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AN ASYMPTOTIC STABILITY AND A UNIFORM ASYMPTOTIC STABILITY FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We consider a system of functional differential equation $x'(t) = F(t, x_t)$ and obtain conditions on a Liapunov functional to ensure the asymptotic stability and the uniform asymptotic stability of the zero solution.

1. INTRODUCTION

The purpose of this paper is to present sufficient conditions, using Liapunov's direct method, to ensure that the zero solution of a system of functional differential equations with infinite delay (including finite delay) is asymptotically stable and that the zero solution of a system of functional differential equations with finite delay is uniformly asymptotically stable. This is, of course, an old problem, and there are many well-known results and applications.

We consider the system

$$x'(t) = F(t, x_t), \tag{1}$$

where x_t is the translation of x on $[t-h, t]$ back to $[-h, 0]$, where $h > 0$ is a fixed constant, and x' denotes the right-hand derivative. The following notation will be used.

For $x \in R^n$, $|x|$ denotes a usual norm in R^n. For $h > 0$, C denotes the space of continuous functions mapping $[-h, 0]$ into R^n, and, for $\phi \in C$, $\|\phi\| = \sup_{-h \le s \le 0} |\phi(s)|$. Also, C_H denotes the set of $\phi \in C$ with $\|\phi\| < H$. If x is a continuous function of u defined for $-h \le u < A$, with $A > 0$, and if t is a fixed number satisfying $0 \le t < A$, then x_t denotes the restriction of x to $[t-h, t]$ so that x_t is an element of C defined by $x_t(\theta) = x(t+\theta)$ for $-h \le \theta \le 0$. We denote by $x(t_0, \phi)$ a solution of (1) with initial condition $\phi \in C$ where $x_{t_0}(t_0, \phi) = \phi$, and we denote by $x(t, t_0, \phi)$ the value of $x(t_0, \phi)$ at t.

It is supposed that $F: R_+ \times C_H \to R^n$ is continuous and takes bounded sets into bounded sets; where $0 < H \le \infty$. It is well known [6, 10] that for each $t_0 \in R_+ = [0, \infty)$ and each $\phi \in C_H$ there is at least one solution

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$x(t_0, \phi)$ defined on an interval $[t_0, t_0 + \alpha)$ and, if there is an $H_1 < H$ with $|x(t, t_0, \phi)| \leq H_1$, then $\alpha = \infty$.

A Liapunov functional is a continuous $V(t, \phi)\colon R_+ \times C_H \to R_+$ whose derivative along a solution of (1) satisfies some specific relation. The derivative of a Liapunov functional $V(t, \phi)$ along a solution $x(t)$ of (1) may be defined in several equivalent ways. If V is differentiable, the natural derivative is obtained using the chain rule. But, in general, $V'_{(1)}(t, \phi)$ denotes the derivative of functional V with respect to (1) defined by

$$V'_{(1)}(t, \phi) = \limsup_{\delta \to 0+} \{V(t+\delta, x_{t+\delta}(t, \phi)) - V(t, \phi)\}/\delta.$$

Definition 1. Let $H > 0$, $S_H = \{x \in R^n \mid |x| < H\}$, and let $U\colon R_+ \times S_H \to R$ be continuous and locally Lipschitz in x. Then the derivative of $U(t, x)$ along a solution x of (1) is defined as

$$U'_{(1)}(t, x) = \limsup_{\delta \to 0+} \{U(t+\delta, x + \delta F(t, x_t)) - U(t, x)\}/\delta.$$

Remark 1. (i) It is easy to check that

$$\limsup_{\delta \to 0+} \frac{1}{\delta}\{U(t+\delta, x(t+\delta)) - U(t, x(t))\} = U'_{(1)}(t, x(t))$$

for any solution $x(t)$ of (1).

(ii) If $U(t, x(t))$ has continuous partial derivatives of the first order,

$$U'_{(1)}(t, x(t)) = \operatorname{grad} U \cdot F + \partial U/\partial t.$$

Definition 2. Let $F(t, 0) = 0$, for all $t \geq 0$.

(a) The zero solution of (1) is said to be stable if for each $\varepsilon > 0$ and $t_0 \geq 0$ there is a $\delta > 0$ such that $[\phi \in C_\delta, t \geq t_0]$ imply $|x(t, t_0, \phi)| < \varepsilon$.

