

MULTIPLICATIVE PERTURBATIONS OF LINEAR VOLTERRA EQUATIONS

ABDELAZIZ RHANDI

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ABSTRACT. We prove that the following problems are essentially equivalent:

$$[\text{VO}]_{CT} \quad u(t) = x + \int_0^t a(t-s)CTu(s) ds,$$

$$[\text{VO}]_{TC} \quad v(t) = y + \int_0^t a(t-s)TCv(s) ds,$$

where T is an unbounded closed linear operator in a Banach space X with dense domain $D(T)$, C is a bounded linear operator on X , and $a \in L^1_{\text{loc}}([0, \infty), \mathbb{R})$, which is exponentially bounded. We give some applications of our abstract theorem to second-order differential operators on the line.

1. INTRODUCTION

The purpose of this note is to study multiplicative perturbations of linear Volterra equations.

Let X be a Banach space and A an unbounded closed linear operator in X with dense domain $D(A)$. Let $a: [0, \infty) \rightarrow \mathbb{R}$ be a locally integrable function. We suppose that $a(t)$ is Laplace-transformable, i.e., there is $\beta \geq 0$ such that $\int_0^\infty e^{-\beta t}|a(t)| dt < \infty$.

We consider the linear Volterra equation

$$[\text{VO}]_A \quad u(t) = x + \int_0^t a(t-s)Au(s) ds \quad (x \in D(A), t \geq 0).$$

Let $(V(t))_{t \geq 0}$ be a family of bounded linear operators in X which is exponentially bounded, i.e., there are constants $M \geq 1$ and $\omega \geq \beta$ such that $\|V(t)\| \leq Me^{\omega t}$ ($t \geq 0$) is satisfied; $(V(t))_{t \geq 0}$ is said to be of type (M, ω) .

The family $(V(t))_{t \geq 0}$ is called a solution family (or a resolvent) for $[\text{VO}]_A$ if the following conditions are satisfied:

- (V₁) $V(t)$ is strongly continuous on \mathbb{R}_+ , and $V(0) = \text{Id}$.
- (V₂) $V(t)$ commutes with A , i.e., $V(t)D(A) \subset D(A)$, and $AV(t)x = V(t)Ax$ for all $x \in D(A)$ and $t \geq 0$.

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(V₃) The following linear Volterra equation holds:

$$V(t)x - x = \int_0^t a(t-s)AV(s)x ds \quad \text{for all } x \in D(A) \text{ and } t \geq 0.$$

We can see that if $[VO]_A$ admits a solution family $(V(t))_{t \geq 0}$ then it is unique and

$$(a * V(t))x := \int_0^t a(t-s)V(s)x ds \in D(A)$$

and

$$V(t)x - x = A \int_0^t a(t-s)V(s)x ds \quad \text{for all } x \in X \text{ and } t \geq 0$$

(cf. [Pr, §1, Proposition 1.1]).

For the special cases $a(t) \equiv 1$ and $a(t) \equiv t$, the solution family $(V(t))_{t \geq 0}$ for $[VO]_A$ becomes the C_0 -semigroup generated by A , respectively, the cosine function generated by A .

Necessary and sufficient conditions for the existence of a solution family for $[VO]_A$ have been considered by DaPrato and Iannelli [DaIa], Arendt and Kellermann [ArKe], Prüss [Pr], and others.

We begin with a fundamental theorem of which we will make use later. By $\rho(A)$ we denote the resolvent set of A .

Theorem (cf. [Pr, §1, Theorem 1.3]). *Assume that $(V(t))_{t \geq 0}$ is strongly continuous and of type (M, ω) . Then $(V(t))_{t \geq 0}$ is a solution family of $[VO]_A$ if and only if the following conditions hold:*

(H₁) $\hat{a}(\lambda) \neq 0$ and $1/\hat{a}(\lambda) \in \rho(A)$ for all $\lambda > \omega$.

