, q_2\}$.

In both cases, $J^* = \overline{\bigcup_{k \in K^*} J_k}$ where $K^* \subset K$ and $f^{2^{m-1}}(J^*) \cap J^* \neq \emptyset$. Because $f^{2^m}(J^*) = J^*$, it follows that $f^{2^{m-1}}(J^*) = J^*$ and for $k \in K^*$ we have $(J_k \cup f^{2^{m-1}}(J_k)) \subset J^*$. Set $J_k^* = \text{conv}(J_k \cup f^{2^{m-1}}(J_k))$. We have that $f^{2^{m-1}}(J_k^*) \supset J_k^*$ and hence there is $x \in J_k^*$ such that $f^{2^{m-1}}(x) = x$. But $x \notin J_k \cup f^{2^{m-1}}(J_k)$ and hence $x \in \text{int}(J_k^*) \subset \text{int}(J^*)$ —a contradiction to the construction of J^* . \square

Lemma 3.5. *Let $f \in C^0(I, I)$ and $E(f) = 0$. Then the following conditions hold:*

- (i) *There is exactly one cover system for f , $\{M_i\}_{i=0}^\infty$.*
- (ii) *$f(M_i) = M_i$.*
- (iii) *$M_{i+1} \subset M_i$ for $i \in \{0, 1, \dots\}$.*
- (iv) *We can say that $M_i = \bigcup_{k \in \mathbb{R}_i} J_k^i$ where $\mathbb{R}_i \subset \mathbb{R}$, $J_{k_1}^i \cap J_{k_2}^i = \emptyset$ for $k_1 \neq k_2$, $J_k^i \subset I$ is a closed interval, and for any interval $J \subset M_i$ there is $k \in \mathbb{R}_i$ such that $J \subset J_k^i$. Now if $|J_k^i| > 0$ then J_k^i is a periodic interval with period 2^i or 2^{i-1} .*

Proof. (i) Let $p \in \text{Per}(f)$ and $i \in \{0, 1, 2, \dots\}$. If $f^{2^i}(p) = p$ then let $I_p^i = \emptyset$ else let I_p^i be the minimal closed periodic interval with period 2^i such

that $p \in I_p^i$. The existence of I_p^i follows from Lemma 3.3 (I_p^i is the intersection of all closed 2^i -periodic intervals which contain p). Now let $M_i = \overline{\bigcup_{p \in \text{Per}(f)} I_p^i}$ and it is easy to see that $\{M_i\}_{i=0}^\infty$ is a unique (from minimality) cover system.

(ii) It is easy to see that $f(I_p^i) = I_{f(p)}^i$ for any $p \in \text{Per}(f)$ and $i \in \{0, 1, \dots\}$. Now because $f(\text{Per}(f)) = \text{Per}(f)$, we have that $f(M_i) = M_i$.

(iii) From the proof of Lemma 3.3 we have that $I_p^{i+1} \subset I_p^i$ and hence $M_{i+1} \subset M_i$.

(iv) The first part is only the decomposition of a closed subset of a real interval into closed connected subsets, and the second part follows from Lemma 3.4. \square

Lemma 3.6. *Let $f \in C^0(I, I)$, $E(f) = 0$, and $\{M_i\}_{i=0}^\infty$ be a cover system for f , where $M_i = \bigcup_{k \in \mathbb{R}_i} J_k^i$ as in Lemma 3.5. Then if $J_k^0 = [a, b]$ where $a < b$, then $f(a) \neq a$ and $f(b) \neq b$.*

Proof. Let $P = \{p \in \text{Per}(f) \cap J_k^0; f(p) \neq p\}$ ($P \neq \emptyset$ from minimality of M_0) and $J_p = \overline{\bigcup_{i=1}^\infty f^i(\text{conv}(\text{orb}(p)))}$ for $p \in P$. Let $p_0, p \in P$. We have that $J_{p_0} \cap \text{orb}(p) = \emptyset$ or $J_p \subset J_{p_0}$.

If $\text{conv}(\text{orb}(p)) \cap J_{p_0} = \emptyset$ then there is a fixed point between $\text{conv}(\text{orb}(p))$ and J_{p_0} (because

$$f(\max(\text{orb}(p))) < \max(\text{orb}(p)), \quad f(\min(\text{orb}(p))) > \min(\text{orb}(p)),$$

and Lemma 3.2) and hence $J_{p_0} \cap J_p = \emptyset$.

If $J_{p_0} \cap J_p \neq \emptyset$ and $|J_p| > |J_{p_0}|$ then $J_{p_0} \subset \text{conv}(\text{orb}(p))$. So we have that $\text{orb}(p) = S_1 \cup S_2$ where $S_1 < J_{p_0} < S_2$, and because there is a fixed point in J_{p_0} from [2, Lemma 8] we have that $f(S_1) = S_2$ and $f(S_2) = S_1$. Hence it is easy to see that if there is a sequence $\{p_i\}_{i=0}^\infty$, $p_i \in P$, such that $J_{p_i} \subset J_{p_{i+1}}$ and $|J_{p_i}| < |J_{p_{i+1}}|$ then $\lim_{i \rightarrow \infty} J_{p_i} = [a^*, b^*]$ where $f(a^*) = b^*$ and $f(b^*) = a^*$.