(b) The zero solution is uniformly stable (U.S.) if it is stable and if δ is independent of t_0.

(c) The zero solution is asymptotically stable (A.S.) if it is stable and if for each $t_0 \geq 0$ there is a $\delta > 0$ such that $\phi \in C_\delta$ implies that $x(t, t_0, \phi) \to 0$ as $t \to \infty$.

(d) The zero solution is uniformly asymptotically stable (U.A.S.) if it is U.S. and if there is an $\eta > 0$ and for each $\gamma > 0$ there exists $T > 0$ such that $[t_0 \in R_+, \phi \in C_\eta, t \geq t_0 + T]$ imply that $|x(t, t_0, \phi)| < \gamma$.

Definition 3. A measurable function $\eta\colon R_+ \to R_+$ is said to be integrally positive with parameter $\delta > 0$ (IP(δ)) if whenever $I = \bigcup_{m=1}^{\infty} [\alpha_m, \beta_m]$ with $\alpha_m < \beta_m < \alpha_{m+1}$ and $\beta_m - \alpha_m \geq \delta$ $(m = 1, 2, 3, \ldots)$, then $\int_I \eta(t)\,dt = \infty$. If a function η is integrally positive for every $\delta > 0$, then it is called integrally positive (IP).

Definition 4. Let $\eta\colon R_+ \to R_+$ be mesaurable.

(a) The function η is said to be weakly integrally positive with parameters $\delta > 0$ and $\Delta > 0$ (WIP(δ, Δ)) if whenever $\{t_i\}$ and $\{\delta_i\}$ satisfy $t_i + \delta_i < t_{i+1} \leq t_i + \delta_i + \Delta$ with $\delta_i \geq \delta$, then

$$\sum_{i=1}^{\infty} \int_{t_i}^{t_i+\delta_i} \eta(t)\,dt = \infty.$$

(b) The function η is said to be uniformly weakly integrally positive with parameters $\delta > 0$ and $\Delta > 0$ (UWIP(δ, Δ)) if (a) holds and for every $M > 0$ there exists $Q > 0$ such that for all $S > Q$ and for all $\{t_i\}$ and $\{\delta_i\}$ satisfying (a), then

$$\int_{[t_1, t_1+S]\cap I} \eta(t)\,dt > M \quad \text{where } I = \bigcup_{i=1}^{\infty}[t_i, t_i + \delta_i].$$

Remark 2. If η is IP(δ), then it is UWIP(δ, Δ) for all $\Delta > 0$. The converse is false. See [4, Remark 4].

In presenting sufficient conditions, the following theorem is basic. Denote by W_i the continuous functions from $R_+ \to R_+$, $W_i(0) = 0$, and $W_i(\gamma)$ strictly increasing (called wedges).

Theorem A (see [6, p. 105]). *Let $H > 0$ and $V: R_+ \times C_H \to R_+$ be continuous. If $\exists$ wedges W_1, W_2, W_3 such that, for all $\phi \in C_H$,*

(i) $W_1(|\phi(0)|) \le V(t, \phi) \le W_2(\|\phi\|)$,
(ii) $V'_{(1)}(t, x_t) \le -W_3(|x(t)|)$, *and*
(iii) $F(t, \phi)$ *is bounded for* ϕ *bounded,*

then $x = 0$ of (1) is uniformly asymptotically stable.

One of our main goals is to eliminate condition (iii) in the above theorem.

2. Main results and some remarks

Let $|x(t)|'$ be the right-hand derivative of $|x(t)|$, let $\{a(t)\}_+ = \max\{a(t), 0\}$, and let $\{a(t)\}_- = \max\{-a(t), 0\}$.

Theorem 1. *Let $H > 0$ and $V: R_+ \times C_H \to R_+$ be continuous and locally Lipschitzian in ϕ and η be* WIP(β, Δ) *for any $\beta > 0$ and $\Delta > 0$. Suppose $U: R_+ \times R^n \to R_+$ is continuous and locally Lipschitz in x such that either $\int_0^t\{U'(s, x)\}_+\,ds$ or $\int_0^t\{U'(s, x)\}_-\,ds$ is uniformly continuous for any bounded solution $x(t)$ of (1) on R_+. Further, suppose $\exists$ wedges W_1, W_2, W_3, and W_4 such that, for all $t \ge 0$ and ϕ in C_H,*

(i) $W_1(|\phi(0)|) \le V(t, \phi)$ *and* $V(t, 0) = 0$,
(ii) $V'_{(1)}(t, x_t) \le -\eta(t)W_2(|x(t)|)$, *and*
(iii) $W_3(|x(t)|) \le U(t, x(t)) \le W_4(|x(t)|)$.

