

UNIVERSAL FILTRATION OF SCHUR COMPLEXES

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ABSTRACT. The Schur complex $L_{\lambda/\mu}\phi$ has proved useful in studying resolutions of determinantal ideals, both in characteristic zero and in a characteristic-free setting. We show here that in every characteristic, $L_{\lambda/\mu}\phi$ is isomorphic, up to a filtration, to a sum of Schur complexes $\sum_{\nu} \gamma(\lambda/\mu; \nu)L_{\nu}\phi$, where $\gamma(\lambda/\mu; \nu)$ is the usual Littlewood-Richardson coefficient. This generalizes a well-known direct sum decomposition of $L_{\lambda/\mu}\phi$ in characteristic zero.

INTRODUCTION

Let F and G be finitely generated free R -modules of ranks m and n , respectively, and $\varphi: G \rightarrow F$ be an R -module homomorphism. To every skew shape λ/μ , associate the Schur complex of φ of shape λ/μ , as defined in [2] and denoted by $L_{\lambda/\mu}\varphi$. It is well known that if R contains a copy of the rationals, then $L_{\lambda/\mu}\varphi$ is isomorphic to $\sum_{\nu} \gamma(\lambda/\mu; \nu)L_{\nu}\varphi$, where $L_{\nu}\varphi$ is the Schur complex associated to the shape ν and $\gamma(\lambda/\mu; \nu)$ is the ordinary Littlewood-Richardson coefficient. In this paper, we construct a universal (= independent of the ring R) filtration of $L_{\lambda/\mu}\varphi$, whose associated graded object precisely coincides with $\sum_{\nu} \gamma(\lambda/\mu; \nu)L_{\nu}\varphi$.

One should remark that the above can be applied to all complexes $L_{\lambda_1/\mu_1}\varphi \otimes L_{\lambda_2/\mu_2}\varphi \otimes \cdots \otimes L_{\lambda_t/\mu_t}\varphi$, which are further instances of Schur complexes associated to skew shapes.

The techniques used in the construction are those introduced in the author's thesis, Brandeis University, 1984 (largely reproduced in [4]), and successfully exploited in [5, 3, 6]. In fact, all the results contained in these four papers can be obtained as corollaries of the result for $L_{\lambda/\mu}\varphi$.

One expects the universal filtration of $L_{\lambda/\mu}\varphi$ to assist in solving some problems akin to those which originated the notion of Schur complex (cf. [1, 7, 9], etc.). For instance, such a filtration is an ingredient for the study of the homology of the complex introduced in [11].

It has to be pointed out that in the very special cases when $\mu = (t)$ and $\mu = (1^t)$, a universal filtration for $L_{\lambda/\mu}\varphi$ has been already described in [8] but in a way which is of little help to us here. (In fact, in his thesis, written before

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[8], Ko believed to have the description of a universal filtration for every Schur complex, but there were some errors.)

Notation and basic facts are freely borrowed from [2].

1. PRELIMINARIES

We closely follow the strategy outlined in [5, §2].

We first define an auxiliary free R -module H ; namely, we set $H = R^s$, where $s = \mu_1$ (the length of the first row of μ). We then call ψ the composite morphism $G \xrightarrow{\varphi} F \xrightarrow{\text{inc}} F \oplus H$.

Choose ordered bases $Y = \{y_1, \dots, y_m\}$ and $X = \{x_1, \dots, x_n\}$ for G and F , respectively (both of them ordered according to the subscripts), and denote by $Z = \{z_1, \dots, z_s\}$ the canonical basis of H . Let $S = Y \cup X \cup Z$. Thinking of S as ordered like $\{z_1, \dots, z_s, x_1, \dots, x_n, y_1, \dots, y_m\}$, it follows easily from the Standard Basis Theorem for Schur complexes and [2, Corollary V.1.14; 2, Theorem II.4.11] that there is an embedding of complexes $i: L_{\lambda/\mu}\varphi \hookrightarrow (L_{\lambda}\psi)_{\tilde{\mu}}$, where $(L_{\lambda}\psi)_{\tilde{\mu}}$ stands for the subcomplex of $L_{\lambda}\psi$ spanned by the tableaux of shape λ which are row-standard mod Y (cf. [2, Definition V.1.8]) and such that each z_i exactly occurs $\tilde{\mu}_i$ times (here, $\tilde{\mu}$ is the conjugate partition of μ).