(H₂) $H(\lambda, A) := (\lambda - \lambda\hat{a}(\lambda)A)^{-1}$ exists, and $H(\lambda, A) = \int_0^\infty e^{\lambda t}V(t) dt$ for all $\lambda > \omega$.

Let T be a closed linear operator in X with dense domain $D(T)$, and let C be a bounded linear operator on X . Consider the operators TC and CT defined by $D(TC) = \{x \in X, Cx \in D(T)\}$ and $D(CT) = D(T)$. Our main goal is to show that the perturbation problems $[VO]_{CT}$ and $[VO]_{TC}$ are essentially equivalent.

Our result extends the one given by Desch and Schappacher for C_0 -semigroups (cf. [DeScha2, Theorem 1]). As an application we consider the following ordinary differential operator of second order:

$$(d/dx)^2(c(x)\cdot) + d(x)(d/dx) + e(x).$$

2. THE MAIN RESULTS

Let X be a Banach space. Let $a: [0, \infty) \rightarrow \mathbb{R}$ be a locally integrable function. We suppose that there exist $\beta \geq 0$ such that

$$\int_0^\infty e^{-\beta t}|a(t)| dt < \infty.$$

Let T be an unbounded closed linear operator in X with dense domain $D(T)$, and let C be a bounded linear operator on X .

Our main theorem is

Theorem 2.1. (a) If $[VO]_{CT}$ admits a solution family $(U(t))_{t \geq 0}$ on X , then $[VO]_{TC}$ admits a solution family $(V(t))_{t \geq 0}$ on X .

(b) If $[VO]_{TC}$ admits a solution family $(V(t))_{t \geq 0}$ on X and $\rho(CT) \neq \emptyset$, then $[VO]_{CT}$ admits a solution family $(U(t))_{t \geq 0}$ on X .

Corollary 2.2. Let A be a closed linear operator in X with dense domain $D(A)$ such that $[VO]_A$ admits a solution family on X . Let $B: X_A \rightarrow X_A$ be a bounded linear operator, where $X_A := (D(A), \|\cdot\|_A)$. Then $[VO]_{A+B}$ admits a solution family on X .

This corollary is a generalization of a perturbation result given by Desch and Schappacher [DeSch1].

Proof of Corollary 2.2. Let $\lambda \in \rho(A)$, and consider $A_\lambda := -\lambda + A + B$. It follows from the proof of Theorem 1.31 [Na] that $\rho(A_\lambda) \neq \emptyset$. We first show that $[VO]_{A_\lambda}$ admits a solution family on X . Since

$$A_\lambda = (I - BR(\lambda, A))(-\lambda + A),$$

it suffices to show that $[VO]_{B_\lambda}$ admits a solution family on X , where $B_\lambda := (-\lambda + A)(I - BR(\lambda, A))$, but this follows from [Rh1, Corollary 1.2] since $B_\lambda = [(-\lambda + A) + (\lambda - A)BR(\lambda, A)]$ and $(\lambda - A)BR(\lambda, A) \in \mathcal{L}(X)$. So again by [Rh1, Corollary 1.2], $[VO]_{A_\lambda}$ admits a solution family on X . \square

Proof of Theorem 2.1. (a) We set

$$V(t)x := x + T \int_0^t a(t-s)U(s)Cx ds.$$

Since $\int_0^t a(t-s)U(s)Cx ds \in D(CT) = D(T)$, $V(\cdot)x$ is well defined for all $x \in X$. On the other hand, T is closed; hence, $V(t)$ is closed and consequently a bounded linear operator.

Since $[VO]_{CT}$ admits a solution family, CT is closed, and therefore the graph norms of CT and T are equivalent on $D(T)$ (cf. [Br, Corollary II.6]). Then there exists $\gamma > 0$ such that

$$\|V(t)x\| \leq \|x\| + \gamma \left\| \int_0^t a(t-s)U(s)Cx ds \right\|_{CT} \quad \text{for all } x \in X,$$

where $\|\cdot\|_{CT}$ denotes the graph norm of CT .

