MULTIPLICATIVE PERTURBATIONS
OF LINEAR VOLterra EQUATIONS

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Abstract. We prove that the following problems are essentially equivalent:

\[ [\text{VO}]_{CT} \quad u(t) = x + \int_0^t a(t-s)CTu(s) \, ds, \]
\[ [\text{VO}]_{TC} \quad v(t) = y + \int_0^t a(t-s)TCv(s) \, ds, \]

where \( T \) is an unbounded closed linear operator in a Banach space \( X \) with dense domain \( D(T) \), \( C \) is a bounded linear operator on \( X \), and \( a \in L^1_{loc}([0, \infty), \mathbb{R}) \), which is exponentially bounded. We give some applications of our abstract theorem to second-order differential operators on the line.

1. Introduction

The purpose of this note is to study multiplicative perturbations of linear Volterra equations.

Let \( X \) be a Banach space and \( A \) an unbounded closed linear operator in \( X \) with dense domain \( D(A) \). Let \( a: [0, \infty) \to \mathbb{R} \) be a locally integrable function. We suppose that \( a(t) \) is Laplace-transformable, i.e., there is \( \beta \geq 0 \) such that \( \int_0^\infty e^{-\beta t} |a(t)| \, dt < \infty \).

We consider the linear Volterra equation

\[ [\text{VO}]_A \quad u(t) = x + \int_0^t a(t-s)Au(s) \, ds \quad (x \in D(A), \ t \geq 0). \]

Let \( (V(t))_{t \geq 0} \) be a family of bounded linear operators in \( X \) which is exponentially bounded, i.e., there are constants \( M \geq 1 \) and \( \omega \geq \beta \) such that \( \|V(t)\| \leq Me^{\omega t} \) \( (t \geq 0) \) is satisfied; \( (V(t))_{t \geq 0} \) is said to be of type \( (M, \omega) \).

The family \( (V(t))_{t \geq 0} \) is called a solution family (or a resolvent) for \( [\text{VO}]_A \) if the following conditions are satisfied:

\begin{itemize}
  \item[(V_1)] \( V(t) \) is strongly continuous on \( \mathbb{R}_+ \), and \( V(0) = \text{Id} \).
  \item[(V_2)] \( V(t) \) commutes with \( A \), i.e., \( V(t)D(A) \subseteq D(A) \), and \( AV(t)x = V(t)Ax \) for all \( x \in D(A) \) and \( t \geq 0 \).
\end{itemize}

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The following linear Volterra equation holds:

\[ V(t)x - x = \int_0^t a(t-s)AV(s)x\,ds \quad \text{for all } x \in D(A) \text{ and } t \geq 0. \]

We can see that if \([VO]_A\) admits a solution family \((V(t))_{t \geq 0}\) then it is unique and

\[(a * V(t))x := \int_0^t a(t-s)V(s)x\,ds \in D(A)\]

and

\[ V(t)x - x = A\int_0^t a(t-s)V(s)x\,ds \quad \text{for all } x \in X \text{ and } t \geq 0 \]

(cf. [Pr, §1, Proposition 1.1]).

For the special cases \(a(t) \equiv 1\) and \(a(t) \equiv t\), the solution family \((V(t))_{t \geq 0}\) for \([VO]_A\) becomes the \(C_0\)-semigroup generated by \(A\), respectively, the cosine function generated by \(A\).

Necessary and sufficient conditions for the existence of a solution family for \([VO]_A\) have been considered by DaPrato and Iannelli [Dala], Arendt and Kellermann [ArKe], Prüss [Pr], and others.

We begin with a fundamental theorem of which we will make use later. By \(\rho(A)\) we denote the resolvent set of \(A\).

