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STABILIZATION OF SOLUTIONS OF WEAKLY SINGULAR QUENCHING PROBLEMS

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ABSTRACT. In this paper we prove that if $0 < \beta < 1$, $D \subset R^N$ is bounded, and $\lambda > 0$, then every element of the ω-limit set of weak solutions of

$$u_t - \Delta u + \lambda u^{-\beta}\chi_{u>0} = 0 \quad \text{in } D \times [0, \infty),$$
$$u = \begin{cases} 1 & \text{on } \partial D \times (0, \infty), \\ u_0 > 0 & \text{on } \overline{D} \times \{0\} \end{cases}$$

is a weak stationary solution of this problem. A consequence of this is that if D is a ball, λ is sufficiently small, and u_0 is a radial, then the set $\{(x, t) | u = 0\}$ is a bounded subset of $D \times [0, \infty)$.

We consider the problem

$$\begin{aligned} &u_t = \Delta u - \lambda u^{-\beta}\chi_{u>0}, && x \in D,\ t > 0; \\ &u = 1, && x \in \partial D,\ t > 0; \\ &u(\cdot, 0) = u_0 > 0, && x \in \overline{D}, \end{aligned}$$

with $\lambda > 0$, $0 < \beta < 1$. We assume that D is a bounded domain in $\mathbb{R}^N$ and that u_0 and ∂D are of class $C^{2+\alpha}$, $u_0 = 1$ on ∂D.

This problem can be considered as a limiting case of models in chemical catalyst kinetics (Langmuir-Hinshelwood model) or of models in enzyme kinetics; see [A, D].

It is known that there are initial functions u_0 for which the corresponding solutions reach zero (quench) in finite time; see [L]. Let $a \in D$ and $0 < T < \infty$ be such that $u > 0$ in $\overline{D} \times [0, T)$ and $u(a, T) = 0$. Under some additional assumptions on u_0 and D, all terms in the equation blow up as $t \to T$ (for blow up of u_t see, e.g., [DL]; for blow up of Δu see [FK]). This means that the solution ceases to exist in the classical sense as $t \to T$. On the other hand, it was shown in [Ph] (see also [BB]) that weak solutions exist globally. They can be obtained as limits of solutions of regularized problems, they are continuous, and the gradient (with respect to x) is continuous (see [Ph]). Uniqueness of

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the weak solutions is an open question. By a solution we mean the (unique) maximal weak solution obtained via regularization.

In this paper we are interested in the behavior of solutions as $t \to \infty$. It follows from [Ph, Lemma 5] that every orbit is relatively compact in $C(\overline{D})$. In other words, the ω-limit set

$$\omega(u_0) := \{v \in W^{1,2}(D) \cap C(\overline{D})\colon \text{there is a } \{t_n\},\ t_n \to \infty,\ \text{such that } u(\cdot, t_n; u_0) \to v \text{ uniformly as } n \to \infty\}$$

is nonempty. We show that $\omega(u_0)$ consists of (weak) stationary solutions.

For a description of the set of stationary solutions see, e.g., [JL, BN]; for the stability properties see [L] in the case $N = 1$ and [FHQ] in the case $N > 1$, D a ball.

One consequence of our result is the eventual positivity of any solution if D is a ball and λ is small enough. More precisely, for any radially symmetric u_0 there is a $\tau(u_0) \geq 0$ such that $u(x, t; u_0) > 0$ on $\overline{D} \times (\tau(u_0), \infty)$. Hence, the frozen zone (dead core, quenching set) $u = 0$, where no reaction takes place, disappears. This follows from the fact that there is a unique positive stationary solution if D is a ball and λ is small enough.

On the other hand, if D is a ball and λ is large enough, then all solutions converge (in the sup-norm) to the unique (nonclassical) stationary solution with a zero set of positive measure.

This shows that the conjecture from [L, §5] is true for large intervals and false for small intervals.

The notation

$$f(u) = u^{-\beta}\chi_{u>0}, \qquad f_\varepsilon(u) = \frac{u}{\varepsilon + u^{\beta+1}}$$

will be used throughout the paper.