Let (M, ω) be the type of $(U(t))_{t \geq 0}$. It follows that

$$\begin{aligned} \|V(t)x\| &\leq \|x\| + M\gamma \left(\int_0^t |a(t-s)|e^{\omega s} ds \right) \|Cx\| \\ &\quad + \gamma \left\| CT \int_0^t a(t-s)U(s)Cx ds \right\| \\ &\leq \|x\| + M\gamma e^{\omega t} \left(\int_0^t |a(s)|e^{-\omega s} ds \right) \|Cx\| + \gamma \|U(t)Cx - Cx\| \\ &\leq \|x\| + M\gamma e^{\omega t} \left(\int_0^\infty |a(s)|e^{-\omega s} ds \right) \|Cx\| + \gamma M e^{\omega t} \|Cx\| + \gamma \|Cx\| \\ &\leq M' e^{\omega t} \|x\| \quad \text{for all } x \in X \text{ and some } M' \geq 1, \end{aligned}$$

so $(V(t))_{t \geq 0}$ is exponentially bounded of type (M', ω) . On the other hand, $V(t)$ is strongly continuous. In fact,

$$\begin{aligned} \|V(t)x - V(t_0)x\| &= \left\| T \left[\int_0^t a(s)U(t-s)Cx ds - \int_0^{t_0} a(s)U(t_0-s)Cx ds \right] \right\| \\ &\leq \alpha \left\| \int_0^t a(s)U(t-s)Cx ds - \int_0^{t_0} a(s)U(t_0-s)Cx ds \right\|_{CT} \\ &= \alpha \left\{ \left\| CT \left[\int_0^t a(s)U(t-s)Cx ds - \int_0^{t_0} a(s)U(t_0-s)Cx ds \right] \right\| \right. \\ &\quad \left. + \left\| \int_0^t a(s)U(t-s)Cx ds - \int_0^{t_0} a(s)U(t_0-s)Cx ds \right\| \right\} \\ &= \alpha \left\{ \|U(t)Cx - U(t_0)Cx\| + \left\| \int_0^t a(s)U(t-s)Cx ds \right. \right. \\ &\quad \left. \left. - \int_0^{t_0} a(s)U(t_0-s)Cx ds \right\| \right\} \\ &\quad \text{for all } x \in X \text{ and some } \alpha > 0. \end{aligned}$$

We only have to show that

$$\int_0^\infty e^{-\lambda t} V(t) dt = (\lambda - \lambda \hat{a}(\lambda)TC)^{-1} \quad \text{for all } \lambda > \omega.$$

For $x \in X$ and $\lambda > \bar{\omega}$,

$$\begin{aligned} \int_0^\infty e^{-\lambda t} V(t)x dt &= \int_0^\infty e^{-\lambda t} \left(x + T \int_0^t a(t-s)U(s)Cx ds \right) dt \\ &= \frac{1}{\lambda} x + T \int_0^\infty e^{-\lambda t} \int_0^t a(t-s)U(s)Cx ds dt. \end{aligned}$$

Applying the Fubini theorem we obtain

$$\begin{aligned} \int_0^\infty e^{-\lambda t} V(t)x dt &= \frac{1}{\lambda} x + T \int_0^\infty \left(\int_s^\infty e^{-\lambda t} a(t-s) dt \right) U(s)Cx ds \\ &= \frac{1}{\lambda} x + T \int_0^\infty e^{-\lambda s} \left(\int_0^\infty e^{-\lambda t} a(t) dt \right) U(s)Cx ds \\ &= \frac{1}{\lambda} x + \frac{\hat{a}(\lambda)}{\lambda} T(I - \hat{a}(\lambda)CT)^{-1} Cx \quad \text{for all } x \in X \text{ and } \lambda > \omega. \end{aligned}$$

