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THE INTEGRATION OPERATOR IN TWO VARIABLES

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ABSTRACT. In this paper we consider the integration operator in two variables on $L_2[0, 1]^2$, determine its multiplicity and reducing subspaces, and make some observations about its invariant and hyperinvariant subspaces.

1. INTRODUCTION

The purpose of this paper is to study the Volterra integration operator W in two variables, that is, the operator defined on $L_2[0, 1]^2$ by

$$(Wf)(x, y) \stackrel{\text{def}}{=} \int_0^y ds \int_0^x f(t, s)\, dt.$$

In particular we find its multiplicity and reducing subspaces and obtain some information on its invariant and hyperinvariant subspaces. It will follow from our results that the properties of W are quite different from the properties of the classical Volterra operator V (defined on $L_2[0, 1]$ by $(Vf)(x) \stackrel{\text{def}}{=} \int_0^x f(t)\, dt$). It is well known that V is compact and quasi-nilpotent. Since $W = V \otimes V$, the same properties are also shared by W. These facts are also easily verified directly.

Before describing the content of this paper, we introduce some notation and recall some definitions. For a complex Banach space X, we will denote by $L(X)$ the algebra of bounded linear operators on X. If A is a subalgebra of $L(X)$ which contains the identity operator, then a subset G of X is called cyclic for A, if the linear span of the set $\{Tx\colon\ x \in G,\ T \in A\}$ is dense in X. The smallest cardinality of a cyclic set for the algebra A is called the multiplicity of A and will be denoted by $m(A)$.

The multiplicity of an operator T in $L(X)$ is defined as the multiplicity of the algebra generated in $L(X)$ by T and the identity operator and will be denoted by $m(T)$.

The commutant of T is defined by

$$T' \stackrel{\text{def}}{=} \{B \in L(X) : TB = BT\}.$$

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A closed subspace M of X is called an invariant subspace of T, if T maps M into itself. If M is also invariant for every operator in T', then M is called hyperinvariant for T. Let $\mathscr{H}$ be a Hilbert space and $T \in L(\mathscr{H})$. A closed subspace E of $\mathscr{H}$ is called a reducing subspace for T, if E and $E^{\perp}$ are both invariant under T.

It is well known (see [2, Theorem 4.14]) that the invariant subspaces of V are exactly the subspaces M_a of the form $M_a = \{f \in L_2[0, 1]: f = 0$ a.e. on $[0, a]\}$ for some $0 < a < 1$. It follows from this, and it is also easily seen directly, that the function $f \equiv 1$ is cyclic for V; hence, $m(V) = 1$. It also follows from this description of invariant subspaces that V has no proper reducing subspaces. Also, since V is unicellular, by a general result (see [2, Corollary 6.27]), every invariant subspace of V is also hyperinvariant.

In §2 we prove that unlike V, the operator W has infinite multiplicity.

In §3 we consider the reducing subspaces of W and prove that the only such subspaces are S_+ and S_-, which consist of the symmetric functions and antisymmetric functions in $L_2[0, 1]^2$ respectively; that is,

$$S_+ = \{f \in L_2[0, 1]^2 : f(x, y) = f(y, x), \text{ a.e. on } [0, 1]^2\},$$
$$S_- = \{f \in L_2[0, 1]^2 : f(x, y) = -f(y, x), \text{ a.e. on } [0, 1]^2\}.$$

In §4 we give some examples of invariant and hyperinvariant subspaces of W; however, the complete characterization of these subspaces remains open.

2. The multiplicity of W

In this section we show that W has infinite multiplicity, that is, we prove

Theorem 1. $m(W) = \infty$.

The proof of the theorem will be based on a result from [1, Proposition 2.1]. For the sake of completeness we include its statement and proof.