Remark 1.1. The embedding i is not $GL(H)$ -equivariant but preserves the $GL(G) \times GL(F)$ structure we care for.

We now explain the reason for introducing the free R -module H .

Let $b^{(k)} = z_1 \wedge z_2 \wedge \dots \wedge z_k$, $r = l(\mu)$, and $q = l(\lambda)$. Consider all sets $\{t_{r1}, \dots, t_{11}, t_{r2}, \dots, t_{12}, \dots, t_{rq}, \dots, t_{1q}\}$, where the elements are nonnegative integers such that

- (a) for every $i = 1, \dots, r$, $\sum_{j=1}^q t_{ij} = \mu_i$,
- (b) if for every $j = 1, \dots, q$, ν_j stands for $\lambda_j - \sum_{i=1}^r t_{ij}$, then (ν_1, \dots, ν_q) is a partition ν

(obviously, $\nu \subseteq \lambda$, and $|\nu| = |\lambda| - |\mu|$).

Definition 1.2. For every partition $\nu \subseteq \lambda$ such that $|\nu| = |\lambda| - |\mu|$, let $B(\lambda/\nu)$ denote the (finite) set consisting of the elements of $\Lambda_{\lambda/\nu}H$ of type

$$\sum_{\beta_r, \dots, \beta_1} b_{\beta_r t_{r1}}^{(\mu_r)} \wedge \dots \wedge b_{\beta_1 t_{11}}^{(\mu_1)} \otimes \dots \otimes b_{\beta_r t_{ri}}^{(\mu_r)} \wedge \dots \wedge b_{\beta_1 t_{1i}}^{(\mu_1)} \otimes \dots \otimes b_{\beta_r t_{rq}}^{(\mu_r)} \wedge \dots \wedge b_{\beta_1 t_{1q}}^{(\mu_1)}$$

for all possible choices of sets $\{t_{r1}, \dots, t_{11}, t_{r2}, \dots, t_{12}, \dots, t_{rq}, \dots, t_{1q}\}$.

Here, $\sum_{\beta_i} b_{\beta_i t_{i1}}^{(\mu_i)} \otimes \dots \otimes b_{\beta_i t_{iq}}^{(\mu_i)} = \Delta(b^{(\mu_i)})$ for every $i = 1, \dots, r$, with Δ the diagonal map $\Lambda^{\mu_i}H \rightarrow \Lambda^{t_{i1}}H \otimes \dots \otimes \Lambda^{t_{iq}}H$.

Remark 1.3. The elements of $B(\lambda/\nu)$ are in fact contained in $(\Lambda_{\lambda/\nu}H)_{\tilde{\mu}}^{U^+(H)}$; that is, they are $U^+(H)$ -invariant and each z_i occurs in them exactly $\tilde{\mu}_i$ times. ($U^+(H) \cong U^+(\mu_1; R)$ is the group of upper unitriangular matrices of order μ_1 with entries in R .)

Lemma 1.4. $\psi = \varphi \oplus \zeta$, where ζ is the zero map $0 \rightarrow H$.

Definition 1.5. For every $b \in B(\lambda/\nu)$, let $\varphi(\nu, b)$ denote the map $\Lambda_{\nu}\varphi \rightarrow (L_{\lambda}\psi)_{\tilde{\mu}}$, which is obtained by first restricting to $\Lambda_{\nu}\varphi \otimes \{b\}$ the (obvious) morphism $\Lambda_{\nu}\varphi \otimes \Lambda_{\lambda/\nu}\zeta \rightarrow \Lambda_{\lambda}\psi$ of [2, Definition V.1.11] and then composing with $d_{\lambda}\psi$.

