OSCILLATION CRITERIA FOR HAMILTONIAN
MATRIX DIFFERENCE SYSTEMS

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ABSTRACT. We obtain some oscillation criteria for the Hamiltonian difference
system
$$\begin{cases} \Delta Y(t) = B(t)Y(t+1) + C(t)Z(t), \\ \Delta Z(t) = -A(t)Y(t+1) - B^*(t)Z(t), \end{cases}$$
where A, B, C, Y, Z are $d \times d$ matrix functions. As a corollary, we establish
the validity of an earlier conjecture for a second-order matrix difference system.

1. INTRODUCTION AND PRELIMINARY RESULTS

Consider the linear Hamiltonian difference system

$$(1.1) \qquad \begin{aligned} \Delta y(t) &= B(t)y(t+1) + C(t)z(t), \\ \Delta z(t) &= -A(t)y(t+1) - B^*(t)z(t), \end{aligned}$$

the corresponding matrix system

$$(1.2) \qquad \begin{aligned} \Delta Y(t) &= B(t)Y(t+1) + C(t)Z(t), \\ \Delta Z(t) &= -A(t)Y(t+1) - B^*(t)Z(t), \end{aligned}$$

and the Riccati equation

$$(1.3) \qquad \begin{aligned} &\Delta W(t) + A(t) + B^*(t)W(t) + W(t)B(t) - B^*(t)W(t)B(t) \\ &+ (I - B(t))^* W(t)(C^{-1}(t) + W(t))^{-1} W(t)(I - B(t)) = 0, \end{aligned}$$

where $A(t)$, $B(t)$, $C(t)$, $W(t)$, $Y(t)$, $Z(t)$ are $d \times d$ matrices with $A(t)$, $C(t)$
Hermitian, $C(t) > 0$, and $I - B(t)$ invertible. Here $y(t)$, $z(t)$ are $d \times 1$ vectors
and t takes on integer values in $[M-1, N+1]$, where M, N are two integers.

In [4, 5] the authors extended many of the results to equations (1.1)–(1.3)
which had been developed for linear Hamiltonian differential systems of the
form

$$(1.4) \qquad \begin{aligned} y'(x) &= B(x)y(x) + C(x)z(x), \\ z'(x) &= -A(x)y(x) - B^*(x)z(x). \end{aligned}$$

Here $x \in I$ is a finite or infinite interval, A, B, C are continuous $d \times d$ matrix-valued functions, and y, z are $d \times 1$ vector functions. Many of the results for (1.4) may be found in the book of Coppel [2], and in [4, 5] it was shown that discrete analogues of many of these results may be obtained. Related work on symmetric three-term recurrences may be found in [1] and the references therein. In this paper, we shall obtain some oscillation and disconjugacy criterion for (1.1), (1.2) and, as a consequence of our results, shall prove a generalization of a conjecture of Peterson and Ridenhour [7]. We recall some notation and definitions.

We say (1.1) is disconjugate on $[M-1, N+1]$ iff for any nontrivial prepared solution $\{y(t), z(t)\}$ of (1.1) there exists at most one integer $p \in [M-1, N]$ such that either $y^*(p)C^{-1}(p)(I-B(p))y(p+1) \leq 0$ when $y(p) \neq 0$ or $y(p) = 0$. Recall that a solution $\{y(t), z(t)\}$ of (1.1) is said to be prepared if $y^*(t)z(t)$ is real valued and that a solution $\{Y(t), Z(t)\}$ of (1.2) is said to be prepared if $Y^*(t)Z(t)$ is Hermitian. We say a prepared solution of (1.2) is a conjoined basis if $\mathrm{Rank}[\begin{smallmatrix}Y(t)\\Z(t)\end{smallmatrix}] \equiv d$, and it is said to be recessive at ∞ if there exists an integer M_0 for which

$$(1.5) \qquad Y^*(t)C^{-1}(t)(I - B(t))Y(t+1) > 0, \qquad t \geq M_0,$$

and

$$(1.6) \qquad \lim_{n \to \infty} \sum_{s=M_0}^{n} u^*(Y^*(s)C^{-1}(s)(I - B(s))Y(s+1))^{-1}u = \infty$$

for every unit vector u. A prepared solution of (1.2) is said to be dominant at ∞ if (1.5) holds for some integer M_0 and

$$(1.7) \qquad \sum_{s=M_0}^{\infty} u^*(Y^*(s)C^{-1}(s)(I - B(s))Y(s+1))^{-1}u$$

converges for every unit vector u.