Let $x \in X$, $\lambda > \omega$, and consider $y := x + T(1/\hat{a}(\lambda) - CT)^{-1}Cx$. We will show that $y \in D(TC)$ and $y = (I - \hat{a}(\lambda)TC)^{-1}x$. We have

$$\begin{aligned} Cy &= Cx + CT \left(\frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} Cx \\ &= Cx + \frac{1}{\hat{a}(\lambda)} \left(\frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} Cx - Cx \\ &= \frac{1}{\hat{a}(\lambda)} \left(\frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} Cx \in D(TC) = D(T); \end{aligned}$$

thus $y \in D(TC)$ and

$$(I - \hat{a}(\lambda)TC)y = y - \hat{a}(\lambda) \cdot \frac{1}{\hat{a}(\lambda)} T \left(\frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} Cx = x.$$

Next, we note that for any $x \in D(TC)$ and $\lambda > \omega$ we have

$$\begin{aligned} & \left(I + T \left(\frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} C \right) (I - \hat{a}(\lambda)TC)x \\ &= x + T \left(\frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} Cx - \hat{a}(\lambda) \left(TCx + T \left(\frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} CTCx \right) \\ &= x + T \left(\frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} Cx - \hat{a}(\lambda) \\ & \quad \times \left(TCx + T \frac{1}{\hat{a}(\lambda)} \left(\frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} Cx - TCx \right) \\ &= x. \end{aligned}$$

Consequently,

$$I + T \left(\frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} C = (I - \hat{a}(\lambda)TC)^{-1}$$

and

$$\int_0^\infty e^{-\lambda t} V(t) dt = \frac{1}{\lambda} (I - \hat{a}(\lambda)TC)^{-1} \quad \text{for } \lambda > \omega.$$

(b) Let $\mu \in \rho(CT)$. We set

$$U(t)x := x + (\mu - CT)C \int_0^t a(t-s)V(s)T(\mu - CT)^{-1}x ds.$$

On the other hand, for all $x \in X$,

$$\int_0^t a(t-s)V(s)x ds \in D(TC).$$

Hence, $U(\cdot)x$ is well defined for all $x \in X$. Since T is closed, $U(t)$ is a bounded linear operator on X . As $t \mapsto TC \int_0^t a(t-s)V(s)x ds = V(t)x - x$ and $t \mapsto \int_0^t a(s)V(t-s)x ds$ are continuous, we conclude that $U(t)$ is strongly continuous. On the other hand, it is clear that $U(\cdot)$ is exponentially bounded of type (M, ω) , where $M \geq M'$ ((M', ω) is the type of $(V(t))_{t \geq 0}$); therefore, it suffices to show that

$$\int_0^\infty e^{-\lambda t} U(t) dt = (\lambda - \lambda \hat{a}(\lambda)CT)^{-1}$$

for all $\lambda > \omega$.

For $x \in X$ and $\lambda > \omega$,

$$\begin{aligned}
 & \int_0^\infty e^{-\lambda t} U(t)x \, dt \\
 &= \frac{1}{\lambda} x + \int_0^\infty e^{-\lambda t} (\mu - CT)C \int_0^t a(t-s)V(s)T(\mu - CT)^{-1}x \, ds \, dt \\
 &= \frac{1}{\lambda} x + (\mu - CT)C \int_0^\infty e^{-\lambda t} \int_0^t a(t-s)V(s)T(\mu - CT)^{-1}x \, ds \, dt \\
 &= \frac{1}{\lambda} x + (\mu - CT)C \int_0^\infty \left(\int_s^\infty e^{-\lambda t} a(t-s) \, dt \right) V(s)T(\mu - CT)^{-1}x \, ds \\
 &= \frac{1}{\lambda} x + (\mu - CT)C \int_0^\infty \left(\int_0^\infty e^{-\lambda t} a(t) \, dt \right) e^{-\lambda s} V(s)T(\mu - CT)^{-1}x \, ds \\
 &= \frac{1}{\lambda} x + \hat{a}(\lambda)(\mu - CT)C(I - \hat{a}(\lambda)TC)^{-1}T(\mu - CT)^{-1}x \\
 &= \frac{1}{\lambda} \left[I + (\mu - CT)C \left(\frac{1}{\hat{a}(\lambda)} - TC \right)^{-1} T(\mu - CT)^{-1} \right] x.
 \end{aligned}$$