Then the zero solution of (1) is asymptotically stable.

Proof. It is evident that the zero solution is stable. By stability, there is a $\delta = \delta(t_0, H)$ such that $[t_0 \ge 0, \phi \in C_\delta, t \ge t_0]$ imply that $|x(t, t_0, \phi)| < H$. Suppose that for some such (t_0, ϕ) the solution $x(t) = x(t, t_0, \phi) \nrightarrow 0$ as $t \to \infty$. First we claim that $\liminf_{t\to\infty} |x(t)| = 0$. If this is false, then there exist constants θ, $T > 0$ such that $|x(t)| \ge \theta$, for $t \ge t_0 + T$. Thus

$$\lim_{t\to\infty} V(t, x_t) \le V(t_0, \phi) - W_2(\theta).\int_{t_0+T}^{\infty} \eta(s)\,ds = -\infty,$$

a contradiction. Suppose that $\int_0^t\{U'(s, x(s))\}_+\,ds$ is uniformly continuous on R_+. Then for some $\gamma > 0$, we can choose a constant $\theta > 0$ and a sequence $t_0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_i < \beta_i < \cdots$ such that $W_4(\theta) < W_3(\gamma)$

and, for $i = 1, 2, 3, \ldots$, $|x(\alpha_i)| = \theta$, $|x(\beta_i)| = \gamma$, and $\theta \le |x(t)|$, for any $t \in [\alpha_i, \beta_i]$. Thus we have

$$W_3(|x(\beta_i)|) \le U(\beta_i, x(\beta_i)) = \int_{\alpha_i}^{\beta_i} U'(s, x(s))\,ds + U(\alpha_i, x(\alpha_i))$$
$$\le \int_{\alpha_i}^{\beta_i} U'(s, x(s))_+\,ds + W_4(|x(\alpha_i)|)$$

and

$$0 < W_3(\gamma) - W_4(\theta) \le \int_{\alpha_i}^{\beta_i} U'(s, x(s))_+\,ds.$$

By assumption there exists $\rho > 0$ such that $\beta_i - \alpha_i \ge \rho$ for $i = 1, 2, 3, \ldots$. Let $I = \bigcup_{i=1}^{\infty}[\alpha_i, \beta_i]$. Then we have

$$\lim_{t\to\infty} V(t, x_t) \le V(t_0, \phi) - W_2(\theta)\int_I \eta(s)\,ds = -\infty,$$

a contradiction. Suppose that $\int_0^t \{U'(s, x(s))\}_-\,ds$ is uniformly continuous on R_+. Then for some $\gamma > 0$, we can choose a constant $\theta > 0$ and a sequence $t_0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_i < \beta_i < \cdots$ such that $W_4(\theta) < W_3(\gamma)$ and, for $i = 1, 2, 3, \ldots$, $|x(\alpha_i)| \ge \gamma$, $|x(\beta_i)| = \theta$, and $\theta \le |x(t)|$, for any $t \in [\alpha_i, \beta_i]$. Thus we have

$$W_4(|x(\beta_i)|) \ge U(\beta_i, x(\beta_i)) = \int_{\alpha_i}^{\beta_i} U'(s, x(s))\,ds + U(\alpha_i, x(\alpha_i))$$
$$\ge -\int_{\alpha_i}^{\beta_i} U'(s, x(s))_-\,ds + W_3(|x(\alpha_i)|)$$

and

$$0 < W_3(\gamma) - W_4(\theta) \le \int_{\alpha_i}^{\beta_i} U'(s, x(s))_-\,ds.$$

By assumption there exists $\rho > 0$ such that $\beta_i - \alpha_i \ge \rho$ for $i = 1, 2, 3, \ldots$. Let $I = \bigcup_{i=1}^{\infty}[\alpha_i, \beta_i]$. Then we have

$$\lim_{t\to\infty} V(t, x_t) \le V(t_0, \phi) - W_2(\theta)\int_I \eta(s)\,ds = -\infty,$$

a contradiction. Thus the proof is complete.