(For the definitions of the complexes Λ_{\cdot} and the map $d_{\lambda}\psi$, cf. [2, p. 262].)

What happens if we apply to a generator of $\text{Im}(\varphi(\nu, b))$ the straightening law going from $\{x_1, \dots, x_n, y_1, \dots, y_m, z_1, \dots, z_s\}$ to $\{z_1, \dots, z_s, x_1, \dots, x_n, y_1, \dots, y_m\}$? Certainly we still have a $U^+(H)$ -invariant element of $(L_{\lambda}\psi)_{\tilde{\mu}}$, but since the tableau

$$\begin{array}{cccccccc} z_1 & z_2 & z_3 & \cdots & \cdots & \cdots & z_{\mu_1} \\ z_1 & z_2 & z_3 & \cdots & \cdots & z_{\mu_2} \\ z_1 & z_2 & z_3 & \cdots & z_{\mu_3} \\ \vdots & & & & & & \end{array}$$

(of shape μ) is the only $U^+(H)$ -invariant standard tableau containing $\tilde{\mu}_i$ copies of z_i for every $i = 1, \dots, s$, we in fact get an element of $\text{Im}(i)$ (cf. [2, Theorem in the appendix]).

Therefore, $\varphi(\nu, b)$ can be thought of as a map from $\Lambda_{\nu}\varphi$ to $L_{\lambda/\mu}\varphi$. This map will be the main ingredient in our construction.

2. THE FILTRATION

Definition 2.1. $M_{\nu}\varphi$ is the subcomplex of the complex $L_{\lambda/\mu}\varphi$ given by the sum $\sum_{\tau \geq \nu, b \in B(\lambda/\tau)} \text{Im}(\varphi(\tau, b))$.

The subcomplexes $M_{\nu}\varphi$, ordered lexicographically ($\nu_1 \leq \nu_2 \Rightarrow M_{\nu_2}\varphi \subseteq M_{\nu_1}\varphi$) will give the filtration of $L_{\lambda/\mu}\varphi$ for which we are looking.

We denote by $\dot{M}_{\nu}\varphi$ the subcomplex immediately preceding $M_{\nu}\varphi$, i.e., $\dot{M}_{\nu}\varphi = \sum_{\tau > \nu, b \in B(\lambda/\tau)} \text{Im}(\varphi(\tau, b))$.

Proposition 2.2. (i) $\{M_{\nu}\varphi\}$ is an exhaustive filtration.

(ii) Each $\varphi(\nu, b): \Lambda_{\nu}\varphi \rightarrow L_{\lambda/\mu}\varphi$ induces a map $L_{\nu}\varphi \rightarrow L_{\lambda/\mu}\varphi/\dot{M}_{\nu}\varphi$, indeed a map $L_{\nu}\varphi \rightarrow M_{\nu}\varphi/\dot{M}_{\nu}\varphi$, denoted by $\bar{\varphi}(\nu, b)$.

Proof. (i) One can mimic the proof of [4, Proposition 3.1], provided one replaces each $\Lambda^i(F \oplus G)$ by $\Lambda^i(\varphi \oplus \zeta)$ and keeps in mind the peculiar nature of ζ , as defined in Lemma 1.4.

(ii) One can mimic the proof of [4, Lemma 3.6], reading $\Lambda^i(\varphi \oplus \zeta)$ instead of $\Lambda^i(F \oplus G)$.

We want to show that each $M_{\nu}\varphi/\dot{M}_{\nu}\varphi$ is isomorphic to the sum of exactly $\gamma(\lambda/\mu; \nu)$ copies of $L_{\nu}\varphi$. The idea is to reduce the case over \mathbf{Z} (whence the case over every R) to the case over \mathbf{Q} , where the result is known to be true (cf., e.g., [10, §0.3; 9]).