Equation (1.1) is said to be eventually disconjugate in case there exists an integer M_0 such that (1.1) is disconjugate on $[M_0 - 1, N_1 + 1]$ for all integers $N_1 > M_0$.

We introduce the following quadratic forms:

$$q[u] = \sum_{t=M}^{N+1} (z^*(t-1)C(t-1)z(t-1) - y^*(t)A(t-1)y(t)),$$

where

$$u = \{y(t), z(t)\} \in \Omega = \{y, z \in C^d : y(M-1) = 0 = y(N+1),$$
$$\Delta y(t) = B(t)y(t+1) + C(t)z(t)\},$$

and

$$Q[U] = \sum_{t=M}^{N+1} (Z^*(t-1)C(t-1)Z(t-1) - Y^*(t)A(t-1)Y(t)),$$

where

$$U = \{Y(t), Z(t)\} \in \Lambda = \{Y, Z \in C^{d \times d} : Y(M-1) = 0 = Y(N+1),$$
$$\Delta Y(t) = B(t)Y(t+1) + C(t)Z(t)\}.$$

We introduce the further notation:

$$\Lambda^+ := \{U \in \Lambda: \text{there is a } t_0, \, M - 1 \leq t_0 \leq N - 1, \text{ such that } Y(t_0) = 0$$
$$\text{and } Y(t_0 + 1) \text{ is nonsingular or there is } M + 1 \leq t_0 \leq N + 1$$
$$\text{such that } Y(t_0) = 0 \text{ and } Y(t_0 - 1) \text{ is nonsingular}\}.$$

We say q is positive on Ω provided $q[u] \geq 0$ for all $u \in \Omega$ and $q = 0$ iff $u \equiv 0$; Q is positive definite on Λ provided, for all $U \in \Lambda$, $Q[U] \geq 0$ and $Q = 0$ iff $U \equiv 0$; Q is strictly positive on Λ^+ if $Q[U] > 0$ for all $U \in \Lambda^+$.

The following results were established in [4, 5]: The first theorem may be regarded as a discrete version of the "Reid Roundabout Theorem" (cf. Ahlbrandt [1]).

Theorem 1. *The following are equivalent*:

   (i) *Equation* (1.1) *is disconjugate on* $[M - 1, N + 1]$.
  (ii) $q[u]$ *is positive definite on* Ω.
 (iii) $Q[U]$ *is positive definite on* Λ *and strictly positive on* Λ^+.
 (iv) *There exists a Hermitian solution of the Riccati equation* (3) *such that* $C^{-1}(t) + W(t) > 0$, $t \in [M - 1, N]$.
  (v) *There exists a solution of equation* (1.2) *such that*

$$Y^*(t)C^{-1}(t)(I - B(t))Y(t + 1) > 0, \qquad t \in [M - 1, N].$$

Theorem 2. *Assume* (1.1) *is eventually disconjugate. Then it follows that*:
(i) *every conjoined basis* $\{Y(t), Z(t)\}$ *satisfies*

$$Y^*(t)C^{-1}(t)(I - B(t))Y(t + 1) > 0, \qquad t \geq M_1 \geq M \; (M_1 \text{ large enough});$$

(ii) *there exists a solution* $\eta_0 = \{Y_0(t), Z_0(t)\}$ *of* (1.2) *which is recessive at* ∞;
(iii) *if* $\eta_1 = \{Y_1(t), Z_1(t)\}$ *is any prepared solution of* (1.2) *such that* $Z_0^*(t)Y_1(t) - Y_0^*(t)Z_1(t)$ *is invertible, then* η_1 *is a dominant solution of* (1.2) *and* $Y_1^{-1}(t)Y_0(t) \to 0$ *(zero matrix) as* $t \to \infty$.