For $x \in X$ and $\lambda > \omega$ we put

$$y := x + (\mu - CT)C \left(\frac{1}{\hat{a}(\lambda)} - TC \right)^{-1} T(\mu - CT)^{-1}x.$$

We will show that $y \in D(CT)$ and $(I - \hat{a}(\lambda)CT)y = x$. An elementary calculation gives

$$\begin{aligned}
 y &= \left(\mu - \frac{1}{\hat{a}(\lambda)} \right) C \left(\frac{1}{\hat{a}(\lambda)} - TC \right)^{-1} T(\mu - CT)^{-1}x + \mu(\mu - CT)^{-1}x \\
 &\in D(CT) = D(T)
 \end{aligned}$$

and

$$\begin{aligned}
 \hat{a}(\lambda)CTy &= \hat{a}(\lambda) \left(\mu - \frac{1}{\hat{a}(\lambda)} \right) \\
 &\quad \times C \left[\frac{1}{\hat{a}(\lambda)} \left(\frac{1}{\hat{a}(\lambda)} - TC \right)^{-1} T(\mu - CT)^{-1}x - T(\mu - CT)^{-1}x \right] \\
 &\quad + \mu \hat{a}(\lambda)CT(\mu - CT)^{-1}x \\
 &= y - x.
 \end{aligned}$$

Consequently, $(I - \hat{a}(\lambda)CT)y = x$.

On the other hand, we have, for any $x \in D(T)$,

$$\begin{aligned}
 & \left[I + (\mu - CT)C \left(\frac{1}{\hat{a}(\lambda)} - TC \right)^{-1} T(\mu - CT)^{-1} \right] CTx \\
 &= CT \left[I + (\mu - CT)C \left(\frac{1}{\hat{a}(\lambda)} - TC \right)^{-1} T(\mu - CT)^{-1} \right] x;
 \end{aligned}$$

hence, $I + (\mu - CT)C(1/\hat{a}(\lambda) - TC)^{-1}T(\mu - CT)^{-1} = (I - \hat{a}(\lambda)CT)^{-1}$. \square

3. APPLICATIONS

In this section we describe two applications of Theorem 2.1 to the following ordinary differential operator of second order

$$\left(\frac{d}{dx}\right)^2 (c(x)\cdot) + d(x) \left(\frac{d}{dx}\right) + e(x)$$

with $a(t) \equiv t$.

Example 3.1. Let I be a bounded and closed subinterval of \mathbb{R} , and consider $X = C(I)$, the Banach space of continuous functions in $x \in I$ with norm $\|u\|_\infty = \max_{x \in I} |u(x)|$.

Assume

- (i) $c(x) > 0$ for all $x \in I$,
- (ii) $c(\cdot)$, $c'(\cdot)$, $d(\cdot)$, and $e(\cdot)$ belong to $C(I)$.

Consider the operator T in X defined by

$$Tu = u'' + d(x)u' + e(x)u, \\ D(T) = \{u \in X; u'' \in X; u'(x) = 0 \text{ for } x \in \partial I\},$$

and let $C: X \rightarrow X$ be given by $Cu = c \cdot u$.