Remark 3. The condition that either $\int_0^t \{U(s,x(s))\}_+ ds$ or $\int_0^t \{U(s,x(s))\}_- ds$ is uniformly continuous for any bounded solution $x(t)$ of (1) on R_+ is satisfied if

$$-p(t) \le U'(t, x(t)) \quad \text{or} \quad U'(t, x(t)) \le q(t),$$

where $p, q\colon R_+ \to R_+$ are measurable functions such that $\int_0^t p(s)\,ds$ and $\int_0^t q(s)\,ds$ are uniformly continuous on R_+.

Corollary 1. *Let $V\colon R_+ \times C_H \to R_+$ be continuous and let η be* WIP(δ, Δ) *for any $\delta > 0$ and $\Delta > 0$. Suppose that*

(i) $W_1(|x(t)|) \le V(t, x_t)$ *and* $V(t, 0) = 0$,
(ii) $V'_{(1)}(t, x_t) \le -\eta(t)W_2(|x(t)|)$, *and*
(iii) $F(t, \phi)$ *is bounded for* ϕ *bounded.*

Then the zero solution of (1) *is asymptotically stable.*

Now, we consider a system of functional differential equations with unbounded delay

$$x' = F(t, x(s); \alpha \le s \le t), \qquad -\infty \le \alpha. \tag{2}$$

To specify a solution of (2) we require a $t_0 \ge \alpha$ and a bounded continuous function $\phi\colon [\alpha, t_0] \to R^n$; we then obtain a solution $x(t, t_0, \phi)$ satisfying (2) on an interval $[t_0, t_0+\beta)$ with $x(t, t_0, \phi) = \phi(t)$ for $\alpha \le t \le t_0$. For details see Driver [5] or Burton [2]. To make the presentation here parallel that for finite delay equations, for each $t > \alpha$ we consider the function space $C(t)$ with $\phi \in C(t)$ if $\phi\colon [\alpha, t] \to R^n$ is bounded and continuous. The norm used is the supremum norm $\|\cdot\|$. Thus, for any $t_0 > \alpha$, our initial function is some $\phi \in C(t_0)$ and our definitions of stability coincide with the one for finite delay. A Liapunov functional is denoted by $V(t, x(\cdot))$. For convenience we may assume that $t_0 \ge 0$.

Theorem 2. *Let $H > 0$ and for each $t_0 > \alpha$ let $C_H(t_0) \subset C(t_0)$ with $\phi \in C_H(t_0)$ if $\|\phi\| < H$, and let $V\colon [t_0, \infty) \times C_H(t_0) \to R_+$ be continuous and locally Lipschitz in ϕ and η be* WIP(β, Δ) *for any $\beta > 0$ and $\Delta > 0$. Suppose $U\colon R_+ \times R^n \to R_+$ is continuous and locally Lipschitz in x such that either $\int_0^t \{U'(s, x)\}_+ \, ds$ or $\int_0^t \{U'(s, x)\}_- \, ds$ is uniformly continuous for any bounded solution $x(t)$ of* (1) *on R_+. Further, suppose $\exists$ wedges W_1, W_2, W_3, and W_4 such that, for all $t \ge t_0$ and ϕ in $C_H(t_0)$,*

(i) $W_1(|\phi(0)|) \le V(t, \phi)$ *and* $V(t, 0) = 0$,
(ii) $V'_{(1)}(t, x_t) \le -\eta(t) W_2(|x(t)|)$, *and*
(iii) $W_4(|x(t)|) \le U(t, x(t)) \le W_4(|x(t)|)$.

Then the zero solution of (2) *is asymptotically stable.*

Proof. The proof requires only slight modifications of the proof of Theorem 1.

Example 1. Consider the scalar equation

$$x'(t) = -a(t)x(t) + b(t)x(t - \lambda t), \tag{A}$$

where $a\colon R_+ \to R_+$ is continuous, $b\colon R_+ \to R$ is continuous, $|b(t-\lambda t)| \ge |b(t)|$ with $0 < \lambda < 1$, and $\eta(t) = a(t) - |b(t)|/(1-\lambda)$ is WIP(δ, Δ) for any $\delta > 0$ and $\Delta > 0$. Then the zero solution of (A) is A.S.

