PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 119, Number 2, October 1993

UNIVERSAL FILTRATION OF SCHUR COMPLEXES

GIANDOMENICO BOFFI

(Communicated by Eric Friedlander)

ABSTRACT. The Schur complex $\mathbf{L}_{\lambda/\mu}\phi$ has proved useful in studying resolutions of determinantal ideals, both in characteristic zero and in a characteristic-free setting. We show here that in every characteristic, $\mathbf{L}_{\lambda/\mu}\phi$ is isomorphic, up to a filtration, to a sum of Schur complexes $\sum_\nu \gamma(\lambda/\mu;\nu)\mathbf{L}_\nu\phi$, where $\gamma(\lambda/\mu;\nu)$ is the usual Littlewood-Richardson coefficient. This generalizes a well-known direct sum decomposition of $\mathbf{L}_{\lambda/\mu}\phi$ in characteristic zero.

INTRODUCTION

Let F and G be finitely generated free R-modules of ranks m and n, respectively, and $\varphi\colon G \to F$ be an R-module homomorphism. To every skew shape λ/μ, associate the Schur complex of φ of shape λ/μ, as defined in [2] and denoted by $\mathbf{L}_{\lambda/\mu}\varphi$. It is well known that if R contains a copy of the rationals, then $\mathbf{L}_{\lambda/\mu}\varphi$ is isomorphic to $\sum_\nu \gamma(\lambda/\mu;\nu)\mathbf{L}_\nu\varphi$, where $\mathbf{L}_\nu\varphi$ is the Schur complex associated to the shape ν and $\gamma(\lambda/\mu;\nu)$ is the ordinary Littlewood-Richardson coefficient. In this paper, we construct a universal (= independent of the ring R) filtration of $\mathbf{L}_{\lambda/\mu}\varphi$, whose associated graded object precisely coincides with $\sum_\nu \gamma(\lambda/\mu;\nu)\mathbf{L}_\nu\varphi$.

One should remark that the above can be applied to all complexes $\mathbf{L}_{\lambda_1/\mu_1}\varphi \otimes \mathbf{L}_{\lambda_2/\mu_2}\varphi \otimes \cdots \otimes \mathbf{L}_{\lambda_t/\mu_t}\varphi$, which are further instances of Schur complexes associated to skew shapes.

The techniques used in the construction are those introduced in the author's thesis, Brandeis University, 1984 (largely reproduced in [4]), and successfully exploited in [5, 3, 6]. In fact, all the results contained in these four papers can be obtained as corollaries of the result for $\mathbf{L}_{\lambda/\mu}\varphi$.

One expects the universal filtration of $\mathbf{L}_{\lambda/\mu}\varphi$ to assist in solving some problems akin to those which originated the notion of Schur complex (cf. [1, 7, 9], etc.). For instance, such a filtration is an ingredient for the study of the homology of the complex introduced in [11].

It has to be pointed out that in the very special cases when $\mu = (t)$ and $\mu = (1^t)$, a universal filtration for $\mathbf{L}_{\lambda/\mu}\varphi$ has been already described in [8] but in a way which is of little help to us here. (In fact, in his thesis, written before

Received by the editors February 10, 1992.
1991 *Mathematics Subject Classification.* Primary 13D25; Secondary 14M12, 15A72.
The author was partially supported by M.U.R.S.T. and is a member of C.N.R.-G.N.S.A.G.A.

[8], Ko believed to have the description of a universal filtration for *every* Schur complex, but there were some errors.)

Notation and basic facts are freely borrowed from [2].

1. Preliminaries

We closely follow the strategy outlined in [5, §2].

We first define an auxiliary free R-module H; namely, we set $H = R^s$, where $s = \mu_1$ (the length of the first row of μ). We then call ψ the composite morphism $G \xrightarrow{\varphi} F \xrightarrow{\text{inc}} F \oplus H$.

Choose ordered bases $Y = \{y_1, \dots, y_m\}$ and $X = \{x_1, \dots, x_n\}$ for G and F, respectively (both of them ordered according to the subscripts), and denote by $Z = \{z_1, \dots, z_s\}$ the canonical basis of H. Let $S = Y \cup X \cup Z$. Thinking of S as ordered like $\{z_1, \dots, z_s, x_1, \dots, x_n, y_1, \dots, y_m\}$, it follows easily from the Standard Basis Theorem for Schur complexes and [2, Corollary V.1.14; 2, Theorem II.4.11] that there is an embedding of complexes $i: \mathbf{L}_{\lambda/\mu}\varphi \hookrightarrow (\mathbf{L}_\lambda\psi)_{\tilde{\mu}}$, where $(\mathbf{L}_\lambda\psi)_{\tilde{\mu}}$ stands for the subcomplex of $\mathbf{L}_\lambda\psi$ spanned by the tableaux of shape λ which are row-standard $\bmod Y$ (cf. [2, Definition V.1.8]) and such that each z_i exactly occurs $\tilde{\mu}_i$ times (here, $\tilde{\mu}$ is the conjugate partition of μ).