By using Theorem 1, one can obtain a comparison theorem between the two systems:

$$(1.8)_i \qquad \begin{aligned} \Delta y(t) &= B_i(t)y(t + 1) + C_i(t)z(t), \\ \Delta z(t) &= -A_i(t)y(t + 1) - B_i(t)z(t), \end{aligned} \qquad i = 1, 2,$$

where we make the assumption on $(1.8)_i$ as (1.1).
Denote

$$(1.9)_i \quad D_i(t) = \begin{bmatrix} C_i^{-1}(t) & -B_i^* C_i^{-1}(t) \\ -C_i^{-1}(t)B_i(t) & B_i^*(t)C_i^{-1}(t)B_i(t) - A_i(t) \end{bmatrix}, \qquad i = 1, 2.$$

The following is then a generalized Sturm Comparison Theorem.

Theorem 3. *If* $(1.8)_1$ *is disconjugate on* $[M - 1, N + 1]$ *and* $D_2(t) \geq D_1(t)$ *on* $[M - 1, N + 1]$, *then* $(1.8)_2$ *is disconjugate also.*

Proof. For $u_1 = \{y_1(t), z_1(t)\}$ with $y_1(M-1) = 0 = y_1(N+1)$ and $\Delta y_1(t) = B_1(t)y_1(t+1) + C_1(t)z_1(t)$, i.e., $z_1(t) = C_1^{-1}(t)(\Delta y_1(t) - B_1(t)y_1(t+1))$, we have

$$q_1[u_1] = \sum_{t=M-1}^{N} (z_1^*(t)C_1(t)z_1(t) - y_1^*(t+1)A_1(t)y_1(t+1))$$

$$= \sum_{t=M-1}^{N} (\Delta y_1^*(t), y_1^*(t+1))D_1(t)\begin{pmatrix} \Delta y_1(t) \\ y_1(t+1) \end{pmatrix},$$

and for $u_2 = \{y_1(t), z_2(t)\}$ with $z_2(t) = C_2^{-1}(t)(\Delta y - 1(t) - B_2(t)y_1(t))$, we have

$$q_2[u_2] = \sum_{t=M-1}^{N} (\Delta y_1^*(t), y_1^*(t+1))D_2(t)\begin{pmatrix} \Delta y_1(t) \\ y_1(t+1) \end{pmatrix}$$

$$\geq \sum_{t=M-1}^{N} (\Delta y_1^*(t), y_1^*(t+1))D_1(t)\begin{pmatrix} \Delta y_1(t) \\ y_1(t+1) \end{pmatrix} = q_1[u_1].$$

By Theorem 1, we know $q_1[u_1] \geq 0$ and so $q_2[u_2] \geq 0$, i.e., $(1.8)_2$ is disconjugate.

Corollary 4. *If* $(1.8)_1$ *is disconjugate and* $B_1(t) = B_2(t)$, $A_1(t) \geq A_2(t)$, $C_1(t) \geq C_2(t)$, *then* $(1.8)_2$ *is disconjugate.*

Proof. This can be shown from Theorem 3 and

$$D_i(t) = \begin{pmatrix} I & 0 \\ -B_i^*(t) & I \end{pmatrix}\begin{pmatrix} C_i^{-1}(t) & 0 \\ 0 & -A_i(t) \end{pmatrix}\begin{pmatrix} I & -B_i(t) \\ 0 & i \end{pmatrix}.$$

Matrix systems. Next we consider the oscillation of solutions of the matrix system (1.2).

Definition. The Hamiltonian matrix difference system (1.2) is said to be non-oscillatory if for each conjoined basis $\{Y(t), Z(t)\}$ there exists an integer t_0 such that, for $t \geq t_0 \geq M - 1$, we have

$$Y^*(t)C^{-1}(t)(I - B(t))Y(t+1) > 0.$$

Otherwise we say it is oscillatory.

