

THE INTEGRATION OPERATOR IN TWO VARIABLES

A. ATZMON AND H. MANOS

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. In this paper we consider the integration operator in two variables on $L_2[0, 1]^2$, determine its multiplicity and reducing subspaces, and make some observations about its invariant and hyperinvariant subspaces.

1. INTRODUCTION

The purpose of this paper is to study the Volterra integration operator W in two variables, that is, the operator defined on $L_2[0, 1]^2$ by

$$(Wf)(x, y) \stackrel{\text{def}}{=} \int_0^y ds \int_0^x f(t, s) dt.$$

In particular we find its multiplicity and reducing subspaces and obtain some information on its invariant and hyperinvariant subspaces. It will follow from our results that the properties of W are quite different from the properties of the classical Volterra operator V (defined on $L_2[0, 1]$ by $(Vf)(x) \stackrel{\text{def}}{=} \int_0^x f(t) dt$). It is well known that V is compact and quasi-nilpotent. Since $W = V \otimes V$, the same properties are also shared by W . These facts are also easily verified directly.

Before describing the content of this paper, we introduce some notation and recall some definitions. For a complex Banach space X , we will denote by $L(X)$ the algebra of bounded linear operators on X . If A is a subalgebra of $L(X)$ which contains the identity operator, then a subset G of X is called cyclic for A , if the linear span of the set $\{Tx : x \in G, T \in A\}$ is dense in X . The smallest cardinality of a cyclic set for the algebra A is called the multiplicity of A and will be denoted by $m(A)$.

The multiplicity of an operator T in $L(X)$ is defined as the multiplicity of the algebra generated in $L(X)$ by T and the identity operator and will be denoted by $m(T)$.

The commutant of T is defined by

$$T' \stackrel{\text{def}}{=} \{B \in L(X) : TB = BT\}.$$

Received by the editors February 24, 1992.

1991 *Mathematics Subject Classification.* Primary 47A15; Secondary 47G10.

Key words and phrases. Integration operator, multiplicity, invariant subspace, hyperinvariant subspace, reducing subspace.

A closed subspace M of X is called an invariant subspace of T , if T maps M into itself. If M is also invariant for every operator in T' , then M is called hyperinvariant for T . Let \mathcal{H} be a Hilbert space and $T \in L(\mathcal{H})$. A closed subspace E of \mathcal{H} is called a reducing subspace for T , if E and E^\perp are both invariant under T .

It is well known (see [2, Theorem 4.14]) that the invariant subspaces of V are exactly the subspaces M_a of the form $M_a = \{f \in L_2[0, 1] : f = 0 \text{ a.e. on } [0, a]\}$ for some $0 < a < 1$. It follows from this, and it is also easily seen directly, that the function $f \equiv 1$ is cyclic for V ; hence, $m(V) = 1$. It also follows from this description of invariant subspaces that V has no proper reducing subspaces. Also, since V is unicellular, by a general result (see [2, Corollary 6.27]), every invariant subspace of V is also hyperinvariant.

In §2 we prove that unlike V , the operator W has infinite multiplicity.

In §3 we consider the reducing subspaces of W and prove that the only such subspaces are S_+ and S_- , which consist of the symmetric functions and antisymmetric functions in $L_2[0, 1]^2$ respectively; that is,

$$S_+ = \{f \in L_2[0, 1]^2 : f(x, y) = f(y, x), \text{ a.e. on } [0, 1]^2\},$$

$$S_- = \{f \in L_2[0, 1]^2 : f(x, y) = -f(y, x), \text{ a.e. on } [0, 1]^2\}.$$

In §4 we give some examples of invariant and hyperinvariant subspaces of W ; however, the complete characterization of these subspaces remains open.

2. THE MULTIPLICITY OF W

In this section we show that W has infinite multiplicity, that is, we prove

Theorem 1. $m(W) = \infty$.

The proof of the theorem will be based on a result from [1, Proposition 2.1]. For the sake of completeness we include its statement and proof.