Remark 1.1. The embedding i is not $\mathrm{GL}(H)$-equivariant but preserves the $\mathrm{GL}(G) \times \mathrm{GL}(F)$ structure we care for.

We now explain the reason for introducing the free R-module H.

Let $b^{(k)} = z_1 \wedge z_2 \wedge \cdots \wedge z_k$, $r = l(\mu)$, and $q = l(\lambda)$. Consider all sets $\{t_{r1}, \dots, t_{11}, t_{r2}, \dots, t_{12}, \dots, t_{rq}, \dots, t_{1q}\}$, where the elements are nonnegative integers such that

(a) for every $i = 1, \dots, r$, $\sum_{j=1}^{q} t_{ij} = \mu_i$,

(b) if for every $j = 1, \dots, q$, ν_j stands for $\lambda_j - \sum_{i=1}^{r} t_{ij}$, then $(\nu_1, \dots, \nu_q)$ is a partition ν

(obviously, $\nu \subseteq \lambda$, and $|\nu| = |\lambda| - |\mu|$).

Definition 1.2. For every partition $\nu \subseteq \lambda$ such that $|\nu| = |\lambda| - |\mu|$, let $B(\lambda/\nu)$ denote the (finite) set consisting of the elements of $\Lambda_{\lambda/\nu}H$ of type

$$\sum_{\beta_r, \dots, \beta_1} b^{(\mu_r)}_{\beta_r t_{r1}} \wedge \cdots \wedge b^{(\mu_1)}_{\beta_1 t_{11}} \otimes \cdots \otimes b^{(\mu_r)}_{\beta_r t_{ri}} \wedge \cdots \wedge b^{(\mu_1)}_{\beta_1 t_{1i}} \otimes \cdots \otimes b^{(\mu_r)}_{\beta_r t_{rq}} \wedge \cdots \wedge b^{(\mu_1)}_{\beta_1 t_{1q}}$$

for all possible choices of sets $\{t_{r1}, \dots, t_{11}, t_{r2}, \dots, t_{12}, \dots, t_{rq}, \dots, t_{1q}\}$.

Here, $\sum_{\beta_i} b^{(\mu_i)}_{\beta_i t_{i1}} \otimes \cdots \otimes b^{(\mu_i)}_{\beta_i t_{iq}} = \Delta(b^{(\mu_i)})$ for every $i = 1, \dots, r$, with Δ the diagonal map $\Lambda^{\mu_i}H \to \Lambda^{t_{i1}}H \otimes \cdots \otimes \Lambda^{t_{iq}}H$.

Remark 1.3. The elements of $B(\lambda/\nu)$ are in fact contained in $(\Lambda_{\lambda/\nu}H)^{U^+(H)}_{\tilde{\mu}}$; that is, they are $U^+(H)$-invariant and each z_i occurs in them exactly $\tilde{\mu}_i$ times. ($U^+(H) \cong U^+(\mu_1; R)$ is the group of upper unitriangular matrices of order μ_1 with entries in R.)

Lemma 1.4. *$\psi = \varphi \oplus \zeta$, where ζ is the zero map $0 \to H$.*

Definition 1.5. For every $b \in B(\lambda/\nu)$, let $\varphi(\nu, b)$ denote the map $\Lambda_\nu\varphi \to (\mathbf{L}_\lambda\psi)_{\tilde{\mu}}$, which is obtained by first restricting to $\Lambda_\nu\varphi \otimes \{b\}$ the (obvious) morphism $\Lambda_\nu\varphi \otimes \Lambda_{\lambda/\nu}\zeta \to \Lambda_\lambda\psi$ of [2, Definition V.1.11] and then composing with $d_\lambda\psi$.

(For the definitions of the complexes $\mathbf{\Lambda}_{\bullet}-$ and the map $d_\lambda\psi$, cf. [2, p. 262].)

What happens if we apply to a generator of $\operatorname{Im}(\varphi(\nu, b))$ the straightening law going from $\{x_1, \dots, x_n, y_1, \dots, y_m, z_1, \dots, z_s\}$ to $\{z_1, \dots, z_s, x_1, \dots, x_n, y_1, \dots, y_m\}$? Certainly we still have a $U^+(H)$-invariant element of $(\mathbf{L}_\lambda\psi)_{\tilde{\mu}}$, but since the tableau

$$\begin{array}{|llllll}
\hline
z_1 & z_2 & z_3 & \cdots\cdots\cdots & z_{\mu_1} \\
z_1 & z_2 & z_3 & \cdots\cdots & z_{\mu_2} \\
z_1 & z_2 & z_3 & \cdots & z_{\mu_3} \\
\vdots & & & & \\
\hline
\end{array}$$

(of shape μ) is the only $U^+(H)$-invariant standard tableau containing $\tilde{\mu}_i$ copies of z_i for every $i = 1, \dots, s$, we in fact get an element of $\operatorname{Im}(i)$ (cf. [2, Theorem in the appendix]).