Lemma 1. *Let $\{t_n\}$ be a sequence for which $u(\cdot, t_n; u_0)$ converges in $L^2(D)$. If v is the limit element and $U_n(x, s) := u(x, t_n + s; u_0)$, $s \in [0, 1]$, then $U_n \to v$ in $L^2(D \times (0, 1))$.*

Proof. For the reader's convenience we prove first that

$$\int_0^\infty \int_D (u_t)^2\, dx\, dt < \infty. \tag{1}$$

Although not explicitly stated, this was in fact shown in [Ph, BB]. Let u_ε be the solution of

$$\begin{aligned} &u_{\varepsilon t} = \Delta u_\varepsilon - \lambda f_\varepsilon(u_\varepsilon), && x \in D,\ t > 0;\\ &u_\varepsilon = 1, && x \in \partial D,\ t > 0;\\ &u_\varepsilon(\cdot, 0) = u_0 && x \in \overline{D}. \end{aligned}$$

Then the following identity holds for $T > 0$:

$$\int_0^T \int_D (u_{\varepsilon t})^2\, dx\, dt + J_\varepsilon(u_\varepsilon(\cdot, T)) = J_\varepsilon(u_0),$$

$$J_\varepsilon(w) := \frac{1}{2}\int_D |\nabla w|^2\, dx + \lambda \int_D \int_0^2 f_\varepsilon(u)\, du\, dx.$$

Obviously, $J_\varepsilon(w) \geq 0$ and

$$J_\varepsilon(u_0) \leq J(u_0) := \frac{1}{2}\int_D |\nabla u_0|^2\,dx + \frac{\lambda}{1-\beta}\int_D u_0^{1-\beta}\,dx\,,$$

therefore

$$\int_0^T \int_D (u_{\varepsilon t})^2\,dx\,dt \leq J(u_0)\,;$$

hence, $u_{\varepsilon t}$ is weakly convergent in $L^2(D \times (0, T))$. Since $u_\varepsilon \to u$ pointwise as $\varepsilon \to 0$, the weak limit of $u_{\varepsilon t}$ is u_t, which yields $\int_0^T \int_D (u_t)^2\,dx\,dt \leq J(u_0)$ for $T > 0$, and (1) follows.

Now we can finish the proof of the lemma using an idea from [LP]:

$$\int_D |u(x\,,\, t_n + s) - u(x\,,\, t_n)|^2\,dx \leq \int_D \int_{t_n}^{t_n+1} (u_t)^2\,dx\,dt\,;$$

hence,

$$\|U_n - u(\cdot\,,\, t_n)\|^2_{L^2(D\times(0\,,\,1))} \leq \int_{t_n}^{\infty} \int_D (u_t)^2\,dx\,dt\,.$$

According to (1), the right-hand side goes to zero as $n \to \infty$. □

Lemma 2. *If* $v \in \omega(u_0)$ *then* $\int_D f(v)\psi\,dx < \infty$ *for any* $\psi \in C^2(\overline{D})$ *vanishing on* ∂D, $\psi \geq 0$.

Proof. Choose $\rho \in C_0^2((0, 1))$, $\rho \geq 0$, $\int_0^1 \rho(s)\,ds = 1$, and set $g(x\,,\, t) = \psi(x)\rho(t + t_n)$. Then

$$\int_{t_n}^{t_n+1} \int_D u(\Delta g + g_t)\,dx\,dt - \lambda \int_{t_n}^{t_n+1} \int_D f(u) g\,dx\,dt - \int_{\partial D} \frac{\partial \psi}{\partial \nu}\,dS = 0\,,$$

and the substitution $t - t_n = s$ leads to

$$\lambda \int_0^1 \int_D f(U_n) g\,dx\,dt = \int_0^1 \int_D U_n(\Delta g + g_t)\,dx\,dt - \int_{\partial D} \frac{\partial \psi}{\partial \nu}\,dS\,.$$

Hence, there is a constant K (which depends only on g) such that

$$\int_0^1 \int_D f(U_n) g\,dx\,dt \leq K \quad \text{for any } n\,. \tag{2}$$

From Lemma 1 we obtain the existence of a subsequence (denoted again by U_n) such that $f_\varepsilon(U_n) \to f_\varepsilon(v)$ pointwise in $D \times (0, 1)$; therefore,

$$\int_0^1 \int_D f_\varepsilon(U_n) g\,dx\,dt \to \int_D f_\varepsilon(v)\psi\,dx\,.$$

Now (2) implies that $\int_D f_\varepsilon(v)\psi\,dx \leq K$ because $f_\varepsilon(U_n) \leq f(U_n)$. Since $f_\varepsilon(v) \to f(v)$ in a monotone way as $\varepsilon \to 0$, the monotone convergence theorem gives the assertion. □

Theorem. *If* $v \in \omega(u_0)$ *then* v *satisfies*

$$\int_D v\Delta\psi\,dx - \lambda\int_D f(v)\psi\,dx - \int_{\partial D} \frac{\partial \psi}{\partial \nu}\,dS = 0$$

for any $\psi \in C^2(\overline{D})$ *vanishing on* ∂D, $\psi \geq 0$.