**Theorem** (cf. [Pr, §1, Theorem 1.3]). *Assume that \((V(t))_{t \geq 0}\) is strongly continuous and of type \((M, \omega)\). Then \((V(t))_{t \geq 0}\) is a solution family of \([VO]_A\) if and only if the following conditions hold:*  

\[(H_1) \quad \dot{a}(\lambda) \neq 0 \quad \text{and} \quad 1/\dot{a}(\lambda) \in \rho(A) \quad \text{for all } \lambda > \omega.\]

\[(H_2) \quad H(\lambda, A) := (\lambda - \lambda \dot{a}(\lambda)A)^{-1} \quad \text{exists, and} \quad H(\lambda, A) = \int_0^\infty e^{\lambda t}V(t)\,dt \quad \text{for all } \lambda > \omega.\]

Let \(T\) be a closed linear operator in \(X\) with dense domain \(D(T)\), and let \(C\) be a bounded linear operator on \(X\). Consider the operators \(TC\) and \(CT\) defined by \(D(TC) = \{x \in X, \ Cx \in D(T)\}\) and \(D(CT) = D(T)\). Our main goal is to show that the perturbation problems \([VO]_{CT}\) and \([VO]_{TC}\) are essentially equivalent.

Our result extends the one given by Desch and Schappacher for \(C_0\)-semigroups (cf. [DeScha2, Theorem 1]). As an application we consider the following ordinary differential operator of second order:

\[ (d/dx)^2(c(x)\cdot) + d(x)(d/dx) + e(x). \]

2. The main results

Let \(X\) be a Banach space. Let \(a: [0, \infty) \to \mathbb{R}\) be a locally integrable function. We suppose that there exist \(\beta \geq 0\) such that

\[ \int_0^\infty e^{-\beta t}|a(t)|\,dt < \infty. \]

Let \(T\) be an unbounded closed linear operator in \(X\) with dense domain \(D(T)\), and let \(C\) be a bounded linear operator on \(X\).
Our main theorem is

**Theorem 2.1.** (a) If \([VO]_{CT}\) admits a solution family \((U(t))_{t \geq 0}\) on \(X\), then \([VO]_{TC}\) admits a solution family \((V(t))_{t \geq 0}\) on \(X\).

(b) If \([VO]_{TC}\) admits a solution family \((V(t))_{t \geq 0}\) on \(X\) and \(\rho(CT) \neq \emptyset\), then \([VO]_{CT}\) admits a solution family \((U(t))_{t \geq 0}\) on \(X\).

**Corollary 2.2.** Let \(A\) be a closed linear operator in \(X\) with dense domain \(D(A)\) such that \([VO]_A\) admits a solution family on \(X\). Let \(B: X_A \to X_A\) be a bounded linear operator, where \(X_A := (D(A), \| \cdot \|_A)\). Then \([VO]_{A+B}\) admits a solution family on \(X\).

This corollary is a generalization of a perturbation result given by Desch and Schappacher [DeSch1].

**Proof of Corollary 2.2.** Let \(\lambda \in \rho(A)\), and consider \(A_\lambda := -\lambda + A + B\). It follows from the proof of Theorem 1.31 [Na] that \(\rho(A_\lambda) \neq \emptyset\). We first show that \([VO]_{A_\lambda}\) admits a solution family on \(X\). Since

\[
A_\lambda = (I - BR(\lambda, A))(-\lambda + A),
\]

it suffices to show that \([VO]_{B_1}\) admits a solution family on \(X\), where \(B_1 := (-\lambda + A)(I - BR(\lambda, A))\), but this follows from [Rh1, Corollary 1.2] since \(B_1 = \left((-\lambda + A) + (\lambda - A)BR(\lambda, A)\right) \in \mathscr{L}(X)\). So again by [Rh1, Corollary 1.2], \([VO]_{A_\lambda + B}\) admits a solution family on \(X\). □

**Proof of Theorem 2.1.** (a) We set

\[
V(t)x := x + T \int_0^t a(t-s)U(s)Cx ds.
\]

Since \(\int_0^t a(t-s)U(s)Cx ds \in D(CT) = D(T)\), \(V(t)x\) is well defined for all \(x \in X\). On the other hand, \(T\) is closed; hence, \(V(t)\) is closed and consequently a bounded linear operator.