Proposition 2. *Let T be an operator in $L(X)$, and assume that for some integer $n \geq 2$ there exists a nonzero continuous n -linear mapping ϕ of X^n into some topological vector space Y , such that, for every n -tuple (x_1, \dots, x_n) in X^n for which $x_i = x_j$ for some $1 \leq i < j \leq n$ and for every pair of nonnegative integers (k_1, k_2) , $\phi(x_1, x_2, \dots, T^{k_1}x_i, x_{i+1}, \dots, T^{k_2}x_j, \dots, x_n) = 0$. Then $n \leq m(T)$.*

Proof. Let A be the subalgebra of $L(X)$ generated by T and the identity operator. First we note that since the set $D = \{T^n : n \geq 0\}$ spans A , the assumption on ϕ implies that for every $(T_1, T_2) \in A \times A$ and for every n -tuple (x_1, \dots, x_n) in X^n for which $x_i = x_j$ for some $1 \leq i < j \leq n$

$$(1) \quad \phi(x_1, x_2, \dots, T_1x_i, x_{i+1}, \dots, T_2x_j, \dots, x_n) = 0.$$

Let G be any subset of X which contains less than n elements, and consider the set $M = \text{span}\{Sx : x \in G, S \in A\}$. The hypothesis that G contains less than n elements implies by (1) that $M^n \subseteq \ker \phi$, and therefore since ϕ is continuous, $\overline{M^n} = \overline{M^n} \subseteq \ker \phi$. Remembering that $\phi \neq 0$, we conclude that $\overline{M^n} \neq X^n$ and therefore $\overline{M} \neq X$. \square

In the proof of the theorem it will be convenient to write W as a convolution

operator. For $f, g \in L_2[0, 1]^2$ the convolution is defined by

$$(g \star f)(x, y) \stackrel{\text{def}}{=} \int_0^y ds \int_0^x f(t, s)g(x - t, y - s) dt.$$

It is known and easily verified that the convolution is commutative, associative, $g \star f \in C([0, 1]^2)$, and $\|f \star g\|_2 \leq \|f\|_2 \|g\|_2$. If we denote by U the function $U \equiv 1$ on $[0, 1]^2$ then it is clear that for every $f \in L_2[0, 1]^2$, $Wf = U \star f$.

In view of Proposition 2, the conclusion of Theorem 1 follows from

Proposition 3. *For every $n \geq 2$ there exists an n -linear mapping ϕ_n that satisfies the assumptions of Proposition 2 for W .*

Proof. For $a \geq 1$, we denote by \square_a the rectangle $[0, 1] \times [0, 1/a]$ and by T_a the operator on $L_2[0, 1]^2$ defined by

$$(2) \quad (T_a f)(x, y) \stackrel{\text{def}}{=} \begin{cases} f(ay, x/a), & (x, y) \in \square_a, \\ 0, & \text{otherwise.} \end{cases}$$

It is easily verified that T_a is a continuous linear operator on $L_2[0, 1]^2$ and that for every $(x, y) \in \square_a$

$$(3) \quad (T_a Wf)(x, y) = (W T_a f)(x, y).$$

Let $n > 1$, and choose $n - 1$ real numbers a_1, a_2, \dots, a_{n-1} such that $a_1 = 1$ and $a_k < a_{k+1}$ for $k = 1, 2, \dots, n - 2$, and define the operator P_n on $L_2[0, 1]^2$ by

$$(P_n f)(x, y) = \begin{cases} f(x, y), & (x, y) \in \square_{a_{n-1}}, \\ 0, & \text{otherwise.} \end{cases}$$

For every f_1, \dots, f_n in $L_2[0, 1]^2$ consider the matrix

$$A(f_1, f_2, \dots, f_n) = \begin{pmatrix} f_1 & \cdots & f_n \\ T_{a_1} f_1 & \cdots & T_{a_1} f_n \\ \vdots & & \vdots \\ T_{a_{n-1}} f_1 & \cdots & T_{a_{n-1}} f_n \end{pmatrix}$$

and define the mapping $\phi_n: (L_2[0, 1]^2)^n \rightarrow L_2[0, 1]^2$ by

$$(4) \quad \phi_n(f_1, \dots, f_n) \stackrel{\text{def}}{=} P_n\{\det[A(f_1, f_2, \dots, f_n)]\},$$

where multiplication in $\det A$ is convolution. Since for every n functions g_1, \dots, g_n in $L_2[0, 1]^2$ we have that

$$\|g_1 \star g_2 \star \cdots \star g_n\|_2 \leq \|g_1\|_2 \|g_2\|_2 \cdots \|g_n\|_2,$$