Proof. We use some ideas from [Ph, LP].

Let $\{t_n\}$ be a sequence such that $U_n \to v$ in $L^2(D \times (0, 1))$ and pointwise in $D \times (0, 1)$. Let $\varphi \in C^\infty(\mathbb{R})$ be such that

$$\varphi(s) = \begin{cases} s-1 & \text{for } s \geq 2, \\ 0 & \text{for } s < \frac{1}{2}, \end{cases}$$

φ', $\varphi'' \geq 0$, and define $\varphi_h(s) := h\varphi(s/h)$ for $h > 0$. Take ρ, ψ as in the proof of Lemma 2 and $g(x, s) = \psi(x)\rho(s)$. Then

$$\int_0^1 \int_D \varphi_h(U_n)(\Delta g + g_t)\, dx\, dt \to \int_0^1 \int_D \varphi_h(v)(\Delta g + g_t)\, dx\, dt \quad \text{as } n \to \infty,$$

$$\int_0^1 \int_D \varphi_h(v)(\Delta g + g_t)\, dx\, dt \to \int_0^1 \int_D v(\Delta g + g_t)\, dx\, dt \quad \text{as } h \to 0,$$

and

$$\int_0^1 \int_D v(\Delta g + g_t)\, dx\, dt - \int_D v\Delta\psi\, dx.$$

On the other hand, we obtain (cf. [Ph])

$$\begin{aligned} &\int_0^1 \int_D \varphi_h(U_n)(\Delta g + g_t)\, dx\, dt \\ &\quad = \lambda \int_0^1 \int_D \varphi_h'(U_n) U_n^{-\beta} g\, dx\, dt \\ &\qquad + \int_0^1 \int_D \varphi_h''(U_n)|\nabla U_n|^2 g\, dx\, dt + \int_0^1 \int_D \varphi_h(1) \frac{\partial g}{\partial \nu}\, dS\, dt. \end{aligned}$$

As $n \to \infty$, we find

$$\int_0^1 \int_D \varphi_h'(U_n) U_n^{-\beta} g\, dx\, dt \to \int_0^1 \int_D \varphi_h'(v) v^{-\beta} g\, dx\, dt = \int_D \varphi_h'(v) v^{-\beta} \psi\, dx.$$

Now $\varphi_h'(v)v^{-\beta}\psi \to f(v)\psi$ pointwise, $\varphi_n'(v)v^{-\beta}\psi \leq f(v)\psi$, and, by Lemma 2, $f(v)\psi$ is integrable. This implies

$$\int_D \varphi_h'(v) v^{-\beta} \psi\, dx \to \int_D f(v)\psi\, dx \quad \text{as } h \to 0.$$

Since $\varphi_h''(s) \leq (c_1/h)\chi_{h/2<s<2h}(s)$ for some constant $c_1 > 0$, we obtain

$$\int_0^1 \int_D \varphi_h''(U_n)|\nabla U_n|^2 g\, dx\, dt \leq \frac{c_1}{h} \int_0^1 \int_D \chi_{h/2<U_n<2h} |\nabla U_n|^2\, dx\, dt.$$

From [Ph, Lemma 5] we get

$$|\nabla U_n|^2 \leq \frac{2}{1-\beta} U_n^{1-\beta} + M U_n \quad \text{for some } M > 0.$$

This implies that

$$\frac{c_1}{h} \int_0^1 \int_D \chi_{h/2<U_n<2h} |\nabla U_n|^2 g\, dx\, dt \leq c_2 \int_0^1 \int_D (U_n^{-\beta} + c_3)\chi_{h/2<U_n<2h}\, g\, dx\, dt.$$

As $n \to \infty$, the right-hand side approaches

$$c_2 \int_0^1 \int_D (v^{-\beta} + c_3)\chi_{h/2<v<2h} g\, dx\, dt,$$

and by Lemma 2 this converges to zero as $h \to 0$. □

Corollary. *If D is a ball, λ is sufficiently small, and u_0 is radial, then there is a $t_0 \geq 0$ such that $u(x, t; u_0) > 0$ for $x \in \overline{D}$, $t > t_0$.*

Proof. If D is a ball and λ is sufficiently small, it is known that in the class of radial functions there is a unique weak stationary solution v, which is positive and hence classical [JL, BN]. The theorem implies then that u must become positive in finite time, because it converges to v uniformly as $t \to \infty$. □

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