Combinatorially, $\gamma(\lambda/\mu; \nu)$ is the number of standard tableaux of shape λ/ν , filled with $\tilde{\mu}_1$ copies of 1, $\tilde{\mu}_2$ copies of 2, $\tilde{\mu}_3$ copies of 3, etc., such that the associated word (formed by listing all entries from bottom to top in each column, starting from the leftmost column) is a lattice permutation (cf. [5, §1]).

To the above standard tableaux of shape λ/ν , one can associate in a one-to-one way $\gamma(\lambda/\mu; \nu)$ elements of $B(\lambda/\nu)$, thus obtaining a subset $B'(\lambda/\nu)$ of $B(\lambda/\nu)$ such that $|B'(\lambda/\nu)| = \gamma(\lambda/\mu; \nu)$ (cf. [5, §3]).

Now notice that if in the first part of the proof of [4, Theorem 4.5] one replaces the standard tableau $d_{\nu}(a) \in L_{\nu}F$ by any linear combination $\sum_j c_j d_{\nu}(a_j)$ of standard tableaux, the very same argument shows:

After ordering the elements of $B'(\lambda/\nu)$ in the way described in [5, §3], for

every $b' \in B'(\lambda/\nu)$ one has that

$$\begin{aligned}
 (\$) \quad & \varphi(\nu, b') \left(\sum c_j a_j \right) \in \dot{\mathbf{M}}_\nu + \sum_{b'' < b'} \text{Im}(\varphi(\nu, b'')) \\
 & \text{implies } \sum_j c_j d_\nu(a_j) = 0, \quad \text{whence } c_j = 0 \text{ for every } j.
 \end{aligned}$$

(The proof of [4, Theorem 4.5] is given in the case μ is a rectangle but does not depend on the assumption $\mu_1 = \dots = \mu_r$, as already remarked in [5, §3].)

Once more, reading $\varphi \oplus \zeta$ instead of $F \oplus G$ in the above, (\$) becomes a statement like the following: After ordering the elements of $B'(\lambda/\nu)$ in the way described in [5, §3], for every $b' \in B'(\lambda/\nu)$,

$$\bar{\varphi}(\nu, b') : \mathbf{L}_\nu \varphi \rightarrow \mathbf{M}_\nu \varphi / \left(\dot{\mathbf{M}}_\nu \varphi + \sum_{b'' < b'} \text{Im}(\varphi(\nu, b'')) \right)$$

is injective over \mathbf{Z} .

Thus, over the field \mathbf{Q} , one is forced to conclude from the known decomposition $\sum_\nu \gamma(\lambda/\mu; \nu) \mathbf{L}_\nu \varphi$ of $\mathbf{L}_{\lambda/\mu} \varphi$ that, for every $b \in B(\lambda/\nu) - B'(\lambda/\nu)$,

$$(\&) \quad \text{Im}(\varphi(\nu, b)) \subseteq \dot{\mathbf{M}}_\nu \varphi + \sum_{b' \in B'(\lambda/\nu)} \text{Im}(\varphi(\nu, b')).$$

Mimicking the second part of the proof of [4, Theorem 4.5] (again ignoring the assumption $\mu_1 = \dots = \mu_r$ and reading $\varphi \oplus \zeta$ instead of $F \oplus G$), it follows that (&) in fact holds over \mathbf{Z} .

Summarizing, we have proven that we can safely discard all maps $\varphi(\nu, b)$ with $b \in B(\lambda/\nu) - B'(\lambda/\nu)$, for they do not add anything to the images of the $\varphi(\nu, b')$ with $b' \in B'(\lambda/\nu)$.

This completes the proof of the following theorem.

Theorem 2.3. $\mathbf{L}_{\lambda/\mu} \varphi$ has a universal filtration whose associated graded object is $\sum_\nu \gamma(\lambda/\mu; \nu) \mathbf{L}_\nu \varphi$.

Remark 2.4. Further observations can be made on $B(\lambda/\nu)$, $B'(\lambda/\nu)$, and their elements, as well as on the quotients $\mathbf{M}_\nu \varphi / \dot{\mathbf{M}}_\nu \varphi$, in a way similar to that of [5, §3].

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