From Theorem 1, we know if (1.2) is nonoscillatory, then (1.1) is eventually disconjugate. If we suppose that there is a Hermitian solution $W(t)$ of (1.3) with $W(t) + C^{-1}(t) > 0$, then this solution satisfies the rewritten Riccati equation (1.3):

(1.10) $$\Delta W(t) + G^*(t)G(t) + h(t) = 0$$

or

(1.11) $$\Delta W(t) + p(t) + h(t) = 0,$$

where

$$G(t) = (C^{-1}(t) + W(t))^{-1/2} W(t)(I - B(t)) + (C^{-1}(t) + W(t))^{1/2} B(t),$$
$$h(t) = A(t) - B^*(t) C^{-1}(t) B(t),$$
$$\rho(t) = (W(t) + C^{-1}(t) B(t))^*(C^{-1}(t) + W(t))^{-1}(W(t) + C^{-1}(t) B(t)).$$

Denote by F the set of all sequences of real numbers $s = \{s(t)\}_{t=0}^{\infty}$ with $0 \leq s(t) \leq 1$ and $\sum_{\tau=0}^{\infty} s(\tau) = +\infty$.

Let $S(t) = \sum_{\tau=0}^{t} s(\tau)$, $S(t, t_0) = \sum_{\tau=t_0}^{t} s(\tau)$, $l(t) = \lambda_d(C^{-1}(t))$, and $L(t) = \lambda_1(C^{-1}(t))$. Here we suppose that the eigenvalues of $C^{-1}(t)$ are ordered with

$$\lambda_1(C^{-1}(t)) \geq \lambda_2(C^{-1}(t)) \geq \cdots \geq \lambda_d(C^{-1}(t)).$$

We introduce the following conditions which will be used in subsequent results:

(S_1) $\limsup_{t \to \infty} S^{-1-\alpha}(t) \sum_{\tau=0}^{t} S(\tau) L(\tau + 1) < +\infty$;

(S_2) $\limsup_{t \to \infty} S^{-\alpha}(t) L(t) < +\infty$; where $\alpha \geq 0$.

Similar to [3], we can prove

Theorem 5. *Let* (S_1) *hold for some* $s \in F$. *Then equation* (1.2) *is oscillatory provided*

$$\limsup_{t \to \infty} S^{-1-\alpha}(t) \lambda_1 \left(\sum_{\tau=0}^{t} S(\tau) \sum_{k=0}^{\tau} (A(k) - B^*(k) C^{-1}(k) B(k)) \right) = +\infty.$$

Theorem 6. *Let* (S_2) *hold for some* $s \in F$. *Then* (1.2) *is oscillatory provided*

$$\limsup_{t \to \infty} S^{-1-\alpha}(t) \lambda_1 \left(\sum_{\tau=0}^{t} (A(\tau) - B^*(\tau) C^{-1}(\tau) B(\tau)) \right) = +\infty.$$

Example. Let

$$C(t) = \begin{pmatrix} t^{-\alpha-1/2} & 0 \\ 0 & t^{-\alpha} \end{pmatrix}, \qquad A(t) = \begin{pmatrix} t^{\alpha+1/2} & \frac{1}{2} \\ \frac{1}{2} & t^{\alpha} \end{pmatrix},$$

$a \geq 0$, $B(t) = \frac{1}{2} I$. From Theorem 6 it follows that (1.2) is oscillatory in this case.

Theorem 7. *If* (1.2) *is nonoscillatory, then there exists* t_0 *such that, for all* $t \geq t_0$, *we have*

$$A(t_0) + \sum_{\tau=t_0+1} (A(\tau) - B^*(\tau) C^{-1}(\tau) B(\tau))$$
$$< C^{-1}(t + 1) + (I - B(t_0))^* C^{-1}(t_0)(I - B(t_0)).$$

Proof. Since (1.2) is nonoscillatory, there exists a sufficiently large integer t_0 and a Hermitian matrix solution of (1.11) with $C^{-1}(t) + W(t) > 0$ for $t \geq t_0$.