and since the operators T_{a_i} are continuous, it follows that ϕ_n is a continuous n -linear mapping of $(L_2[0, 1]^2)^n$ into $L_2[0, 1]^2$. Next, let $(f_1, \dots, f_n) \in (L_2[0, 1]^2)^n$ and assume that there exist $1 \leq i < j \leq n$ such that $f_i = W^r f$, $f_j = W^m f$ for some $f \in L_2[0, 1]^2$. We have to show that $\phi_n(f_1, \dots, f_n) \equiv 0$. First if $(x, y) \notin \square_{a_{n-1}}$ then, for every $f \in L_2[0, 1]^2$, $(P_n f)(x, y) = 0$ and so

$$[\phi_n(f_1, \dots, f_n)](x, y) = [P_n(\det A)](x, y) = 0.$$

It remains to prove that this holds also for $(x, y) \in \square_{a_{n-1}}$. By changing the order of the columns of A we may assume that $i = 1$ and $j = 2$, namely, $f_1 = W^r f$ and $f_2 = W^m f$. Then on $\square_{a_{n-1}}$

$$(5) \quad \begin{aligned} & \phi_n(W^r f, W^m f, f_3, \dots, f_n) \\ &= \det \begin{pmatrix} W^r f & W^m f & f_3 & \cdots & f_n \\ T_{a_1} W^r f & T_{a_1} W^m f & T_{a_1} f_3 & \cdots & T_{a_1} f_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ T_{a_{n-1}} W^r f & T_{a_{n-1}} W^m f & T_{a_{n-1}} f_3 & \cdots & T_{a_{n-1}} f_n \end{pmatrix}. \end{aligned}$$

For every $1 \leq k < n - 1$, $a_{n-1} > a_k$; hence, $\square_{a_{n-1}} \subset \square_{a_k}$. Therefore, using (3) and remembering that $Wf = U \star f$ and that convolution is associative and commutative, we obtain that on $\square_{a_{n-1}}$

$$(6) \quad \begin{aligned} & \phi_n(W^r f, W^m f, f_3, \dots, f_n) \\ &= \det \begin{pmatrix} W^r f & W^m f & f_3 & \cdots & f_n \\ W^r T_{a_1} f & W^m T_{a_1} f & T_{a_1} f_3 & \cdots & T_{a_1} f_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ W^r T_{a_{n-1}} f & W^m T_{a_{n-1}} f & T_{a_{n-1}} f_3 & \cdots & T_{a_{n-1}} f_n \end{pmatrix} \\ &= W^{r+m} \det \begin{pmatrix} f & f & f_3 & \cdots & f_n \\ T_{a_1} f & T_{a_1} f & T_{a_1} f_3 & \cdots & T_{a_1} f_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ T_{a_{n-1}} f & T_{a_{n-1}} f & T_{a_{n-1}} f_3 & \cdots & T_{a_{n-1}} f_n \end{pmatrix}. \end{aligned}$$

Now in the last matrix the first two columns are the same on $\square_{a_{n-1}}$ and, therefore, the determinant vanishes on $\square_{a_{n-1}}$. Noticing that if g is any function that vanishes on some rectangle of the form $[0, a] \times [0, b]$ —that is included in $[0, 1]^2$ —then $W^k g$ also vanishes on that rectangle for every k , we conclude that $\phi_n(W^r f, W^m f, f_3, \dots, f_n) = 0$ on $\square_{a_{n-1}}$.

It remains to show that ϕ_n is not identically zero. For this consider the functions $g_1(x, y) = 1$, $g_2(x, y) = x$, \dots , $g_n(x, y) = x^{n-1}$. We claim that $\phi_n(g_1, \dots, g_n) \neq 0$. Indeed, by definition (2) and the fact that $\square_{a_{n-1}} \subseteq \square_{a_k}$ we have for (x, y) in the rectangle $\square_{a_{n-1}}$ and for every $1 \leq k \leq n - 1$ that $(T_{a_k} g_m)(x, y) = g_m(a_k y, x/a_k) = a_k^{m-1} y^{m-1}$. By the definition of ϕ_n we get that on $\square_{a_{n-1}}$

$$\phi_n(g_1, \dots, g_n) = \det \begin{pmatrix} 1 & x & \cdots & x^{n-1} \\ 1 & y & \cdots & y^{n-1} \\ 1 & a_2 y & \cdots & a_2^{n-1} y^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & a_{n-1} y & \cdots & a_{n-1}^{n-1} y^{n-1} \end{pmatrix}$$