Proposition 2. *Let T be an operator in $L(X)$, and assume that for some integer $n \geq 2$ there exists a nonzero continuous n-linear mapping ϕ of X^n into some topological vector space Y, such that, for every n-tuple $(x_1, \dots, x_n)$ in X^n for which $x_i = x_j$ for some $1 \leq i < j \leq n$ and for every pair of nonnegative integers (k_1, k_2), $\phi(x_1, x_2, \dots, T^{k_1}x_i, x_{i+1}, \dots, T^{k_2}x_j, \dots, x_n) = 0$. Then $n \leq m(T)$.*

Proof. Let A be the subalgebra of $L(X)$ generated by T and the identity operator. First we note that since the set $D = \{T^n : n \geq 0\}$ spans A, the assumption on ϕ implies that for every $(T_1, T_2) \in A \times A$ and for every n-tuple $(x_1, \dots, x_n)$ in X^n for which $x_i = x_j$ for some $1 \leq i < j \leq n$

$$\phi(x_1, x_2, \dots, T_1x_i, x_{i+1}, \dots, T_2x_j, \dots, x_n) = 0. \tag{1}$$

Let G be any subset of X which contains less than n elements, and consider the set $M = \operatorname{span}\{Sx : x \in G, S \in A\}$. The hypothesis that G contains less than n elements implies by (1) that $M^n \subseteq \ker\phi$, and therefore since ϕ is continuous, $\overline{M}^n = \overline{M^n} \subseteq \ker\phi$. Remembering that $\phi \not\equiv 0$, we conclude that $\overline{M}^n \neq X^n$ and therefore $\overline{M} \neq X$. $\square$

In the proof of the theorem it will be convenient to write W as a convolution operator. For $f, g \in L_2[0, 1]^2$ the convolution is defined by

$$(g \star f)(x, y) \stackrel{\text{def}}{=} \int_0^y ds \int_0^x f(t, s) g(x-t, y-s)\, dt.$$

It is known and easily verified that the convolution is commutative, associative, $g \star f \in C([0, 1]^2)$, and $\|f \star g\|_2 \leq \|f\|_2 \|g\|_2$. If we denote by U the function $U \equiv 1$ on $[0, 1]^2$ then it is clear that for every $f \in L_2[0, 1]^2$, $Wf = U \star f$.

In view of Proposition 2, the conclusion of Theorem 1 follows from

Proposition 3. *For every $n \geq 2$ there exists an n-linear mapping ϕ_n that satisfies the assumptions of Proposition 2 for W.*

Proof. For $a \geq 1$, we denote by $\square_a$ the rectangle $[0, 1] \times [0, 1/a]$ and by T_a the operator on $L_2[0, 1]^2$ defined by

$$(T_a f)(x, y) \stackrel{\text{def}}{=} \begin{cases} f(ay, x/a), & (x, y) \in \square_a, \\ 0, & \text{otherwise}. \end{cases} \tag{2}$$

It is easily verified that T_a is a continuous linear operator on $L_2[0, 1]^2$ and that for every $(x, y) \in \square_a$

$$(T_a W f)(x, y) = (W T_a f)(x, y). \tag{3}$$

Let $n > 1$, and choose $n-1$ real numbers $a_1, a_2, \dots, a_{n-1}$ such that $a_1 = 1$ and $a_k < a_{k+1}$ for $k = 1, 2, \dots, n-2$, and define the operator P_n on $L_2[0, 1]^2$ by

$$(P_n f)(x, y) = \begin{cases} f(x, y), & (x, y) \in \square_{a_{n-1}}, \\ 0, & \text{otherwise}. \end{cases}$$

For every $f_1, \dots, f_n$ in $L_2[0, 1]^2$ consider the matrix

$$A(f_1, f_2, \dots, f_n) = \begin{pmatrix} f_1 & \cdots & f_n \\ T_{a_1} f_1 & \cdots & T_{a_1} f_n \\ \vdots & & \vdots \\ T_{a_{n-1}} f_1 & \cdots & T_{a_{n-1}} f_n \end{pmatrix}$$

and define the mapping $\phi_n \colon (L_2[0, 1]^2)^n \to L_2[0, 1]^2$ by

$$\phi_n(f_1, \dots, f_n) \stackrel{\text{def}}{=} P_n\{\det[A(f_1, f_2, \dots, f_n)]\}, \tag{4}$$

where multiplication in $\det A$ is convolution. Since for every n functions $g_1, \dots, g_n$ in $L_2[0, 1]^2$ we have that

$$\|g_1 \star g_2 \star \cdots \star g_n\|_2 \leq \|g_1\|_2 \|g_2\|_2 \cdots \|g_n\|_2,$$