Taking the summation of both sides of (1.11) from t_0 to t, we obtain

$$-W(t+1) = \sum_{\tau=t_0+1}^{t} h(\tau) + \sum_{\tau=t_0+1}^{t} \rho(\tau) + A(t_0) - W(t_0) + W(t_0)B(t_0)$$
$$+ B^*(t_0)W(t_0) - B^*(t_0)W(t_0)B(t_0) + (I - B(t_0))^* W(t_0)$$
$$\times (C^{-1}(t_0) + W(t_0))^{-1} W(t_0)(I - B(t_0))$$
$$\geq \sum_{\tau=t_0+1}^{t} h(\tau) + A(t_0)$$
$$+ (I - B(t_0))^*(W(t_0)(C^{-1} + W(t_0))^{-1}W(t_0) - W(t_0))(I - B(t_0))$$
$$= \sum_{\tau=t_0+1}^{t} h(\tau) + A(t_0)$$
$$- (I - B(t_0))^* C^{-1}(t_0)(C^{-1}(t_0) + W(t_0))^{-1}W(t_0)(I - B(t_0))$$
$$= \sum_{\tau=t_0+1}^{t} h(\tau) + A(t_0) - (I - B(t_0))^* C^{-1}(t_0)(I - B(t_0))$$
$$+ (I - B(t_0))^*(C^{-1}(t_0) - C^{-1}(t_0)$$
$$\times (C^{-1}(t_0) + w(t_0))^{-1}W(t_0))(I - B(t_0))$$
$$= \sum_{\tau=t_0+1}^{t} h(\tau) + A(t_0) - (I - B(t_0))^* C^{-1}(t_0)(I - B(t_0))$$
$$+ (I - B(t_0))^* C^{-1}(t_0)(C^{-1}(t_0) + W(t_0))^{-1}C^{-1}(t_0)(I - B(t_0))$$
$$> \sum_{\tau=t_0+1}^{t} h(\tau) + A(t_0) - (I - B(t_0))^* C^{-1}(t_0)(I - B(t_0)).$$

From $-W(t+1) < C^{-1}(t+1)$, the result follows.

Note. Taking $t = t_0$, we get [5, Proposition 2.1]. If $B(t) = 0$, $C(t) \equiv I$, we get [7, Theorem 1].

From [4] we may express $Q[U]$ in the equivalent form

$$(1.12) \qquad Q[U] = Y^*(t)W(t)Y(t)|_{M-1}^{N+1} + \sum_{t=M-1}^{N} F^*(t)F(t),$$

where

$$F(t) = (C^{-1}(t) + W(t))^{-1/2}W(t)(I - B(t))Y(t+1) - (C^{-1}(t) + W(t))^{1/2}C(t)Z(t),$$

$U = \{Y(t), Z(t)\} \in \Lambda$, and (1.2) is nonoscillatory.

Theorem 8. *Suppose* $C(t) \equiv I$, $B^*(t) + B(t) \leq B^*(t)B(t)$, *and there exists* $U = \{Y(t), Z(t)\}$ *with* $\Delta Y(t) = B(t)Y(t+1) + Z(t)$ *such that*

$$(1.13) \qquad \limsup_{N\to\infty} \lambda_1(Q[U]) = -\infty.$$

Then (1.2) *is oscillatory.*

Note. If $B(t) \equiv 0$, then this is [7, Theorem 5].