Since \([VO]_{CT}\) admits a solution family, \(CT\) is closed, and therefore the graph norms of \(CT\) and \(T\) are equivalent on \(D(T)\) (cf. [Br, Corollary II.6]). Then there exists \(\gamma > 0\) such that

\[
\|V(t)x\| \leq \|x\| + \gamma \left\| \int_0^t a(t-s)U(s)Cx ds \right\|_{CT}\]

for all \(x \in X\),

where \(\| \cdot \|_{CT}\) denotes the graph norm of \(CT\).

Let \((M, \omega)\) be the type of \((U(t))_{t \geq 0}\). It follows that

\[
\|V(t)x\| \leq \|x\| + M\gamma \left( \int_0^t |a(t-s)|e^{\omega s} ds \right) \|Cx\|
\]

\[
+ \gamma \left\| CT \int_0^t a(t-s)U(s)Cx ds \right\|
\]

\[
\leq \|x\| + M\gamma e^{\omega t} \left( \int_0^t |a(s)|e^{-\omega s} ds \right) \|Cx\| + \gamma \|U(t)Cx - Cx\|
\]

\[
\leq \|x\| + M\gamma e^{\omega t} \left( \int_0^\infty |a(s)|e^{-\omega s} ds \right) \|Cx\| + \gamma M e^{\omega t} \|Cx\| + \gamma \|Cx\|
\]

\[
\leq M' e^{\omega t}\|x\| \quad \text{for all } x \in X \text{ and some } M' \geq 1,
\]
so \((V(t))_{t \geq 0}\) is exponentially bounded of type \((M', \omega)\). On the other hand, \(V(t)\) is strongly continuous. In fact,

\[
\|V(t)x - V(t_0)x\| = \left\| T \left[ \int_0^t a(s)U(t-s)Cx \, ds - \int_0^{t_0} a(s)U(t_0-s)Cx \, ds \right] \right\| \\
\leq \alpha \left\| \int_0^t a(s)U(t-s)Cx \, ds - \int_0^{t_0} a(s)U(t_0-s)Cx \, ds \right\|_{CT} \\
= \alpha \left\{ \left\| CT \left[ \int_0^t a(s)U(t-s)Cx \, ds - \int_0^{t_0} a(s)U(t_0-s)Cx \, ds \right] \right\| \\
+ \left\| \int_0^t a(s)U(t-s)Cx \, ds - \int_0^{t_0} a(s)U(t_0-s)Cx \, ds \right\| \right\} \\
= \alpha \left\{ \|U(t)Cx - U(t_0)Cx\| + \left\| \int_0^t a(s)U(t-s)Cx \, ds \\
- \int_0^{t_0} a(s)U(t_0-s)Cx \, ds \right\| \right\}
\]

for all \(x \in X\) and some \(\alpha > 0\).

We only have to show that

\[
\int_0^\infty e^{-\lambda t}V(t) \, dt = (\lambda - \lambda \hat{a}(\lambda)TC)^{-1} \quad \text{for all } \lambda > \omega.
\]

For \(x \in X\) and \(\lambda > \omega\),

\[
\int_0^\infty e^{-\lambda t}V(t)x \, dt = \int_0^\infty e^{-\lambda t} \left( x + T \int_0^t a(t-s)U(s)Cx \, ds \right) \, dt \\
= \frac{1}{\lambda} x + T \int_0^\infty e^{-\lambda t} \int_0^t a(t-s)U(s)Cx \, ds \, dt.
\]