and since the operators T_{a_i} are continuous, it follows that ϕ_n is a continuous n-linear mapping of $(L_2[0, 1]^2)^n$ into $L_2[0, 1]^2$. Next, let $(f_1, \dots, f_n) \in (L_2[0, 1]^2)^n$ and assume that there exist $1 \leq i < j \leq n$ such that $f_i = W^r f$, $f_j = W^m f$ for some $f \in L_2[0, 1]^2$. We have to show that $\phi_n(f_1, \dots, f_n) \equiv 0$. First if $(x, y) \notin \square_{a_{n-1}}$ then, for every $f \in L_2[0, 1]^2$, $(P_n f)(x, y) = 0$ and so

$$[\phi_n(f_1, \dots, f_n)](x, y) = [P_n(\det A)](x, y) = 0.$$

It remains to prove that this holds also for $(x, y) \in \square_{a_{n-1}}$. By changing the order of the columns of A we may assume that $i = 1$ and $j = 2$, namely, $f_1 = W^r f$ and $f_2 = W^m f$. Then on $\square_{a_{n-1}}$

$$\phi_n(W^r f, W^m f, f_3, \ldots, f_n) \tag{5}$$
$$= \det \begin{pmatrix} W^r f & W^m f & f_3 & \cdots & f_n \\ T_{a_1} W^r f & T_{a_1} W^m f & T_{a_1} f_3 & \cdots & T_{a_1} f_n \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \\ T_{a_{n-1}} W^r f & T_{a_{n-1}} W^m f & T_{a_{n-1}} f_3 & \cdots & T_{a_{n-1}} f_n \end{pmatrix}.$$

For every $1 \le k < n-1$, $a_{n-1} > a_k$; hence, $\square_{a_{n-1}} \subset \square_{a_k}$. Therefore, using (3) and remembering that $Wf = U \star f$ and that convolution is associative and commutative, we obtain that on $\square_{a_{n-1}}$

$$\begin{aligned} &\phi_n(W^r f, W^m f, f_3, \ldots, f_n) \\ &= \det \begin{pmatrix} W^r f & W^m f & f_3 & \cdots & f_n \\ W^r T_{a_1} f & W^m T_{a_1} f & T_{a_1} f_3 & \cdots & T_{a_1} f_n \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \\ W^r T_{a_{n-1}} f & W^m T_{a_{n-1}} f & T_{a_{n-1}} f_3 & \cdots & T_{a_{n-1}} f_n \end{pmatrix} \\ &= W^{r+m} \det \begin{pmatrix} f & f & f_3 & \cdots & f_n \\ T_{a_1} f & T_{a_1} f & T_{a_1} f_3 & \cdots & T_{a_1} f_n \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \\ T_{a_{n-1}} f & T_{a_{n-1}} f & T_{a_{n-1}} f_3 & \cdots & T_{a_{n-1}} f_n \end{pmatrix}. \end{aligned} \tag{6}$$

Now in the last matrix the first two columns are the same on $\square_{a_{n-1}}$ and, therefore, the determinant vanishes on $\square_{a_{n-1}}$. Noticing that if g is any function that vanishes on some rectangle of the form $[0, a] \times [0, b]$—that is included in $[0, 1]^2$—then $W^k g$ also vanishes on that rectangle for every k, we conclude that $\phi_n(W^r f, W^m f, f_3, \ldots, f_n) = 0$ on $\square_{a_{n-1}}$.

It remains to show that ϕ_n is not identically zero. For this consider the functions $g_1(x, y) = 1$, $g_2(x, y) = x, \ldots, g_n(x, y) = x^{n-1}$. We claim that $\phi_n(g_1, \ldots, g_n) \not\equiv 0$. Indeed, by definition (2) and the fact that $\square_{a_{n-1}} \subseteq \square_{a_k}$ we have for (x, y) in the rectangle $\square_{a_{n-1}}$ and for every $1 \le k \le n-1$ that $(T_{a_k} g_m)(x, y) = g_m(a_k y, x/a_k) = a_k^{m-1} y^{m-1}$. By the definition of ϕ_n we get that on $\square_{a_{n-1}}$

$$\phi_n(g_1, \ldots, g_n) = \det \begin{pmatrix} 1 & x & \cdots & x^{n-1} \\ 1 & y & \cdots & y^{n-1} \\ 1 & a_2 y & \cdots & a_2^{n-1} y^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & a_{n-1} y & \cdots & a_{n-1}^{n-1} y^{n-1} \end{pmatrix}$$