Proof. Consider the functional

$$V(t, x_t) = |x(t)| + \frac{1}{1-\lambda} \int_{(1-\lambda)t}^{t} |b(s)|\,|x(s)|\, ds.$$

Then we have

$$\begin{aligned} V'(t, x_t) \le & -a(t)|x(t)| + |b(t)|\,|x(t-\lambda t)| \\ & + \frac{1}{1-\lambda}|b(t)|\,|x(t)| - |b(t-\lambda t)|\,|x(t-\lambda t)| \\ \le & -\left(a(t) - \frac{1}{1-\lambda}|b(t)|\right)|x(t)| - (|b(t-\lambda t)| - |b(t)|)|x(t-\lambda t)| \\ \le & -\left(a(t) - \frac{1}{1-\lambda}|b(t)|\right)|x(t)|. \end{aligned}$$

Since $|b(t-\lambda t)| \geq |b(t)|$ for any $t \geq 0$ with $0 < \lambda < 1$, $|b(t)|$ is bounded on R_+. Also, we have

$$U'(t, x(t)) = |x(t)|' \leq -a(t)|x(t)| + |b(t)|\,|x(t-\lambda t)|.$$

That is, $U(t, x(t))$ is bounded above on R_+, where $U(t, x(t)) = |x(t)|$. Hence, it follows from Theorem 2 that the zero solution of (A) is A.S.

Example 2. Consider the scalar equation

$$\text{(B)} \qquad x'(t) = -a(t)f(x(t)) + \int_{-\infty}^{t} C(t-s)g(x(s))\,ds,$$

where $a\colon R_+ \to R_+$ is continuous, $C\colon R_+ \to R$ is continuous with $\int_0^\infty |C(u)|\,du < \infty$, $\eta(t) = a(t) - M\int_0^\infty |C(u)|\,du$ is WIP(δ, Δ) for any $\delta > 0$ and $\Delta > 0$, $f\colon R \to R$ is continuous and strictly increasing with $f(0) = 0$, and $g\colon R \to R$ is continuous with $|g(x)| \leq M|f(x)|$ for some $M \geq 0$. Then $x = 0$ of (B) is A.S.

Proof. Consider the functional

$$V(t, x(\cdot)) = |x(t)| + \int_{-\infty}^{t}\int_{t}^{\infty} |C(u-s)|\,du|g(x(s))|\,ds.$$

Then we have

$$\begin{aligned} V'(t, x(\cdot)) \leq &-a(t)|f(x(t))| + \int_{-\infty}^{t} |C(t-s)|\,|g(x(s))|\,ds \\ &+ \int_{t}^{\infty} |C(u-t)|\,du|g(x(t))| - \int_{\infty}^{t} |C(t-s)|\,|g(x(s))|\,ds \\ \leq &-a(t)|f(x(t))| + M\int_0^\infty |C(u)|\,du|f(x(t))| \\ = &-\left\{a(t) - M\int_0^\infty |C(u)|\,du\right\}|f(x(t))|. \end{aligned}$$

Also, we have

$$\begin{aligned} U'(t, x(t)) = |x(t)|' &\leq -a(t)|f(x(t))| + \int_{-\infty}^{t} |C(t-s)|\,|g(x(s))|\,ds \\ &\leq -a(t)|f(x(t))| + M\int_0^\infty |C(u)|\,du f(\|x_t\|), \end{aligned}$$

where $\|x_t\| = \sup_{-\infty \leq s \leq t} |x(s)|$. That is, $|x(t)|'$ is bounded above. Thus, it follows from Theorem 2 that $x = 0$ of (B) is A.S.

Now, we prove a uniform asymptotic stability theorem for a system of functional differential equations with finite delay.

Theorem 3. *Let $H > 0$ and $V\colon R_+ \times C_H \to R_+$ be continuous and locally Lipschitzian in ϕ and η be* UWIP(β, h) *for any $\beta > 0$. Suppose $U\colon R_+ \times R^n \to R_+$ is continuous and locally Lipschitz in x such that either $\int_0^t \{U'(s, x)\}_+\,ds$ or $\int_0^t \{U'(s, x)\}_-\,ds$ is uniformly continuous for any bounded solution $x(t)$ of* (1) *on R_+. Further, suppose $\exists$ wedges W_1, W_2, W_3, W_4, and W_5 such that, for all $t \geq 0$ and ϕ in C_H,*

(i) $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(\|\phi\|)$,

(ii) $V'_{(1)}(t, x_t) \leq -\eta(t) W_3(|x(t)|)$, *and*

(iii) $W_4(|x(t)|) \leq U(t, x(t)) \leq W_5(|x(t)|)$.