Hence we can assume that $p_0 \in P$ such that for any $p \in P$ either $J_p \subset J_{p_0}$ or $J_p \cap J_{p_0} = \emptyset$. Now if $J_k^0 = J_{p_0}$ then we are done (see Lemma 3.2). So we can assume that $J_{p_0} = [a_0, b_0]$ and $a < a_0$. Let $J^* = \overline{\bigcup_{p \in P} J_p}$. If $P \cap [a, a_0] = \emptyset$ then $J^* \cap [a, a_0] = \emptyset$. If $P \cap [a, a_0] \neq \emptyset$ then there is a fixed point $c \in (a, a_0)$ (the same arguments as above) and we can assume that it is maximal. Then $P \cap [c, a_0] = \emptyset$ and hence (Lemma 3.2) $J^* \cap (c, a_0) = \emptyset$. So in both cases we can replace J_k^0 by J^* , which is a contradiction to minimality of M_0 . \square

4. REDUCTION FUNCTIONS

Definition 4.1. Let $I = [a, b]$ be an interval and $\mathcal{A} = \bigcup_{i \in K} I_i$ where $I_i \subset I$ is a closed interval and $I_i \cap I_j = \emptyset$ for all $i \neq j$. We will call a function $h \in C^0(I, I)$ a *reduction function* for \mathcal{A} if

$$h(x) = x - |[a, x] \cap I_{j_x}| - \sum_{I_i \subset [a, x]} |I_i| \quad \text{where } x \in \text{int}(I_{j_x})$$

(semihomeomorphism h reduces the intervals I_i to the points).

Lemma 4.2. *Let $f \in C^0(I, I)$ be a nondegenerate function, $E(f) = 0$, $\{M_i\}_{i=0}^\infty$ be a cover system for f , where $M_i = \bigcup_{k \in \mathbb{R}_i} J_k^i$ as in Lemma 3.5, h be a*

reduction function for M_i , and $J = h(I)$. Then there is $g \in C^0(J, J)$ such that $h \circ f = g \circ h$ and g is a nondegenerate function.

Proof. The existence of a $g \in C^0(J, J)$ such that $h \circ f = g \circ h$ follows from Lemma 3.5.

We can assume that $i = 0$. If not, we only replace function f by f^{2^i} and g by g^{2^i} (it is easy to see that g is nondegenerate if and only if g^{2^i} is).

Let $J_k^0 = [a, b]$, where $a < b$. Lemma 3.6 implies that there are $a^*, b^* \in I$ such that $a^* < a$, $b < b^*$, and if $x \in [a^*, a]$ ($x \in [b, b^*]$) then $f(x) \geq a$ ($f(x) \leq b$). Hence if $x \in h([a^*, a])$ ($x \in h([b, b^*])$) then $g(x) \geq h([a, b])$ ($g(x) \leq h([a, b])$). But we have that $\text{Per}(g) = \text{Fix}(g)$ and $|h([a^*, a])| \neq 0 \neq |h([b, b^*])|$ (see Lemma 3.5). Hence we have that

(6) if $|J_k^0| > 0$ then there is no g -nontrapping neighborhood of $h(J_k^0)$.

Let $x \in J$, O_x be a neighborhood of x , $p \in \text{Per}(g)$, $[p, q]^*$ be an m - g -nontrapping neighborhood of p , $\lim_{n \rightarrow \infty} g^{mn}(x) = p$, and $z_1, z_2 \in (p, q)^*$.

We will show that $m = 1$. Let, for example, $p < q$. If $[p, q]$ is not a g -nontrapping neighborhood of p , then there is $z \in [p, q]$ such that for all $x \in [p, z]$ we have $g(x) < z$. Now from [14, Theorem 1] $g^m(x) < z$ for all $m \in \mathbb{N}$ and $x \in [p, z]$. Hence $[p, q]$ is not an m - g -nontrapping neighborhood of p for any $m \in \mathbb{N}$, which is a contradiction.

Hence we have that $m = 1$ and from (6) that $h^{-1}(p) = p_f \in \text{Fix}(f)$.

Let $h^{-1}([p, q]^*) = [p_f, q_f]^*$. We can assume that $p < q$ and hence $p_f < q_f$. If $[p_f, q_f]$ is not an f -nontrapping neighborhood of p_f , then there are $z_f, z_f^* \in [p_f, q_f]$ such that $z_f^* < z_f$ and $f([p_f, z_f]) \cap [p_f, z_f] \subset [p_f, z_f^*]$. Because $[p, q]$ is a g -nontrapping neighborhood of p , there is $J_k^0 = [a, b] \subset M_0$ such that $z_f, z_f^* \in J_k^0$. We have that $p_f < a$ and hence $f([a, z_f]) \subset [a, z_f^*]$, and similarly as in the proof of Lemma 3.6 we can replace J_k^0 by the set J^* such that $J^* \cap (z_f^*, z_f) = \emptyset$, which is a contradiction to the minimality of M_0 . Hence we have that $[p_f, q_f]^*$ is an f -nontrapping neighborhood of p_f .