where multiplication in the determinant is convolution. Denoting by M_{ij} the minors of the determinant, we obtain that on $\square_{a_{n-1}}$

$$\phi_n(g_1, \dots, g_n) = \sum_{k=1}^{n} (-1)^k x^{k-1} \star M_{1k}.$$

The highest power of x appears when $k = n$, namely, in the term $x^{n-1} \star M_{1n}$. Therefore, to show that $\phi_n(g_1, \dots, g_n) \not\equiv 0$, it suffices to show that $x^{n-1} \star M_{1n} \not\equiv 0$; but since M_{1n} is a polynomial, this is obviously true if $M_{1n} \not\equiv 0$. So it suffices to prove that

$$M_{1n} = \det \begin{pmatrix} 1 & y & \cdots & y^{n-2} \\ 1 & a_2 y & \cdots & a_2^{n-2} y^{n-2} \\ \vdots & \vdots & & \vdots \\ 1 & a_{n-1} y & \cdots & a_{n-1}^{n-2} y^{n-2} \end{pmatrix} \not\equiv 0.$$

It is easy to see that

$$M_{1n} = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & a_2 & \cdots & a_2^{n-2} \\ \vdots & \vdots & & \vdots \\ 1 & a_{n-1} & \cdots & a_{n-1}^{n-2} \end{pmatrix} U \star y \star \cdots \star y^{n-2}$$

where the multiplication in the last determinant is the usual multiplication. But the last determinant is the Van-der-Monde determinant of $a_1 = 1, a_2, \dots, a_{n-1}$ and hence, is equal to $\prod_{i>j\geq 1}(a_i - a_j)$, which is not zero since $a_i > a_j$, for $i > j$, so $M_{1n} \not\equiv 0$.

This completes the proof of Proposition 3 and, hence, also of Theorem 1. □

3. The reducing subspaces of W

We recall that a subspace E of $L_2[0, 1]^2$ is reducing for an operator T on $L_2[0, 1]^2$ if E and $E^\perp$ are both invariant under T, or equivalently if E is invariant under T and T^*. We denote by S_+ the symmetric functions in $L_2[0, 1]^2$, namely,

$$S_+ = \{f \in L_2[0, 1]^2 : f(x, y) = f(y, x) \text{ a.e. on } [0, 1]^2\},$$

and by S_- the antisymmetric functions in $L_2[0, 1]^2$, namely,

$$S_- = \{f \in L_2[0, 1]^2 : f(x, y) = -f(y, x) \text{ a.e. on } [0, 1]^2\}.$$

Consider the operator τ defined on $L_2[0, 1]^2$ by

$$(\tau f)(x, y) = f(y, x).$$

It is easily verified that τ commutes with W. Therefore, if $f \in S_+$ then

$$\tau(Wf) = W(\tau f) = Wf;$$

hence, $Wf \in S_+$. Similarly if $f \in S_-$ then $Wf \in S_-$; that is, S_+ and S_- are invariant under W. It is easy to see that S_- is the orthogonal complement of S_+, and, therefore, S_+, S_- are reducing subspaces of W.

The main result of this section is the following theorem.

Theorem 4. *The only nontrivial reducing subspaces of W are S_+ and S_-.*

For the proof of the theorem we shall need several lemmas.

Lemma 5. *Let T be an operator in a Hilbert space $\mathscr{H}$, and assume M is a reducing subspace for T. If λ is an eigenvalue of multiplicity one and x a corresponding eigenvector, then $x \in M$ or $x \in M^\perp$.*

Proof. Let λ be an eigenvalue of multiplicity one and x a corresponding eigenvector. Suppose $x = x_1 + x_2$, where $x_1 \in M$ and $x_2 \in M^\perp$. Then

$$\lambda x = Tx = Tx_1 + Tx_2. \tag{7}$$

Since M and $M^\perp$ are invariant under T, $Tx_1 \in M$ and $Tx_2 \in M^\perp$, hence by (7), $Tx_1 = \lambda x_1$ and $Tx_2 = \lambda x_2$. Since λ is of multiplicity one, this implies that $x_1 \equiv 0$ or $x_2 \equiv 0$, hence $x \in M$ or $x \in M^\perp$. □