Then the zero solution of (1) *is uniformly asymptotically stable.*

Proof. It is evident that the zero solution is U.S. Let $0 < H' < H$ and take $\delta_0 = \delta_0(H')$ of U.S. For any $\varepsilon > 0$, we try to show that there is a $T = T(\varepsilon) > 0$ such that any solution $x(t, t_0, \phi)$ of (1) with $\|\phi\| < \delta_0$ satisfies $|x(t, t_0, \phi)| < \varepsilon$ for any $t \geq t_0 + T$. Let $\delta = \delta(\varepsilon)$ be the above constant for uniform stability. Suppose that a solution $x = x(t, t_0, \phi)$, $\|\phi\| < \delta_0$, satisfies $\|x_t(t_0, \phi)\| \geq \delta$, for any $t \geq t_0$. Then we have $t^* \in [t, t+h]$ for each $t \geq t_0$ such that $|x(t^*)| \geq \delta$. Now we can choose a constant $\theta > 0$ with $W_4(\delta) > W_5(\theta)$. By assumption on η there is an $L = L(\varepsilon) > 0$ such that there is at least $t' \in [t, t+L]$ with $|x(t')| \leq \theta$ for any $t \geq t_0$.

By Definition 4 there is an $L = L(\varepsilon) > 0$ such that

$$\int_{t_0}^{t_0+L} \eta(s)\, ds > W_2(\delta_0)/W_3(\theta).$$

If $|x(t)| > \theta$ were true for all $t \in [t_0, t_0 + L]$, then we would have

$$\begin{aligned} 0 \leq V(t_0 + L) &\leq V(t_0, \phi) - \int_{t_0}^{t_0+L} \eta(s) W_3(|x(s)|)\, ds \\ &\leq W_2(\delta_0) - W_3(\theta) \int_{t_0}^{t_0+L} \eta(s)\, ds < 0, \end{aligned}$$

a contradiction.

Now, we shall assume that $\int_0^t \{U'(s, x(s))\}_+ ds$ is uniformly continuous on R_+ and $L \geq h$. Then we can choose a sequence

$$t_0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_i < \beta_i < \cdots$$

such that, for $i = 1, 2, 3, \ldots$,

$$|x(\alpha_i)| = \theta, \quad |x(\beta_i)| \geq \delta, \quad \theta \leq |x(t)| \qquad \text{for any } t \in [\alpha_i, \beta_i],$$

$\alpha_i \in [t_0 + (2i-2)h,\ t_0 + (2i-1)h] \cup I_i$, and $\beta_i \in I_i$, where $I_i = [t_0 + (2i-1)h,\ t_0 + 2ih]$. Thus we have

$$\begin{aligned} W_4(|x(\beta_i)|) \leq U(\beta_i, x(\beta_i)) &= \int_{\alpha_i}^{\beta_i} U'(s, x(s))\, ds + U(\alpha_i, x(\alpha_i)) \\ &\leq \int_{\alpha_i}^{\beta_i} U'(s, x(s))_+\, ds + W_5(|x(\alpha_i)|) \end{aligned}$$

and

$$0 < W_4(\delta) - W_5(\theta) \leq \int_{\alpha_i}^{\beta_i} U'(s, x(s))_+\, ds.$$

By assumption there exists a $\rho > 0$ with $\beta_i - \alpha_i \geq \rho$ for $i = 1, 2, 3, \ldots$. Let $\mu = \min\{h, \rho\}$ and $I = \bigcup_{i=1}^{\infty} [\beta_i - \mu, \beta_i]$. Then we have

$$\lim_{t \to \infty} V(t, x_t) \leq V(t_0, \phi) - W_3(\theta) \int_I \eta(s)\, ds = -\infty,$$

a contradiction. Let N be the smallest positive integer such that

$$W_2(\delta_0) - W_3(\theta)\sum_{i=1}^{N}\int_{\beta_i-\mu}^{\beta_i}\eta(s)\,ds < 0.$$

Then N only depends on ε and we can take $T = 2Nh$ such that, at some $\tau \in [t_0, t_0+T]$ $\|x_\tau(t_0, \phi)\|$. Thus the proof is complete.