Now let $x_f \in h^{-1}(x)$ and $z_{i_f} \in h^{-1}(z_i)$ for $i = 1, 2$.

We have that $z_{1_f}, z_{2_f} \in (p_f, q_f)^*$, $\lim_{n \rightarrow \infty} f^n(x_f) = p_f$, and there is a neighborhood O_{x_f} of x_f such that $h(O_{x_f}) \subset O_x$.

Because f is nondegenerate, there is $n_0 \in \mathbb{N}$ such that $[z_{1_f}, z_{2_f}] \subset f^{n_0}(O_{x_f})$ and hence we have that $[z_1, z_2] \subset g^{n_0}(O_x)$. \square

Lemma 4.3. Let $f \in C^0(I, I)$ be a shrink function, $E(f) = 0$, and $\{M_i\}_{i=0}^\infty$ be a cover system for f . Then for all $\varepsilon > 0$ there is $\varepsilon^* > 0$ and $i \in \mathbb{N}$ such that if h_i is a reduction function for M_i and $J \subset I$ is an interval with $|J| > \varepsilon$ then $|h_i(J)| > \varepsilon^*$. (In other words, h_i^{-1} blows up small intervals to small intervals.)

Proof. Assume the contrary.

Let there be $\varepsilon > 0$, an increasing sequence $\{n_i\}_{i=1}^\infty$, and a set of intervals $\{I_i\}_{i=1}^\infty$ such that $|I_i| > \varepsilon$ and $|h_{n_i}(I_i)| \leq \frac{1}{i}$ for all $i \in \mathbb{N}$.

We can assume that $\bigcap_{i=1}^\infty I_i = J$ such that $|J| \geq \frac{\varepsilon}{2}$ (if necessary we can consider proper subsequences). Hence $\lim_{i \rightarrow \infty} |h_{n_i}(J)| = 0$, and from Lemma 3.5 we have $|h_i(A)| \geq |h_j(A)|$ for $i > j$ and $A \subset I$. Hence we have that $|h_i(J)| = 0$ for $i \geq 0$ and so $J \subset M_i$ for $i \geq 0$. So there is a sequence $\{J_k^i\}_{i=0}^\infty$ such that

$J \subset J_{k_i}^i$, and from Lemma 3.5 we obtain a contradiction to the assumption that f is a shrink function. \square

5. PROOF OF MAIN THEOREM

Theorem 5.1 (Gedeon and Kuchta [8]). *Let $f \in C^0(I, I)$, and let there be an $m \in \mathbb{N}$ such that if $p \in \text{Per}(f)$ then $f^m(p) = p$. Then f has the shadowing property if and only if f is a nondegenerate function.*

Remark 5.2. This is a slight variation of [8, Theorem 2]. In [8] there is an assumption that $\text{Per}(f)$ is nowhere dense in I and there is a different definition of nondegenerate function.

But from [8, Lemma 7] it is easy to see that we can change the used definition of a nondegenerate function for Definition 1.4.

Moreover, if $\text{Per}(f)$ is dense in some interval $J \subset I$ then we have $J \subset \text{Per}(f)$ (see [13, 4.1 Lemma]), and hence it is easy to see that f is a degenerate function and does not have the shadowing property. So this assumption is not necessary.

Main Theorem. *Let $f \in C^0(I, I)$ and $E(f) = 0$. Then f has the shadowing property if and only if f is a nondegenerate shrink function.*

Proof. \Rightarrow If f has the shadowing property then Propositions 2.2 and 2.3 imply that f is a nondegenerate shrink function.

\Leftarrow Let f be a nondegenerate shrink function and $\varepsilon > 0$. From Lemma 3.5 there is a cover system $\{M_i\}_{i=0}^\infty$ of f . From Lemma 4.3 there is $\varepsilon^* > 0$ and $i \in \mathbb{N}$ such that if h is a reduction function for M_i and $x, y \in I$ then

$$(7) \quad \text{if } |h(x) - h(y)| < \varepsilon^* \text{ then } |x - y| < \varepsilon.$$

Let $J = h(I)$. From Lemma 4.2 there is $g \in C^0(J, J)$ such that $h \circ f = g \circ h$ and g is a nondegenerate function. From Theorem 5.1 we have that g has the shadowing property. Hence there is $\delta > 0$ such that every δ -chain of g is ε^* -shadowed by some $y^* \in J$.

Now let $\mathbf{X}_\delta = \{\mathbf{x}_i\}_{i=0}^\infty$ be a δ -chain of f . Then $\mathbf{X}_\delta^* = \{h(\mathbf{x}_i)\}_{i=0}^\infty$ is a δ -chain of g and there is $y^* \in J$ such that $|g^i(y^*) - h(\mathbf{x}_i)| < \varepsilon^*$ for all $i \geq 0$.

Let $y \in h^{-1}(y^*)$. We have that $h(f^i(y)) = g^i(h(y)) = g^i(y^*)$ and from (7) we have $|f^i(y) - \mathbf{x}_i| < \varepsilon$ for all $i \geq 0$. \square

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