Proof. Suppose not, i.e., suppose (1.2) is nonoscillatory. Then by Theorem 2 there exists a solution $\{Y(t), Z(t)\}$ of (1.2) and an integer t_0 such that

$$(1.14) \qquad \sum_{t=t_0}^{\infty} (Y^*(t)(I - B(t))Y(t+1))^{-1} = \tau \quad \text{(constant matrix)}.$$

We are going to prove that $Y^*(t)(I - B(t))Y(t+1)$ is decreasing. To see this, observe that

$$(1.15)$$
$$\begin{aligned}
\Delta(Y^*(t)&(I - B(t))Y(t+1)) \\
&= Y^*(t+1)(\Delta Y(t) + \Delta Z(t)) + \Delta Y^*(t)(I - B(t))Y(t+1) \\
&= Y^*(t+1)[I - (I + W(t))^{-1}(I - B(t)) + W(t+1) \\
&\qquad - W(t)(I + W(t))^{-1}(I - B(t)) \\
&\qquad\qquad \times (I - (I - B^*(t))(I + W(t))^{-1})(I - B(t))]Y(t+1) \\
&= Y^*(t+1)[I + W(t+1) - (I - B^*(t))(I + W(t))^{-1}(I - B(t))]Y(t+1).
\end{aligned}$$

From (1.12)–(1.14) we know that if t is sufficiently large, we have $W(t) < 0$, i.e., $0 < I + W(t) < I$. Furthermore,

$$(1.16) \qquad \begin{aligned}
-(I - B(t))^*(I + W(t))^{-1}(I - B(t)) &< -(I - B(t))^*(I - B(t)) \\
&= -I + (B^*(t) + B(t)) - B^*(t)B(t).
\end{aligned}$$

Combining (1.15), (1.16), and the condition, we see that

$$\Delta(Y^*(t)(I - B(t))Y(t+1)) \le 0,$$

i.e., $(Y^*(t)(I - B(t))Y(t+1))^{-1}$ is increasing. This contradicts (1.14) and completes the proof.

Next we consider two matrix systems:

$$(1.17)_i \qquad \begin{aligned}
\Delta Y(t) &= B_i(t)Y(t+1) + C_i(t)Z(t), \\
\Delta Z(t) &= -A_i(t)Y(t+1) - B_i^*(t)Z(t),
\end{aligned} \qquad i = 1, 2.$$

We make the same assumption on $(1.17)_i$ as $(1.8)_i$. Using Theorem 3, it is easy to show:

Theorem 9. *If $(1.17)_1$ is nonoscillatory and $D_2(t) \ge D_1(t)$ for $t \ge t_0 \ge M - 1$ for some integer t_0, then $(1.17)_2$ is nonoscillatory as well.*

Next we wish to consider certain subsystems of (1.2). To this end we denote $R = \{i_1, i_2, \ldots, i_k\}$, $1 \le i_1 < i_2 < \cdots < i_k \le d$, $A(t) = (a_{ij})_{d \times d}$, $B(t) = (b_{ij})_{d \times d}$, and $C^{-1}(t) = (c_{ij})_{d \times d}$. We suppose that $B(t)$ satisfies $b_{ij} = 0$ if $i \notin R$ and $j \in R$.
Let

$$\widetilde{A}(t) = (\tilde{a}_{ij})_{k \times k}, \qquad \widetilde{B}(t) = (\tilde{b}_{ij})_{k \times k}, \qquad \widetilde{C}(t) = (\tilde{c}_{ij})_{k \times k}^{-1},$$

where $\tilde{a}_{ij} = a_{l_i l_j}$, $\tilde{b}_{ij} = b_{l_i l_j}$, $\tilde{c}_{ij} = c_{l_i l_j}$ if $l_i, l_j \in R$.
For the $k \times k$ matrix system,

$$(1.18) \qquad \begin{aligned}
\Delta \widetilde{Y}(t) &= \widetilde{B}(t)\widetilde{Y}(t+1) + \widetilde{C}(t)\widetilde{Z}(t), \\
\Delta \widetilde{Z}(t) &= -\widetilde{A}(t)\widetilde{Y}(t+1) - \widetilde{B}^*(t)\widetilde{Z}(t).
\end{aligned}$$

We have

Theorem 10. *If* (1.18) *is oscillatory, so is* (1.2).