where multiplication in the determinant is convolution. Denoting by M_{ij} the

minors of the determinant, we obtain that on $\square_{a_{n-1}}$

$$\phi_n(g_1, \dots, g_n) = \sum_{k=1}^n (-1)^k x^{k-1} \star M_{1k}.$$

The highest power of x appears when $k = n$, namely, in the term $x^{n-1} \star M_{1n}$. Therefore, to show that $\phi_n(g_1, \dots, g_n) \neq 0$, it suffices to show that $x^{n-1} \star M_{1n} \neq 0$; but since M_{1n} is a polynomial, this is obviously true if $M_{1n} \neq 0$. So it suffices to prove that

$$M_{1n} = \det \begin{pmatrix} 1 & y & \dots & y^{n-2} \\ 1 & a_2 y & \dots & a_2^{n-2} y^{n-2} \\ \vdots & \vdots & & \vdots \\ 1 & a_{n-1} y & \dots & a_{n-1}^{n-2} y^{n-2} \end{pmatrix} \neq 0.$$

It is easy to see that

$$M_{1n} = \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & a_2 & \dots & a_2^{n-2} \\ \vdots & \vdots & & \vdots \\ 1 & a_{n-1} & \dots & a_{n-1}^{n-2} \end{pmatrix} U \star y \star \dots \star y^{n-2}$$

where the multiplication in the last determinant is the usual multiplication. But the last determinant is the Van-der-Monde determinant of $a_1 = 1, a_2, \dots, a_{n-1}$ and hence, is equal to $\prod_{i>j \geq 1} (a_i - a_j)$, which is not zero since $a_i > a_j$, for $i > j$, so $M_{1n} \neq 0$.

This completes the proof of Proposition 3 and, hence, also of Theorem 1. \square

3. THE REDUCING SUBSPACES OF W

We recall that a subspace E of $L_2[0, 1]^2$ is reducing for an operator T on $L_2[0, 1]^2$ if E and E^\perp are both invariant under T , or equivalently if E is invariant under T and T^* . We denote by S_+ the symmetric functions in $L_2[0, 1]^2$, namely,

$$S_+ = \{f \in L_2[0, 1]^2 : f(x, y) = f(y, x) \text{ a.e. on } [0, 1]^2\},$$

and by S_- the antisymmetric functions in $L_2[0, 1]^2$, namely,

$$S_- = \{f \in L_2[0, 1]^2 : f(x, y) = -f(y, x) \text{ a.e. on } [0, 1]^2\}.$$

Consider the operator τ defined on $L_2[0, 1]^2$ by

$$(\tau f)(x, y) = f(y, x).$$

It is easily verified that τ commutes with W . Therefore, if $f \in S_+$ then

$$\tau(Wf) = W(\tau f) = Wf;$$

hence, $Wf \in S_+$. Similarly if $f \in S_-$ then $Wf \in S_-$; that is, S_+ and S_- are invariant under W . It is easy to see that S_- is the orthogonal complement of S_+ , and, therefore, S_+, S_- are reducing subspaces of W .

The main result of this section is the following theorem.

Theorem 4. *The only nontrivial reducing subspaces of W are S_+ and S_- .*

For the proof of the theorem we shall need several lemmas.

Lemma 5. *Let T be an operator in a Hilbert space \mathcal{H} , and assume M is a reducing subspace for T . If λ is an eigenvalue of multiplicity one and x a corresponding eigenvector, then $x \in M$ or $x \in M^\perp$.*

Proof. Let λ be an eigenvalue of multiplicity one and x a corresponding eigenvector. Suppose $x = x_1 + x_2$, where $x_1 \in M$ and $x_2 \in M^\perp$. Then

$$(7) \quad \lambda x = Tx = Tx_1 + Tx_2.$$

Since M and M^\perp are invariant under T , $Tx_1 \in M$ and $Tx_2 \in M^\perp$, hence by (7), $Tx_1 = \lambda x_1$ and $Tx_2 = \lambda x_2$. Since λ is of multiplicity one, this implies that $x_1 \equiv 0$ or $x_2 \equiv 0$, hence $x \in M$ or $x \in M^\perp$. \square