Applying the Fubini theorem we obtain

\[
\int_0^\infty e^{-\lambda t}V(t)x \, dt = \frac{1}{\lambda} x + T \int_0^\infty \left( \int_s^\infty e^{-\lambda t}a(t-s) \, dt \right) U(s)Cx \, ds \\
= \frac{1}{\lambda} x + T \int_0^\infty \left( \int_s^\infty e^{-\lambda t}a(t) \, dt \right) U(s)Cx \, ds \\
= \frac{1}{\lambda} x + \frac{\hat{a}(\lambda)}{\lambda} (I - \hat{a}(\lambda)CT)^{-1}Cx \quad \text{for all } x \in X \text{ and } \lambda > \omega.
\]

Let \(x \in X\), \(\lambda > \omega\), and consider \(y := x + T(1/\hat{a}(\lambda) - CT)^{-1}Cx\). We will show that \(y \in D(TC)\) and \(y = (I - \hat{a}(\lambda)TC)^{-1}x\). We have

\[
Cy = Cx + CT \left( \frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} Cx \\
= Cx + \frac{1}{\hat{a}(\lambda)} \left( \frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} Cx - Cx \\
= \frac{1}{\hat{a}(\lambda)} \left( \frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} Cx \in D(CT) = D(T);
\]
thus \( y \in D(TC) \) and

\[
(I - \hat{a}(\lambda)TC)y = y - \hat{a}(\lambda) \cdot \frac{1}{\hat{a}(\lambda)} T \left( \frac{1}{\hat{a}(\lambda)} - CT \right)^{-1}Cx = x.
\]

Next, we note that for any \( x \in D(TC) \) and \( \lambda > \omega \) we have

\[
\left( I + T \left( \frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} \right) (I - \hat{a}(\lambda)TC)x = x + T \left( \frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} Cx - \hat{a}(\lambda) \left( TCx + T \left( \frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} CTCx \right)
\]

\[
= x + T \left( \frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} Cx - \hat{a}(\lambda) \left( TCx + T \frac{1}{\hat{a}(\lambda)} \left( \frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} Cx - TCTx \right)
\]

Consequently,

\[
I + T \left( \frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} C = (I - \hat{a}(\lambda)TC)^{-1}
\]

and

\[
\int_0^\infty e^{-\lambda t}V(t) dt = \frac{1}{\lambda} (I - \hat{a}(\lambda)TC)^{-1} \quad \text{for } \lambda > \omega.
\]

(b) Let \( \mu \in \rho(CT) \). We set

\[
U(t)x := x + (\mu - CT)C \int_0^t a(t - s)V(s)T(\mu - CT)^{-1}x ds.
\]

On the other hand, for all \( x \in X \),

\[
\int_0^t a(t - s)V(s)x ds \in D(TC).
\]

Hence, \( U(\cdot)x \) is well defined for all \( x \in X \). Since \( T \) is closed, \( U(t) \) is a bounded linear operator on \( X \). As \( t \mapsto TC \int_0^t a(t - s)V(s)x ds = V(t)x - x \) and \( t \mapsto \int_0^t a(s)V(t - s)x ds \) are continuous, we conclude that \( U(t) \) is strongly continuous. On the other hand, it is clear that \( U(\cdot) \) is exponentially bounded of type \((M, \omega)\), where \( M \geq M' \) \((M', \omega) \) is the type of \((V(t))_{t \geq 0}\) ; therefore, it suffices to show that

\[
\int_0^\infty e^{-\lambda t}U(t) dt = (\lambda - \lambda a(\lambda)CT)^{-1}
\]

for all \( \lambda > \omega \).
For $x \in X$ and $\lambda > \omega$,
\[
\int_0^\infty e^{-\lambda t} U(t)x \, dt = \frac{1}{\lambda} x + \int_0^\infty e^{-\lambda t}(\mu - CT)C \int_0^t a(t-s)V(s)T(\mu - CT)^{-1}x \, ds \, dt \\
= \frac{1}{\lambda} x + (\mu - CT)C \int_0^\infty \left( \int_s^\infty e^{-\lambda t}a(t-s) \, dt \right) V(s)T(\mu - CT)^{-1}x \, ds \\
= \frac{1}{\lambda} x + (\mu - CT)C \int_0^\infty \left( \int_s^\infty e^{-\lambda t}a(t-s) \, dt \right) e^{-\lambda s}V(s)T(\mu - CT)^{-1}x \, ds \\
= \frac{1}{\lambda} x + A(T)(\mu - CT)C(I - \hat{a}(\lambda)(\mu - CT)^{-1})T(\mu - CT)^{-1}x \\
= \frac{1}{\lambda} \left[ I + (\mu - CT)C \left( \frac{1}{\hat{a}(\lambda)} - TC \right)^{-1} T(\mu - CT)^{-1} \right] x.
\]