Therefore, $\varphi(\nu, b)$ can be thought of as a map from $\mathbf{\Lambda}_\nu\varphi$ to $\mathbf{L}_{\lambda/\mu}\varphi$. This map will be the main ingredient in our construction.

2. The filtration

Definition 2.1. $\mathbf{M}_\nu\varphi$ is the subcomplex of the complex $\mathbf{L}_{\lambda/\mu}\varphi$ given by the sum $\sum_{\tau\geq\nu,\ b\in B(\lambda/\tau)} \operatorname{Im}(\varphi(\tau, b))$.

The subcomplexes $\mathbf{M}_\nu\varphi$, ordered lexicographically ($\nu_1 \leq \nu_2 \Rightarrow \mathbf{M}_{\nu_2}\varphi \subseteq \mathbf{M}_{\nu_1}\varphi$) will give the filtration of $\mathbf{L}_{\lambda/\mu}\varphi$ for which we are looking.

We denote by $\dot{\mathbf{M}}_\nu\varphi$ the subcomplex immediately preceding $\mathbf{M}_\nu\varphi$, i.e., $\dot{\mathbf{M}}_\nu\varphi = \sum_{\tau>\nu,\ b\in B(\lambda/\tau)} \operatorname{Im}(\varphi(\tau, b))$.

Proposition 2.2. (i) *$\{\mathbf{M}_\nu\varphi\}$ is an exhaustive filtration.*

(ii) *Each $\varphi(\nu, b)\colon \mathbf{\Lambda}_\nu\varphi \to \mathbf{L}_{\lambda/\mu}\varphi$ induces a map $\mathbf{L}_\nu\varphi \to \mathbf{L}_{\lambda/\mu}\varphi/\dot{\mathbf{M}}_\nu\varphi$, indeed a map $\mathbf{L}_\nu\varphi \to \mathbf{M}_\nu\varphi/\dot{\mathbf{M}}_\mu\varphi$, denoted by $\overline{\varphi}(\nu, b)$.*

Proof. (i) One can mimic the proof of [4, Proposition 3.1], provided one replaces each $\mathbf{\Lambda}^i(F \oplus G)$ by $\mathbf{\Lambda}^i(\varphi \oplus \zeta)$ and keeps in mind the peculiar nature of ζ, as defined in Lemma 1.4.

(ii) One can mimic the proof of [4, Lemma 3.6], reading $\mathbf{\Lambda}^i(\varphi \oplus \zeta)$ instead of $\mathbf{\Lambda}^i(F \oplus G)$.

We want to show that each $\mathbf{M}_\nu\varphi/\dot{\mathbf{M}}_\nu\varphi$ is isomorphic to the sum of exactly $\gamma(\lambda/\mu; \nu)$ copies of $\mathbf{L}_\nu\varphi$. The idea is to reduce the case over $\mathbf{Z}$ (whence the case over every R) to the case over $\mathbf{Q}$, where the result is known to be true (cf., e.g., [10, §0.3; 9]).

Combinatorially, $\gamma(\lambda/\mu; \nu)$ is the number of standard tableaux of shape λ/ν, filled with $\tilde{\mu}_1$ copies of 1, $\tilde{\mu}_2$ copies of 2, $\tilde{\mu}_3$ copies of 3, etc., such that the associated word (formed by listing all entries from bottom to top in each column, starting from the leftmost column) is a lattice permutation (cf. [5, §1]).

To the above standard tableaux of shape λ/ν, one can associate in a one-to-one way $\gamma(\lambda/\mu; \nu)$ elements of $B(\lambda/\nu)$, thus obtaining a subset $B'(\lambda/\nu)$ of $B(\lambda/\nu)$ such that $|B'(\lambda/\nu)| = \gamma(\lambda/\mu; \nu)$ (cf. [5, §3]).