A simple computation shows that the adjoint of W is given by

$$(W^*g)(t, s) = \int_s^1 dy \int_t^1 g(x, y) dx, \quad g \in L_2[0, 1]^2.$$

Lemma 6. *For every integer $n \neq 0$, $r_n = i/2\pi n$ is an eigenvalue of multiplicity one of the operator $W - W^*$, and the corresponding eigenfunctions are constant multiples of the function $f_n(x, y) = e^{-2\pi i n x} - e^{-2\pi i n y}$.*

Proof. Let $\lambda \neq 0$ be an eigenvalue of $W - W^*$ and $F(x, y)$ a corresponding eigenfunction. Then $(W - W^*)F = \lambda F$. This implies that

$$(8) \quad \int_0^y ds \int_0^x F(t, s) dt - \int_y^1 ds \int_x^1 F(t, s) dt = \lambda F(x, y).$$

The left-hand side of (8) is a continuous function, so F is continuous, and therefore, the left-hand side is a differentiable function. Differentiating (8) with respect to y , we get that

$$(9) \quad \int_0^1 F(t, y) dt = \lambda \frac{\partial F}{\partial y}.$$

Differentiating (9) with respect to x we obtain the differential equation

$$(10) \quad \lambda \frac{\partial^2 F}{\partial x \partial y} = 0.$$

From the assumption that $\lambda \neq 0$, (10) implies that

$$(11) \quad F(x, y) = f(x) + g(y),$$

where f and g are differentiable functions on $[0, 1]$. Substituting this in (9) we obtain that

$$\int_0^1 [f(t) + g(y)] dt = \lambda \frac{dg}{dy};$$

hence,

$$C_1 + g(y) = \lambda \frac{dg}{dy},$$

where $C_1 = \int_0^1 f(t) dt$. The solution of this differential equation is

$$(12) \quad g(y) = Be^{y/\lambda} + C_1,$$

where B is a constant. Similarly we obtain that $f(x) = Ae^{x/\lambda} + C_2$, where A and C_2 are constants, and therefore we obtain that for some constant C

$$(13) \quad F(x, y) = Ae^{x/\lambda} + Be^{y/\lambda} + C,$$

and substituting again (13) in the equation $(W - W^*)F = \lambda F$, we obtain

$$\begin{aligned} & \int_0^y dt \int_0^x (Ae^{t/\lambda} + Be^{s/\lambda} + C) dt - \int_y^1 ds \int_x^1 (Ae^{t/\lambda} + Be^{s/\lambda} + C) dt \\ & = \lambda(Ae^{x/\lambda} + Be^{y/\lambda} + C). \end{aligned}$$

This implies that

$$\begin{aligned} & y[-\lambda A + \lambda Ae^{1/\lambda} + C] + x[-\lambda B + \lambda Be^{1/\lambda} + C] \\ & + [-\lambda Ae^{1/\lambda} - \lambda Be^{1/\lambda} - C - \lambda C] \equiv 0; \end{aligned}$$

hence, we obtain the following three equations:

- (1) $-\lambda A + \lambda Ae^{1/\lambda} + C = 0.$
- (2) $-\lambda B + \lambda Be^{1/\lambda} + C = 0.$
- (3) $-\lambda Ae^{1/\lambda} - \lambda Be^{1/\lambda} - C - \lambda C = 0.$

By subtracting (2) from (1) we get $(B - A)(1 - e^{1/\lambda}) = 0.$

Possibility a. $e^{1/\lambda} - 1 = 0.$ The solutions are: $\lambda_n = i/2\pi n$ for a nonzero integer n and then $C = 0$ and $A = -B$; namely, the eigenvalues are $r_n = i/2\pi n$ and the corresponding eigenfunctions are constant multiples of the functions $f_n(x, y) = e^{-2\pi i n x} - e^{-2\pi i n y}.$

Possibility b. $A = B.$ In this case, $C = -2\lambda A/(\lambda - 1)$ where λ is the solution of the equation $e^{1/\lambda} = (\lambda + 1)/(\lambda - 1).$ It is easily verified that $r_n = i/2\pi n$ is not a solution of the last equation. (The solutions of this equation give other eigenvalues, in which we are not interested here.) So for any $n \neq 0,$ r_n is an eigenvalue of multiplicity one and f_n is a corresponding eigenfunction. \square

Lemma 7. Let $P_{km}(x, y) = x^k y^m - x^m y^k$ and $Q_{km}(x, y) = x^k y^m + x^m y^k.$ Then:

- (1) $\overline{\text{span}}\{Q_{km}(x, y), k \geq m\} = S_+,$ and
- (2) $\overline{\text{span}}\{P_{km}(x, y), k > m\} = S_-.$

Proof. Let Q denote the set of symmetric polynomials in two variables—that is, $Q = \{q; q(x, y) = q(y, x), \forall(x, y) \in [0, 1]^2\}$ —and P the set of antisymmetric polynomials in two variables—that is, $P = \{p; p(x, y) = -p(y, x), \forall(x, y) \in [0, 1]^2\}.$ It is easily seen that Q is the linear span of the polynomials Q_{km} and P is the linear span of the polynomials $P_{km}.$ This implies the lemma by observing that Q is dense in S_+ and P is dense in $S_-.$

Lemma 8. If M is a reducing subspace for W then $S_- \subseteq M$ or $S_- \subseteq M^\perp.$

Proof. Since M is a reducing subspace for $W,$ it is also reducing for $W - W^*.$ By Lemmas 5 and 6 we get that, for any $n \neq 0,$ $f_n(x, y) = e^{-2\pi i n x} - e^{-2\pi i n y}$ belongs either to M or to $M^\perp.$ In particular, $f(x, y) = e^{2\pi i x} - e^{2\pi i y}$ belongs either to M or to $M^\perp.$

We now show that if $f \in M$ then $S_- \subseteq M.$ For every $n \geq 2$ consider the polynomial P_n defined by

$$P_n(x, y) = y^n - x^n + x - y.$$

First we claim that $P_n \in M,$ $n = 1, 2, \dots.$ We prove this by induction. Since

$$f(x, y) = e^{2\pi i x} - e^{2\pi i y} \in M$$

and M is invariant under W and W^* ,

$$f_1 = 4\pi i W^* W f + \frac{1}{2\pi i} f \in M$$

and

$$f_2 = 4\pi i W W^* f + 2W f \in M.$$

A direct computation shows that

$$f_1(x, y) = x^2 \left[\frac{1}{2\pi i} - 1 - \frac{1}{2\pi i} e^{2\pi i y} + y \right] - y^2 \left[\frac{1}{2\pi i} - 1 - \frac{1}{2\pi i} e^{2\pi i x} + x \right] + (x - y)$$

and

$$f_2(x, y) = x^2 \left[\frac{1}{2\pi i} - \frac{1}{2\pi i} e^{2\pi i y} + y \right] - y^2 \left[\frac{1}{2\pi i} - \frac{1}{2\pi i} e^{2\pi i x} + x \right].$$

Therefore,

$$f_1(x, y) - f_2(x, y) = x - y - x^2 + y^2 = P_2(x, y) \in M.$$

A simple computation shows that

$$P_{n+1} = (n + 1)[W P_n - W^* P_n + \frac{1}{2} P_2],$$

and therefore if we assume that $P_n \in M$, we obtain that also $P_{n+1} \in M$, and the claim is proved.

Next we claim that, for every $n \geq 1$, $x^n - y^n \in M$. Indeed, since M is closed and

$$\|P_n - (x - y)\|_2 = \|y^n - x^n\|_2 \leq \|y^n\|_2 + \|x^n\|_2 = 2 \frac{1}{\sqrt{2n + 1}} \rightarrow_{n \rightarrow \infty} 0,$$

we conclude that $x - y \in M$. Since for every $n \geq 2$

$$P_n(x, y) = y^n - x^n + x - y \in M,$$

this implies that for every $n \geq 1$

$$(14) \quad x^n - y^n \in M$$

and, therefore, for $k > m$

$$W^m(x^{k-m} - y^{k-m}) = \frac{1}{m!(k - m + 1)(k - m + 2) \dots k} P_{km}(x, y) \in M.$$

Hence $P_{km} \in M$ and by Lemma 6, this implies that $S_- \subseteq M$. Similarly one shows that if $f \in M^\perp$ then $S_- \subseteq M^\perp$. \square

For every pair of nonnegative integers $n, m \geq 0$ denote

$$f_{nm}(x, y) = \cos\left(\frac{2n + 1}{2}\pi x\right) \cos\left(\frac{2m + 1}{2}\pi y\right).$$

Lemma 9. *The only eigenfunctions of the operator W^*W are constant multiples of the functions $\{f_{nm}\}_{n,m \geq 0}$, and the corresponding eigenvalues are $\lambda_{nm} = 16/(2n + 1)^2(2m + 1)^2\pi^4$.*