Proof. Suppose (1.18) is oscillatory. Then by Theorem 1 we can find two integers M, N such that there exists a nonzero vector sequence $\tilde{u}(t) = \binom{\tilde{y}(t)}{\tilde{z}(t)} \in \tilde{\Omega}$, $\tilde{y}(t) = (\tilde{y}_i)_k$, $\tilde{z}(t) = (\tilde{z}_i)_k$, with $\tilde{q}[\tilde{u}] \leq 0$. (Here $\tilde{q}$, $\tilde{\Omega}$ correspond to (1.18).)

Let
$$u = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}, \qquad y(t) = (y_j)_d, \; z(t) = (z_j)_d,$$

with
$$y_{i_j} = \begin{cases} \tilde{y}_j & \text{if } i_j \in R, \\ 0 & \text{otherwise}, \end{cases} \qquad z_{i_j} = \begin{cases} \tilde{z}_j & \text{if } i_j \in R, \\ 0 & \text{otherwise}, \end{cases}$$

Then from $q[u] = \tilde{q}[\tilde{u}] \leq 0$ we conclude that (1.2) is oscillatory.

Remark. If $B(t) \equiv 0$, then Theorem 10 establishes the conjecture of [7].

Theorem 11. *If $C^{-1}(t) \leq M$ (constant Hermitian matrix), $\sum_{t=M}^{\infty} h(t)$ exists, and* (1.2) *is nonoscillatory, then*
$$\lim_{t \to \infty} C^{-1}(t)B(t) = L \quad (constant \; Hermitian \; matrix);$$
furthermore, $L \leq M$.

Proof. Since (1.2) is nonoscillatory, there exists a Hermitian solution $W(t)$ for $t \geq t_0$ (some integer $t_0 \geq M$) of (1.3) with $W(t) + C^{-1}(t) > 0$.

Taking the summation of both sides of (1.3), we have

$$(1.19) \qquad -W(t+1) = W(t_0) + \sum_{\tau=t_0}^{t} \rho(\tau) + \sum_{\tau=t_0}^{t} h(\tau).$$

Now since $-W(t+1) \leq C^{-1}(t+1) \leq M$, and since $\sum_{\tau=t_0}^{t} h(\tau)$ exists, it follows that $\sum_{\tau=t_0}^{\infty} \rho(\tau)$ exists, so $\lim_{t \to \infty} W(t) = W_0$, a constant Hermitian matrix with $W_0 \geq -M$.

Since $\rho(t) \geq 0$, we have $\lim_{t \to \infty} \rho(t) = 0$. From (1.19) and $C^{-1}(t) \leq M$, it follows that $W(t)$ is bounded, i.e., $\lambda_d((C^{-1}(t) + W(t))^{-1})$ does not go to zero as $t \to \infty$.

From the Courant-Fisher Theorem [6] we get
$$\lambda_1(\rho(t)) \geq \lambda_1[(W(t) + C^{-1}(t)B(t))^*(W(t) + C^{-1}(t)B(t))]\lambda_d[(C^{-1}(t) + W(t))^{-1}].$$
Now let $t \to \infty$ to obtain
$$\lim_{t \to \infty} \lambda_1((W(t) + C^{-1}(t)B(t))^*(W(t) + C^{-1}(t)B(t))) = 0,$$

i.e.,
$$\lim_{t \to \infty} (W(t) + C^{-1}(t)B(t)) = 0,$$

i.e.,
$$\lim_{t \to \infty} C^{-1}(t)B(t) = \lim_{t \to \infty} (-W(t)) = -W_0 \leq M.$$

This completes the proof.

Corollary 12. *Suppose $C^{-1}(t) \leq M$ (constant Hermitian matrix) and $\liminf_{t \to \infty} \sum_{\tau=M}^{t} \lambda_d(h(\tau)) > -\infty$. Then there exists a Hermitian solution $W(t)$ for $t \geq t_0 \geq M$ of (1.3) which satisfies $\lim_{t \to \infty}(W(t) + C^{-1}(t)B(t)) = 0$; furthermore, $C^{-1}(t)B(t)$ is bounded.*

References

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