It is known that CT generates a cosine function on X (cf. [WaSe, §3]); hence, by Theorem 2.1, TC also generates a cosine function on X . The above argument gives an operator-theoretical approach to the following initial boundary value problem:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2}(c \cdot u) + d(x) \frac{\partial}{\partial x}(c \cdot u) + e(x)c(x) \cdot u, \quad t \in \mathbb{R}, x \in I, \\ u(0, x) = f(x), \quad \frac{\partial}{\partial t} u(0, x) = g(x), \quad x \in I (f, g \in D(TC)). \quad \square$$

Example 3.2. Let $X = L^2(I)$, where $I = (0, 1)$, and define T on X by $Tu = u''$ with $D(T) = \{f \in H^2(0, 1): f(0) = f(1) = 0\}$ (Dirichlet boundary conditions) or $D(T) = \{f \in H^2(0, 1): f'(0) = f'(1) = 0\}$ (Neumann boundary conditions). Then T is selfadjoint and form-negative, so T generates a cosine function on X (cf. [Fa, Theorems 2.2 and 5.1]).

Now let $c \in W^{1,\infty}(I)$ such that $c(x) \geq \delta$ for all $x \in I$ and some constant $\delta > 0$.

We consider the operator C defined by $Cu = c \cdot u$. Then CT generates a cosine function on X . In fact, the function $\varphi(x) = \int_0^x 1/\sqrt{c(s)} ds$ is a homeomorphism of I onto another interval J . Moreover, φ induces an isomorphism between $L^2(I)$ and $L^2(J)$ defined by $V: L^2(J) \rightarrow L^2(I)$, $Vf := f \circ \varphi$.

For $u \in D(CT) = D(T)$ we have

$$(V^{-1}CTV)u = (c \circ \varphi^{-1} \cdot \varphi'' \circ \varphi^{-1})u' + u'' := b(\cdot)u' + u'',$$

where $b(\cdot) := c \circ \varphi^{-1}(\cdot) \cdot \varphi'' \circ \varphi^{-1}(\cdot)$. On the other hand, it follows from [Fa, Theorem 2.2, p. 105 and Theorem 5.1, p. 116] that

$$E := \{x \in L^2(J): U(t)x \text{ is continuously differentiable in } t \in \mathbb{R}\} \\ = \begin{cases} H_0^1(J) & \text{in the case of Dirichlet boundary conditions,} \\ H^1(J) & \text{in the case of Neumann boundary conditions,} \end{cases}$$

where $(U(t))_{t \geq 0}$ is a cosine function on $L^2(J)$ with generator \tilde{T} defined by $\tilde{T}u = u''$ ($u \in L^2(J)$). It follows from the Kisynski theorem (cf. [Ki] or [Wa]) that the matrix operator

$$\mathcal{T} := \begin{pmatrix} 0 & I \\ \tilde{T} & 0 \end{pmatrix} \quad \text{with domain } D(\tilde{T}) \times E$$

is the infinitesimal generator of a strongly continuous group on $E \times L^2(J)$. Since $\mathcal{B} \in \mathcal{L}(E \times L^2(J))$, $\mathcal{B} + \mathcal{T}$ generates a strongly continuous group in $E \times L^2(J)$, where

$$\mathcal{B} := \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} \quad (Bu = b(\cdot)u' \text{ for } u \in E).$$

By [Wa, Theorem], $(V^{-1}CTV)$ generates a cosine function on $L^2(J)$. Consequently, CT generates a cosine function on X . Applying Theorem 2.1, we get that TC generates a cosine function on X , i.e., the following problem is well posed:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2}{\partial x^2}(c \cdot u), & t \in \mathbb{R}, \quad x \in I, \\ u(0, x) &= f(x), \quad \frac{\partial}{\partial t} u(0, x) = g(x), & x \in I \quad (f, g \in D(TC)). \quad \square \end{aligned}$$

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LABORATOIRE DE MATHÉMATIQUES, UNIVERSITÉ DE FRANCHE-COMTÉ, 25030 BESANÇON-CEDEX,
FRANCE

Current address: Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, W-7400 Tübingen, Germany

E-mail address: abr@micelangelo.mathematik.uni-tuebingen.de