Proof. It is easy to verify that f_{nm} are eigenfunctions of W^*W and that λ_{nm} are the corresponding eigenvalues. Since it is well known that $\{f_{nm}\}_{n,m \geq 0}$ is

a complete orthogonal system in $L_2[0, 1]^2$, it follows that there are no other eigenfunctions. \square

Lemma 10. *If M is a reducing subspace for W then either $S_+ \subseteq M$ or $S_+ \subseteq M^\perp$.*

Proof. If M is reducing for W then M is also reducing for W^*W . Lemma 9 implies that $\lambda = 16/\pi^4$ is an eigenvalue of multiplicity one of the operator W^*W , and a corresponding eigenfunction is $f(x, y) = \cos(\pi x/2) \cos(\pi y/2)$. Hence by Lemma 5, f belongs either to M or to M^\perp .

We will show that if $f \in M$ then $S_+ \subseteq M$. For every $n \geq 1$ define $g_n(x, y) = 1 - x^n - y^n$. First we claim that $g_n \in M$ for $n = 1, 2, \dots$. We prove this by induction. Since $f(x, y) = \cos(\pi x/2) \cos(\pi y/2) \in M$ and M is invariant under W and W^* ,

$$f_1 = \frac{\pi^4}{16} W^2 f - f \in M$$

and

$$f_2 = \frac{\pi^3}{4(\pi - 2)} [(W^*)^2 f - W W^* f] \in M.$$

By a direct computation

$$f_1(x, y) = 1 - \cos\left(\frac{\pi}{2}x\right) - \cos\left(\frac{\pi}{2}y\right)$$

and

$$f_2(x, y) = 1 + \frac{2}{\pi} \left[1 - \cos\left(\frac{\pi}{2}x\right) - \cos\left(\frac{\pi}{2}y\right) \right] - (y + x).$$

This implies that

$$g_1 = f_2 - \frac{2}{\pi} f_1 \in M.$$

A simple computation shows that

$$g_{n+1} = (n+1) \left[\frac{n}{n+1} g_1 - W^* g_n + W g_n \right].$$

Therefore, if we assume that $g_n \in M$, we obtain that also $g_{n+1} \in M$, and the claim is proved. It is easy to see that $g_n \rightarrow 1$ in $L_2[0, 1]^2$, and therefore since M is closed, the function $U \equiv 1$ belong to M . Since $g_n \in M$, this implies that, for every $n \geq 0$, $x^n + y^n \in M$, and therefore, for every $0 \leq m \leq k$,

$$W^m(x^{k-m} + y^{k-m}) = \frac{1}{m!(k-m+1)\dots k} Q_{km} \in M.$$

Hence $Q_{km} \in M$, and by Lemma 7 we conclude that $S_+ \subseteq M$. A similar argument shows that if $f \in M^\perp$ then $S_- \subseteq M^\perp$. \square

Proof of Theorem 4. Let M be a reducing subspace for W . It follows from Lemmas 8 and 10 that there are four possibilities:

- (1) $S_- \subseteq M$ and $S_+ \subseteq M$.
- (2) $S_- \subseteq M^\perp$ and $S_+ \subseteq M^\perp$.
- (3) $S_- \subseteq M$ and $S_+ \subseteq M^\perp$.
- (4) $S_- \subseteq M^\perp$ and $S_+ \subseteq M$.

Since $S_- \oplus S_+ = L_2[0, 1]^2$, possibility (1) implies that $M = L_2[0, 1]^2$ and possibility (2) implies that $M^\perp = L_2[0, 1]^2$; hence, $M = \{0\}$. Since $S_+^\perp = S_-$, possibility (3) implies that $M = S_-$ and possibility (4) implies that $M = S_+$. This concludes the proof of the theorem. \square

4. INVARIANT AND HYPERINVARIANT SUBSPACES FOR W

It is easy to see that if E is a measurable subset of $[0, 1]^2$, which satisfies the condition

$$(x, y) \in E \Rightarrow [0, x] \times [0, y] \subseteq E,$$

then the subspace

$$(15) \quad M_E \stackrel{\text{def}}{=} \{f \in L_2[0, 1]^2 : f = 0 \text{ a.e. on } E\}$$

is an invariant subspace for W . These subspaces are in a sense analogous to the invariant subspaces of the classical Volterra operator V ; however, there are many other invariant subspaces for W . For example, such are the subspaces S_+ and S_- considered in §3; and if G is any finite subset of $L_2[0, 1]^2$, then in view of Theorem 1 the cyclic subspaces generated by G —that is, the closed span of the set $\{W^n f : f \in G, n \geq 0\}$ —is a proper invariant subspace of W . In particular, if G consists of the single function $U \equiv 1$, then it is easily verified that this subspace consists of all functions f in $L_2[0, 1]^2$, which are of the form $f(x, y) = g(xy)$, where g is a measurable function on $[0, 1]$.