Now notice that if in the first part of the proof of [4, Theorem 4.5] one replaces the standard tableau $d_\nu(a) \in L_\nu F$ by any linear combination $\sum_j c_j d_\nu(a_j)$ of standard tableaux, the very same argument shows:

After ordering the elements of $B'(\lambda/\nu)$ in the way described in [5, §3], for every $b' \in B'(\lambda/\nu)$ one has that

$$(\$)\qquad \begin{gathered}\varphi(\nu\,,\,b')\left(\sum c_j a_j\right) \in \dot{\mathbf{M}}_\nu + \sum_{b''<b'} \mathrm{Im}(\varphi(\nu\,,\,b'')) \\ \text{implies} \quad \sum_j c_j d_\nu(a_j) = 0\,, \quad \text{whence } c_j = 0 \text{ for every } j\,.\end{gathered}$$

(The proof of [4, Theorem 4.5] is given in the case μ is a rectangle but does not depend on the assumption $\mu_1 = \cdots = \mu_r$, as already remarked in [5, §3].)

Once more, reading $\varphi \oplus \zeta$ instead of $F \oplus G$ in the above, ($) becomes a statement like the following: After ordering the elements of $B'(\lambda/\nu)$ in the way described in [5, §3], for every $b' \in B'(\lambda/\nu)$,

$$\overline{\varphi}(\nu\,,\,b')\colon \mathbf{L}_\nu\varphi \to \mathbf{M}_\nu\varphi \Big/ \left(\dot{\mathbf{M}}_\nu\varphi + \sum_{b''<b'} \mathrm{Im}(\varphi(\nu\,,\,b''))\right)$$

is injective over $\mathbf{Z}$.

Thus, over the field $\mathbf{Q}$, one is forced to conclude from the known decomposition $\sum_\nu \gamma(\lambda/\mu\,;\,\nu)\mathbf{L}_\nu\varphi$ of $\mathbf{L}_{\lambda/\mu}\varphi$ that, for every $b \in B(\lambda/\nu) - B'(\lambda/\nu)$,

$$(\&)\qquad \mathrm{Im}(\varphi(\nu\,,\,b)) \subseteq \dot{\mathbf{M}}_\nu\varphi + \sum_{b' \in B'(\lambda/\nu)} \mathrm{Im}(\varphi(\nu\,,\,b'))\,.$$

Mimicking the second part of the proof of [4, Theorem 4.5] (again ignoring the assumption $\mu_1 = \cdots = \mu_r$ and reading $\varphi \oplus \zeta$ instead of $F \oplus G$), it follows that (&) in fact holds over $\mathbf{Z}$.

Summarizing, we have proven that we can safely discard all maps $\varphi(\nu\,,\,b)$ with $b \in B(\lambda/\nu) - B'(\lambda/\nu)$, for they do not add anything to the images of the $\varphi(\nu\,,\,b')$ with $b' \in B'(\lambda/\nu)$.

This completes the proof of the following theorem.

Theorem 2.3. *$\mathbf{L}_{\lambda/\mu}\varphi$ has a universal filtration whose associated graded object is $\sum_\nu \gamma(\lambda/\mu\,;\,\nu)\mathbf{L}_\nu\varphi$.*

Remark 2.4. Further observations can be made on $B(\lambda/\nu)$, $B'(\lambda/\nu)$, and their elements, as well as on the quotients $\mathbf{M}_\nu\varphi/\dot{\mathbf{M}}_\nu\varphi$, in a way similar to that of [5, §3].

References

1. K. Akin, D. A. Buchsbaum, and J. Weyman, *Resolutions of determinantal ideals: The submaximal minors*, Adv. in Math. **39** (1981), 1–30.
2. ——, *Schur functors and Schur complexes*, Adv. in Math. **44** (1982), 207–278.
3. M. Barnabei and A. Brini, *The Littlewood-Richardson rule for co-Schur modules*, Adv. in Math. **67** (1988), 143–173.
4. G. Boffi, *The universal form of the Littlewood-Richardson rule*, Adv. in Math. **68** (1988), 40–63.
5. ——, *Characteristic-free decomposition of skew Schur functors*, J. Algebra **125** (1989), 288–297.
6. ——, *A remark on a paper by Barnabei and Brini*, J. Algebra **139** (1991), 458–467.
7. T. Jozefiak, P. Pragacz, and J. Weyman, *Resolutions of determinantal varieties and tensor complexes associated with symmetric and antisymmetric matrices*, Astérisque **87-88** (1981), 109–189.

8. H. J. Ko, *The decomposition of Schur complexes*, Trans. Amer. Math. Soc. **324** (1991), 255–270.

9. H. A. Nielsen, *Tensor functors of complexes*, Aarhus Univ. Preprint Series, no. 15, 1978.

10. P. Pragacz and J. Weyman, *Complexes associated with trace and evaluation. Another approach to Lascoux's resolution*, Adv. in Math. **57** (1985), 163–207.

11. M. Artale and G. Boffi, *On a subcomplex of the Schur complex*, preprint, 1991.

Dipartimento di Matematica, II Università di Roma, Via O. Raimondo, 00173 Roma, Italia