Example 3. Consider the scalar equation

$$\text{(C)} \qquad x'(t) = -a(t)x(t) + b(t)x(t-h),$$

where $a\colon R_+ \to R_+$ is continuous, $b\colon R_+ \to R$ is continuous, $\int_0^t |b(s)|\,ds$ is uniformly continuous on R_+, and $\eta(t) = a(t) - |b(t+h)|$ is UWIP(δ, h) for $\delta > 0$. Then $x = 0$ of (C) is U.A.S.

Proof. Consider the functional

$$V(t, x_t) = |x(t)| + \int_{t-h}^{t}|b(u+h)|\,|x(u)|\,du.$$

Then we have

$$\begin{aligned}|x(t)| \le V(t, x_t) &= |x(t)| + \int_{t-h}^{t}|b(u+h)|\,|x(u)|\,du\\ &\le \|x_t\| + \|x_t\|\int_{t-h}^{t}|b(u+h)|\,du\\ &\le \|x_t\| + M\|x_t\| \quad \text{for some } M>0 \text{ with } \int_{t-h}^{t}|b(u+h)|\,du \le M\end{aligned}$$

on R_+ (by Remark 6). Also, we have

$$\begin{aligned}V'(t, x_t) &\le -a(t)|x(t)| + |b(t)|\,|x(t-h)| + |b(t+h)|\,|x(t)| - |b(t)|\,|x(t-h)|\\ &\le -\{a(t) - |b(t+h)|\}|x(t)|.\end{aligned}$$

Furthermore,

$$U'(t, x(t)) = |x(t)|' \le -a(t)|x(t)| + |b(t)|\,|x(t-h)| \le |b(t)|\,|x(t-h)|$$

and

$$\int_0^t |b(s)|\,ds \quad \text{is uniformly continuous on } R_+.$$

Therefore, it follows from Theorem 3 that $x = 0$ of (C) is U.A.S.

Remark 3. The above example generalizes the results of the example in [9].

Remark 4. Consider the scalar equation

$$\text{(D)} \qquad x'(t) = -a(t)x(t) + b(t)x(t-\gamma(t)),$$

where $a(t)\colon R_+ \to R_+$ is continuous, $\gamma(t) \ge 0$ with $\gamma'(t) \le \alpha < 1$ and $\gamma(t) < h$ for some $h > 0$, $b(t)\colon R_+ \to R$ is continuous with $|b(t-\gamma(t))| \ge |b(t)|$, $a_0(1-\alpha) \ge 1$ for some $a_0 > 0$, and $\eta(t) = a(t) - a_0|b(t)|$ is UWIP(δ, h) for any $\delta > 0$. Then $x = 0$ of (D) is U.A.S.

Proof. Consider the functional

$$V(t, x(\cdot)) = |x(t)| + a_0 \int_{t-\gamma(t)}^{t} |b(s)|\,|x(s)|\,ds.$$

Then we have

$$\begin{aligned} V'(t, x(\cdot)) \le & -a(t)|x(t)| + |b(t)|\,|x(t-\gamma(t))| + a_0|b(t)|\,|x(t)| \\ & - a_0(1-\alpha)|b(t-\gamma(t))|\,|x(t-\gamma(t))| \\ \le & -\{a(t) - a_0|b(t)|\}|x(t)|. \end{aligned}$$

Since

$$U'(t, x(t)) = |x(t)|' \le -a(t)|x(t)| + |b(t)|\,|x(t-\gamma(t))|$$

and $|b(t)|$ is bounded, $|x(t)|'$ is bounded above. Thus $x = 0$ of (D) is U.A.S.

Now, we shall compare Theorem 2 with the following theorem proved by Burton and Hatvani (see [4]).

Definition 5. A measurable function $\eta: R_+ \to R_+$ is said to be positive in measure (PIM) if for every $\varepsilon > 0$ there are $T \in R_+$ and $\delta > 0$ such that $[t \ge T, Q \subset [t-h, t]$ is open, $\mu(Q) \ge \varepsilon]$ imply that $\int_Q \eta(t)\,dt \ge \delta$.