These examples indicate that W has a very rich and varied supply of invariant subspaces, and a characterization of all of them might be a hopeless task. On the other hand, it might be easier to characterize all the hyperinvariant subspaces of W .

First we note, that unlike for V , not every invariant subspace of W is also a hyperinvariant subspace. Indeed, since the operator τ (introduced in §3) commutes with W , every hyperinvariant subspace for W must be invariant for τ . This implies, in particular, that a necessary condition for an invariant subspace of the form M_E to be hyperinvariant is that E should be a symmetric set (that is, if $(x, y) \in E$ then $(y, x) \in E$ for almost all $(x, y) \in E$). Thus, for example, if $0 < a, b < 1$ and $a \neq b$ then the subspace $M_{[0, a] \times [0, b]}$ is an invariant subspace for W which is not hyperinvariant.

It should be observed that not every invariant subspace of W which is also invariant for τ is hyperinvariant for W . Such examples are provided by the subspaces S_+ and S_- which are not hyperinvariant for W , since they are not invariant for the convolution operator defined on $L_2[0, 1]^2$ by $L_h f = h \star f$, with $h(x, y) = x$, which commutes with W .

We conclude with two problems.

Problem 1. Let E be a measurable subset of $[0, 1]^2$ which satisfies

- (1) $(x, y) \in E \Rightarrow [0, x] \times [0, y] \subseteq E$, and
- (2) $(x, y) \in E \Rightarrow (y, x) \in E$.

Is M_E a hyperinvariant subspace for W ? In particular, is the answer positive if $E = [0, a]^2$ for some $0 < a < 1$?

Problem 2. Is every hyperinvariant subspace for W of the form M_E , where E is a subset as in Problem 1?

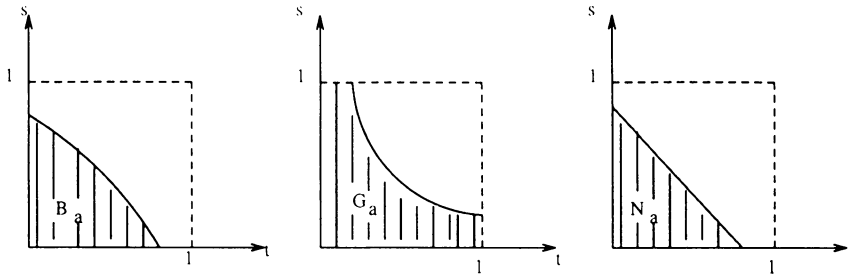


FIGURE 1. M_{B_a} , M_{G_a} , M_{N_a} consist of all functions that vanish in the domains B_a , G_a , and N_a respectively

We mention without proof that one can show that if for $0 < a < 1$ we denote $B_a = \{(t, s) \in [0, 1]^2 : (1-t)(1-s) \geq a\}$, $G_a = \{(t, s) \in [0, 1]^2 : ts \leq a\}$, and, for $0 < a < 2$, $N_a = \{(t, s) \in [0, 1]^2 : s + t \leq a\}$, then all the subspaces M_{B_a} , M_{G_a} , and M_{N_a} are hyperinvariant subspaces for W . (See Figure 1.) Thus we obtain a positive answer to the first part of Problem 1 in these particular cases.

REFERENCES

1. A. Atzmon, *Multilinear mappings and estimates of multiplicity*, *Integral Equations Operator Theory* **10** (1987), 1–16.
2. H. Radjavi and P. Rosenthal, *Invariant subspaces*, Springer-Verlag, New York, 1973.

SCHOOL OF MATHEMATICAL SCIENCES, RAYMOND AND BEVERLY SACKLER FACULTY OF EXACT SCIENCES, TEL-AVIV UNIVERSITY, TEL-AVIV 69978, ISRAEL
E-mail address, A. Atzmon: aatzmon@math.tau.ac.il