A simple computation shows that the adjoint of W is given by

$$(W^*g)(t, s) = \int_s^1 dy \int_t^1 g(x, y)\, dx, \qquad g \in L_2[0, 1]^2.$$

Lemma 6. *For every integer $n \neq 0$, $r_n = i/2\pi n$ is an eigenvalue of multiplicity one of the operator $W - W^*$, and the corresponding eigenfunctions are constant multiples of the function $f_n(x, y) = e^{-2\pi i n x} - e^{-2\pi i n y}$.*

Proof. Let $\lambda \neq 0$ be an eigenvalue of $W - W^*$ and $F(x, y)$ a corresponding eigenfunction. Then $(W - W^*)F = \lambda F$. This implies that

$$\int_0^y ds \int_0^x F(t, s)\, dt - \int_y^1 ds \int_x^1 F(t, s)\, dt = \lambda F(x, y). \tag{8}$$

The left-hand side of (8) is a continuous function, so F is continuous, and therefore, the left-hand side is a differentiable function. Differentiating (8) with respect to y, we get that

$$\int_0^1 F(t, y)\, dt = \lambda \frac{\partial F}{\partial y}. \tag{9}$$

Differentiating (9) with respect to x we obtain the differential equation

$$\lambda \frac{\partial^2 F}{\partial x \partial y} = 0. \tag{10}$$

From the assumption that $\lambda \neq 0$, (10) implies that

$$F(x, y) = f(x) + g(y), \tag{11}$$

where f and g are differentiable functions on $[0, 1]$. Substituting this in (9) we obtain that

$$\int_0^1 [f(t) + g(y)]\, dt = \lambda \frac{dg}{dy};$$

hence,

$$C_1 + g(y) = \lambda \frac{dg}{dy},$$

where $C_1 = \int_0^1 f(t)\, dt$. The solution of this differential equation is

$$g(y) = Be^{y/\lambda} + C_1, \tag{12}$$

where B is a constant. Similarly we obtain that $f(x) = Ae^{x/\lambda} + C_2$, where A and C_2 are constants, and therefore we obtain that for some constant C

$$F(x, y) = Ae^{x/\lambda} + Be^{y/\lambda} + C, \tag{13}$$

and substituting again (13) in the equation $(W - W^*)F = \lambda F$, we obtain

$$\int_0^y dt \int_0^x (Ae^{t/\lambda} + Be^{s/\lambda} + C)\,dt - \int_y^1 ds \int_x^1 (Ae^{t/\lambda} + Be^{s/\lambda} + C)\,dt$$
$$= \lambda(Ae^{x/\lambda} + Be^{y/\lambda} + C).$$

This implies that

$$y[-\lambda A + \lambda A e^{1/\lambda} + C] + x[-\lambda B + \lambda B e^{1/\lambda} + C]$$
$$+ [-\lambda A e^{1/\lambda} - \lambda B e^{1/\lambda} - C - \lambda C] \equiv 0;$$

hence, we obtain the following three equations:

(1) $-\lambda A + \lambda A e^{1/\lambda} + C = 0$.
(2) $-\lambda B + \lambda B e^{1/\lambda} + C = 0$.
(3) $-\lambda A e^{1/\lambda} - \lambda B e^{1/\lambda} - C - \lambda C = 0$.

By subtracting (2) from (1) we get $(B - A)(1 - e^{1/\lambda}) = 0$.

Possibility a. $e^{1/\lambda} - 1 = 0$. The solutions are: $\lambda_n = i/2\pi n$ for a nonzero integer n and then $C = 0$ and $A = -B$; namely, the eigenvalues are $r_n = i/2\pi n$ and the corresponding eigenfunctions are constant multiples of the functions $f_n(x, y) = e^{-2\pi i n x} - e^{-2\pi i n y}$.