Theorem B. *Let* η *be PIM and* $V: R_+ \times C_H \to R_+$ *be continuous with*

(i) $W_1(|\phi(\cdot)|) \le V(t, \phi) \le W_2(|\phi(\cdot)|) + W_3(|||\phi|||)$ *and*

(ii) $V'(t, x_t) \le -\eta(t) W_4(|x(t)|)$,

where $|||\phi||| = [\int_{-h}^{0} |\phi(s)|^2\,ds]^{1/2}$. *Then* $x = 0$ *of* (1) *is U.A.S.*

Remark 4. We obtain more general results when we apply Theorem 3 to Example 5 than when we apply Theorem B to Example 5, because the PIM condition is stronger than the UWIP(δ, h) condition for any $\delta > 0$ (cf. Theorem 11 and Remark 4 in [4]) and $|b(t)|$ should be bounded in order to have $V'(t, x_t) \le -\eta(t) W(|x(t)|)$ for some coefficient function η and wedge W.

Example 5. Consider the scalar equation

$$\text{(E)} \qquad x'(t) = -a(t)x(t) + b(t)\int_{t-h}^{t} x(u)\,du,$$

where $a(t): R_+ \to R_+$ is continuous, $b(t): R_+ \to R$ is continuous, $\int_t^{t+h} |b(s)|\,ds$ is bounded on R_+, and $a(t) - \int_t^{t+h} |b(s)|\,ds$ is PIM.

By Theorem B the zero solution of (E) is U.A.S. (see [4]), but we can also show that $x = 0$ of (E) is U.A.S. under the condition that $a(t) - \int_t^{t+h} |b(s)|\,ds$ is UWIP(δ, h) for any $\delta > 0$ and $\int_0^t |b(s)|\,ds$ is uniformly continuous on R_+.

Proof. We can use the functional, which was used in an example in [4] and we note that $\int_t^{t+h} |b(s)|\,ds$ is bounded on R_+. For details see [4].

Remark 5. Theorem 2 generalizes Theorem 2.1 in [6, p. 105] and Theorem 8.7.4 in [2, p. 301].

Remark 6. If $b(t): R_+ \to R_+$ is measurable on R_+ and $\alpha(t) = \int_0^t b(s)\,ds$ is uniformly continuous on R_+, then $f(t) = \int_t^{t+b} b(s)\,ds$ is bounded on R_+ for some $\delta > 0$.

Proof. Since $\alpha(t) = \int_0^t b(s)\,ds$ is uniformly continuous on $[0, \infty)$, there exists $\delta^* = \delta^*(\delta) > 0$ such that $\int_t^{t+\delta^*} b(s)\,ds \leq \delta$ for any $t \geq 0$. Now we can choose the smallest positive integer N such that $N\delta^* \geq \delta$. Then we have

$$f(t) = \int_t^{t+\delta} b(s)\,ds \leq \int_t^{t+N\delta^*} b(s)\,ds \leq N\delta \quad \text{for any } t \geq 0.$$

Remark 7. If $b(t)\colon R_+ \to R_+$ is bounded and continuous on R_+, then, obviously, $\alpha(t) = \int_0^t b(s)\,ds$ is uniformly continuous on R_+. If $b(t)\colon R_+ \to R_+$ is unbounded and continuous on R_+ and $\int_0^\infty b(s)\,ds < \infty$, then $\alpha(t) = \int_0^t b(s)\,ds$ is uniformly continuous on R_+ (by [8, Proposition 14, p. 88]). But the next example shows that $\alpha(t) = \int_0^t b(s)\,ds$ is uniformly continuous on R_+ even though $b(t)\colon R_+ \to R_+$ is unbounded, continuous, and not integrable on R_+.

Example 6. Consider the function

$$b(t) = \begin{cases} n & \text{if } t = n,\ n \geq 2, \\ \text{linear} & \text{if } t \in [n - 1/n^2, n] \cup [n, n + 1/n^2],\ n \geq 2, \\ 0 & \text{if } t \in [0, \frac{7}{4}] \cup [n + 1/n^2, (n+1) - 1/(n+1)^2],\ n \geq 2. \end{cases}$$

Then we have

$$\int_0^\infty b(s)\,ds = \sum_{n=2}^\infty \frac{1}{n} = \infty$$

and

$$\int_{n-1/n^2}^{n+1/n^2} b(s)\,ds = \frac{1}{2}\left(\frac{2}{n^2}\right)(n) = \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

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