For $x \in X$ and $\lambda > \omega$ we put
\[
y := x + (\mu - CT)C \left( \frac{1}{\hat{a}(\lambda)} - TC \right)^{-1} T(\mu - CT)^{-1}x.
\]

We will show that $y \in D(CT)$ and $(I - \hat{a}(\lambda)CT)y = x$. An elementary calculation gives
\[
y = \left( \mu - \frac{1}{\hat{a}(\lambda)} \right) C \left( \frac{1}{\hat{a}(\lambda)} - TC \right)^{-1} T(\mu - CT)^{-1}x + \mu(\mu - CT)^{-1}x \\
\in D(CT) = D(T)
\]

and
\[
\hat{a}(\lambda)CTy = \hat{a}(\lambda) \left( \mu - \frac{1}{\hat{a}(\lambda)} \right) \\
\times C \left[ \frac{1}{\hat{a}(\lambda)} \left( \frac{1}{\hat{a}(\lambda)} - TC \right)^{-1} T(\mu - CT)^{-1}x - T(\mu - CT)^{-1}x \right] \\
+ \mu\hat{a}(\lambda)CT(\mu - CT)^{-1}x \\
= y - x.
\]

Consequently, $(I - \hat{a}(\lambda)CT)y = x$.

On the other hand, we have, for any $x \in D(T)$,
\[
\left[ I + (\mu - CT)C \left( \frac{1}{\hat{a}(\lambda)} - TC \right)^{-1} T(\mu - CT)^{-1} \right] CTx \\
= CT \left[ I + (\mu - CT)C \left( \frac{1}{\hat{a}(\lambda)} - TC \right)^{-1} T(\mu - CT)^{-1} \right] x;
\]

hence, $I + (\mu - CT)C(1/\hat{a}(\lambda) - TC)^{-1}T(\mu - CT)^{-1} = (I - \hat{a}(\lambda)CT)^{-1}$.
3. Applications

In this section we describe two applications of Theorem 2.1 to the following ordinary differential operator of second order

\[ \left( \frac{d}{dx} \right)^2 (c(x)u) + d(x) \left( \frac{d}{dx} \right) e(x) \]

with \( a(t) \equiv t \).

**Example 3.1.** Let \( I \) be a bounded and closed subinterval of \( \mathbb{R} \), and consider \( X = C(I) \), the Banach space of continuous functions in \( x \in I \) with norm \( \|u\|_\infty = \max_{x \in I} |u(x)| \).

Assume

(i) \( c(x) > 0 \) for all \( x \in I \),
(ii) \( c(\cdot), c'(\cdot), d(\cdot), \) and \( e(\cdot) \) belong to \( C(I) \).

Consider the operator \( T \) in \( X \) defined by

\[ Tu = u'' + d(x)u' + e(x)u, \quad D(T) = \{ u \in X; u'' \in X; u'(x) = 0 \text{ for } x \in \partial I \}, \]

and let \( C : X \to X \) be given by \( Cu = c \cdot u \).