Possibility b. $A = B$. In this case, $C = -2\lambda A/(\lambda - 1)$ where λ is the solution of the equation $e^{1/\lambda} = (\lambda + 1)/(\lambda - 1)$. It is easily verified that $r_n = i/2\pi n$ is not a solution of the last equation. (The solutions of this equation give other eigenvalues, in which we are not interested here.) So for any $n \neq 0$, r_n is an eigenvalue of multiplicity one and f_n is a corresponding eigenfunction. □

Lemma 7. *Let* $P_{km}(x, y) = x^k y^m - x^m y^k$ *and* $Q_{km}(x, y) = x^k y^m + x^m y^k$. *Then*:

(1) $\overline{\text{span}}\{Q_{km}(x, y),\ k \geq m\} = S_+$, *and*
(2) $\overline{\text{span}}\{P_{km}(x, y),\ k > m\} = S_-$.

Proof. Let Q denote the set of symmetric polynomials in two variables—that is, $Q = \{q;\ q(x, y) = q(y, x),\ \forall (x, y) \in [0, 1]^2\}$—and P the set of antisymmetric polynomials in two variables—that is, $P = \{p;\ p(x, y) = -p(y, x),\ \forall (x, y) \in [0, 1]^2\}$. It is easily seen that Q is the linear span of the polynomials Q_{km} and P is the linear span of the polynomials P_{km}. This implies the lemma by observing that Q is dense in S_+ and P is dense in S_-.

Lemma 8. *If* M *is a reducing subspace for* W *then* $S_- \subseteq M$ *or* $S_- \subseteq M^\perp$.

Proof. Since M is a reducing subspace for W, it is also reducing for $W - W^*$. By Lemmas 5 and 6 we get that, for any $n \neq 0$, $f_n(x, y) = e^{-2\pi i n x} - e^{-2\pi i n y}$ belongs either to M or to $M^\perp$. In particular, $f(x, y) = e^{2\pi i x} - e^{2\pi i y}$ belongs either to M or to $M^\perp$.

We now show that if $f \in M$ then $S_- \subseteq M$. For every $n \geq 2$ consider the polynomial P_n defined by

$$P_n(x, y) = y^n - x^n + x - y.$$

First we claim that $P_n \in M$, $n = 1, 2, \ldots$. We prove this by induction. Since

$$f(x, y) = e^{2\pi i x} - e^{2\pi i y} \in M$$

and M is invariant under W and W^*,

$$f_1 = 4\pi i W^* W f + \frac{1}{2\pi i} f \in M$$

and

$$f_2 = 4\pi i W W^* f + 2Wf \in M.$$

A direct computation shows that

$$f_1(x, y) = x^2\left[\frac{1}{2\pi i} - 1 - \frac{1}{2\pi i}e^{2\pi i y} + y\right] - y^2\left[\frac{1}{2\pi i} - 1 - \frac{1}{2\pi i}e^{2\pi i x} + x\right] + (x - y)$$

and

$$f_2(x, y) = x^2\left[\frac{1}{2\pi i} - \frac{1}{2\pi i}e^{2\pi i y} + y\right] - y^2\left[\frac{1}{2\pi i} - \frac{1}{2\pi i}e^{2\pi i x} + x\right].$$

Therefore,

$$f_1(x, y) - f_2(x, y) = x - y - x^2 + y^2 = P_2(x, y) \in M.$$

A simple computation shows that

$$P_{n+1} = (n+1)[WP_n - W^* P_n + \tfrac{1}{2}P_2],$$

and therefore if we assume that $P_n \in M$, we obtain that also $P_{n+1} \in M$, and the claim is proved.

Next we claim that, for every $n \geq 1$, $x^n - y^n \in M$. Indeed, since M is closed and

$$\|P_n - (x - y)\|_2 = \|y^n - x^n\|_2 \leq \|y^n\|_2 + \|x^n\|_2 = 2\frac{1}{\sqrt{2n+1}} \to_{\to\infty} 0,$$

we conclude that $x - y \in M$. Since for every $n \geq 2$

$$P_n(x, y) = y^n - x^n + x - y \in M,$$

this implies that for every $n \geq 1$

$$x^n - y^n \in M \tag{14}$$

and, therefore, for $k > m$

$$W^m(x^{k-m} - y^{k-m}) = \frac{1}{m!\,(k-m+1)(k-m+2)\cdots k} P_{km}(x, y) \in M.$$

Hence $P_{km} \in M$ and by Lemma 6, this implies that $S_- \subseteq M$. Similarly one shows that if $f \in M^\perp$ then $S_- \subseteq M^\perp$. □