It is known that \( CT \) generates a cosine function on \( X \) (cf. [WaSe, §3]); hence, by Theorem 2.1, \( TC \) also generates a cosine function on \( X \). The above argument gives an operator-theoretical approach to the following initial boundary value problem:

\[ \begin{align*}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2}{\partial x^2} (c \cdot u) + d(x) \frac{\partial}{\partial x} (c \cdot u) + e(x) c(x) \cdot u, \quad t \in \mathbb{R}, \ x \in I, \\
u(0, x) &= f(x), \quad \frac{\partial}{\partial t} u(0, x) = g(x), \quad x \in I \ (f, g \in D(T)) \quad \square
\end{align*} \]

**Example 3.2.** Let \( X = L^2(I) \), where \( I = (0, 1) \), and define \( T \) on \( X \) by \( Tu = u'' \) with \( D(T) = \{ f \in H^2(0, 1): f(0) = f(1) = 0 \} \) (Dirichlet boundary conditions) or \( D(T) = \{ f \in H^2(0, 1): f'(0) = f'(1) = 0 \} \) (Neumann boundary conditions). Then \( T \) is selfadjoint and form-negative, so \( T \) generates a cosine function on \( X \) (cf. [Fa, Theorems 2.2 and 5.1]).

Now let \( c \in W^{1, \infty}(I) \) such that \( c(x) \geq \delta \) for all \( x \in I \) and some constant \( \delta > 0 \).

We consider the operator \( C \) defined by \( Cu = c \cdot u \). Then \( CT \) generates a cosine function on \( X \). In fact, the function \( \phi(x) = \int_0^x 1/\sqrt{c(s)} \, ds \) is a homeomorphism of \( I \) onto another interval \( J \). Moreover, \( \phi \) induces an isomorphism between \( L^2(I) \) and \( L^2(J) \) defined by \( V : L^2(J) \to L^2(I), \ V f := f \circ \phi \).

For \( u \in D(CT) = D(T) \) we have

\[ (V^{-1}CTV)u = (c \circ \phi^{-1} \cdot \phi'' \circ \phi^{-1}) u' + u'' := b(\cdot) u' + u'', \]

where \( b(\cdot) := c \circ \phi^{-1}(\cdot) \cdot \phi'' \circ \phi^{-1}(\cdot) \). On the other hand, it follows from [Fa, Theorem 2.2, p. 105 and Theorem 5.1, p. 116] that

\[ E := \{ x \in L^2(J): U(t)x \text{ is continuously differentiable in } t \in \mathbb{R} \} \]

\[ = \begin{cases} 
H^1_0(J) & \text{in the case of Dirichlet boundary conditions,} \\
H^1(J) & \text{in the case of Neumann boundary conditions,}
\end{cases} \]
where \((U(t))_{t \geq 0}\) is a cosine function on \(L^2(J)\) with generator \(\tilde{T}\) defined by 
\[\tilde{T}u = u''\quad (u \in L^2(J)).\]
It follows from the Kisynski theorem (cf. [Ki] or [Wa]) that the matrix operator
\[
\mathcal{F} := \begin{pmatrix}
0 & I \\
\tilde{T} & 0
\end{pmatrix}
\]
with domain \(D(\tilde{T}) \times E\) is the infinitesimal generator of a strongly continuous group on \(E \times L^2(J)\).
Since \(\mathcal{B} \in \mathcal{L}(E \times L^2(J))\), \(\mathcal{B} + \mathcal{F}\) generates a strongly continuous group in \(E \times L^2(J)\), where
\[
\mathcal{B} := \begin{pmatrix}
0 & 0 \\
B & 0
\end{pmatrix}
\quad (Bu = b(\cdot)u'\text{ for } u \in E).
\]
By [Wa, Theorem], \((V^{-1}CTV)\) generates a cosine function on \(L^2(J)\). Consequently, \(CT\) generates a cosine function on \(X\). Applying Theorem 2.1, we get that \(TC\) generates a cosine function on \(X\), i.e., the following problem is well posed:
\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2}(c \cdot u), \quad t \in \mathbb{R}, \quad x \in I,
\]
\[
u(0, x) = f(x), \quad \frac{\partial}{\partial t}u(0, x) = g(x), \quad x \in I \quad (f, g \in D(TC)).\]

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**References**


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