For every pair of nonnegative integers $n, m \geq 0$ denote

$$f_{nm}(x, y) = \cos\left(\frac{2n+1}{2}\pi x\right)\cos\left(\frac{2m+1}{2}\pi y\right).$$

Lemma 9. *The only eigenfunctions of the operator W^*W are constant multiples of the functions $\{f_{nm}\}_{n,m\geq 0}$, and the corresponding eigenvalues are $\lambda_{nm} = 16/(2n+1)^2(2m+1)^2\pi^4$.*

Proof. It is easy to verify that f_{nm} are eigenfunctions of W^*W and that λ_{nm} are the corresponding eigenvalues. Since it is well known that $\{f_{nm}\}_{n,m\geq 0}$ is a complete orthogonal system in $L_2[0, 1]^2$, it follows that there are no other eigenfunctions. □

Lemma 10. *If M is a reducing subspace for W then either $S_+ \subseteq M$ or $S_+ \subseteq M^\perp$.*

Proof. If M is reducing for W then M is also reducing for W^*W. Lemma 9 implies that $\lambda = 16/\pi^4$ is an eigenvalue of multiplicity one of the operator W^*W, and a corresponding eigenfunction is $f(x, y) = \cos(\pi x/2)\cos(\pi y/2)$. Hence by Lemma 5, f belongs either to M or to $M^\perp$.

We will show that if $f \in M$ then $S_+ \subseteq M$. For every $n \geq 1$ define $g_n(x, y) = 1 - x^n - y^n$. First we claim that $g_n \in M$ for $n = 1, 2, \ldots$. We prove this by induction. Since $f(x, y) = \cos(\pi x/2)\cos(\pi y/2) \in M$ and M is invariant under W and W^*,

$$f_1 = \frac{\pi^4}{16}W^2 f - f \in M$$

and

$$f_2 = \frac{\pi^3}{4(\pi - 2)}[(W^*)^2 f - WW^* f] \in M.$$

By a direct computation

$$f_1(x, y) = 1 - \cos\left(\frac{\pi}{2}x\right) - \cos\left(\frac{\pi}{2}y\right)$$

and

$$f_2(x, y) = 1 + \frac{2}{\pi}\left[1 - \cos\left(\frac{\pi}{2}x\right) - \cos\left(\frac{\pi}{2}y\right)\right] - (y + x).$$

This implies that

$$g_1 = f_2 - \frac{2}{\pi}f_1 \in M.$$

A simple computation shows that

$$g_{n+1} = (n+1)\left[\frac{n}{n+1}g_1 - W^* g_n + W g_n\right].$$

Therefore, if we assume that $g_n \in M$, we obtain that also $g_{n+1} \in M$, and the claim is proved. It is easy to see that $g_n \to 1$ in $L_2[0, 1]^2$, and therefore since M is closed, the function $U \equiv 1$ belong to M. Since $g_n \in M$, this implies that, for every $n \geq 0$, $x^n + y^n \in M$, and therefore, for every $0 \leq m \leq k$,

$$W^m(x^{k-m} + y^{k-m}) = \frac{1}{m!\,(k-m+1)\cdots k}Q_{km} \in M.$$

Hence $Q_{km} \in M$, and by Lemma 7 we conclude that $S_+ \subseteq M$. A similar argument shows that if $f \in M^\perp$ then $S_- \subseteq M^\perp$. □

Proof of Theorem 4. Let M be a reducing subspace for W. It follows from Lemmas 8 and 10 that there are four possibilities:

(1) $S_- \subseteq M$ and $S_+ \subseteq M$.
(2) $S_- \subseteq M^\perp$ and $S_+ \subseteq M^\perp$.
(3) $S_- \subseteq M$ and $S_+ \subseteq M^\perp$.
(4) $S_- \subseteq M^\perp$ and $S_+ \subseteq M$.

Since $S_- \oplus S_+ = L_2[0, 1]^2$, possibility (1) implies that $M = L_2[0, 1]^2$ and possibility (2) implies that $M^\perp = L_2[0, 1]^2$; hence, $M = \{0\}$. Since $S_+^\perp = S_-$, possibility (3) implies that $M = S_-$ and possibility (4) implies that $M = S_+$. This concludes the proof of the theorem. □

4. Invariant and hyperinvariant subspaces for W

It is easy to see that if E is a measurable subset of $[0, 1]^2$, which satisfies the condition

$$(x, y) \in E \Rightarrow [0, x] \times [0, y] \subseteq E,$$

then the subspace

$$M_E \stackrel{\text{def}}{=} \{f \in L_2[0, 1]^2 : f = 0 \text{ a.e. on } E\} \tag{15}$$

is an invariant subspace for W. These subspaces are in a sense analogous to the invariant subspaces of the classical Volterra operator V; however, there are many other invariant subspaces for W. For example, such are the subspaces S_+ and S_- considered in §3; and if G is any finite subset of $L_2[0, 1]^2$, then in view of Theorem 1 the cyclic subspaces generated by G—that is, the closed span of the set $\{W^n f\colon\ f \in G,\ n \geq 0\}$—is a proper invariant subspace of W. In particular, if G consists of the single function $U \equiv 1$, then it is easily verified that this subspace consists of all functions f in $L_2[0, 1]^2$, which are of the form $f(x, y) = g(xy)$, where g is a measurable function on $[0, 1]$.

These examples indicate that W has a very rich and varied supply of invariant subspaces, and a characterization of all of them might be a hopeless task. On the other hand, it might be easier to characterize all the hyperinvariant subspaces of W.

First we note, that unlike for V, not every invariant subspace of W is also a hyperinvariant subspace. Indeed, since the operator τ (introduced in §3) commutes with W, every hyperinvariant subspace for W must be invariant for τ. This implies, in particular, that a necessary condition for an invariant subspace of the form M_E to be hyperinvariant is that E should be a symmetric set (that is, if $(x, y) \in E$ then $(y, x) \in E$ for almost all $(x, y) \in E$). Thus, for example, if $0 < a, b < 1$ and $a \neq b$ then the subspace $M_{[0,a]\times[0,b]}$ is an invariant subspace for W which is not hyperinvariant.

It should be observed that not every invariant subspace of W which is also invariant for τ is hyperinvariant for W. Such examples are provided by the subspaces S_+ and S_- which are not hyperinvariant for W, since they are not invariant for the convolution operator defined on $L_2[0, 1]^2$ by $L_h f = h \star f$, with $h(x, y) = x$, which commutes with W.

We conclude with two problems.

Problem 1. Let E be a measurable subset of $[0, 1]^2$ which satisfies

(1) $(x, y) \in E \Rightarrow [0, x] \times [0, y] \subseteq E$, and
(2) $(x, y) \in E \Rightarrow (y, x) \in E$.

Is M_E a hyperinvariant subspace for W? In particular, is the answer positive if $E = [0, a]^2$ for some $0 < a < 1$?

Problem 2. Is every hyperinvariant subspace for W of the form M_E, where E is a subset as in Problem 1?

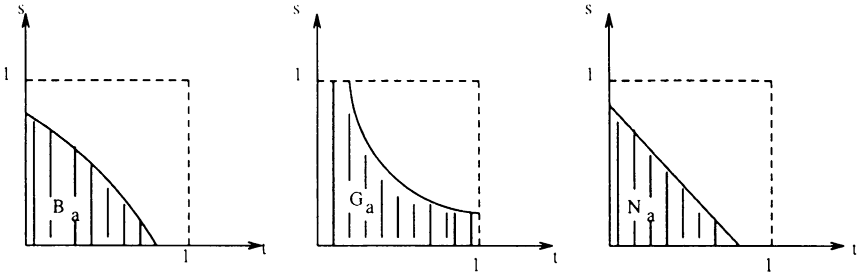

FIGURE 1. M_{B_a}, M_{G_a}, M_{N_a} consist of all functions that vanish in the domains B_a, G_a, and N_a respectively

We mention without proof that one can show that if for $0 < a < 1$ we denote $B_a = \{(t, s) \in [0, 1]^2 : (1-t)(1-s) \geq a\}$, $G_a = \{(t, s) \in [0, 1]^2 : ts \leq a\}$, and, for $0 < a < 2$, $N_a = \{(t, s) \in [0, 1]^2 : s + t \leq a\}$, then all the subspaces M_{B_a}, M_{G_a}, and M_{N_a} are hyperinvariant subspaces for W. (See Figure 1.) Thus we obtain a positive answer to the first part of Problem 1 in these particular cases.

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