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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF POINCARÉ DIFFERENCE EQUATIONS

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ABSTRACT. It is shown that if the zeros $\lambda_1, \lambda_2, \dots, \lambda_n$ of the polynomial

$$q(\lambda) = \lambda^n + a_1\lambda^{n-1} + \dots + a_n$$

are distinct and r is an integer in $\{1, 2, \dots, n\}$ such that $|\lambda_s| \neq |\lambda_r|$ if $s \neq r$, then the Poincaré difference equation

$$y(n+m) + (a_1 + p_1(m))y(n+m-1) + \dots + (a_n + p_n(m))y(m) = 0$$

has a solution y_r such that (A) $y_r(m) = \lambda_r^m(1 + o(1))$ as $m \to \infty$, provided that the sums $\sum_{j=m}^{\infty} p_i(j)$ $(1 \leq i \leq n)$ converge sufficiently rapidly. Our results improve over previous results in that these series may converge conditionally, and we give sharper estimates of the $o(1)$ terms in (A).

1. INTRODUCTION

We consider the Poincaré difference equation

$$(1)\qquad y(n+m) + (a_1 + p_1(m))y(n+m-1) + \dots + (a_n + p_n(m))y(m) = 0,$$

where $a_n \neq 0$, the polynomial

$$q(\lambda) = \lambda^n + a_1\lambda^{n-1} + \dots + a_n$$

has distinct zeros $\lambda_1, \lambda_2, \dots, \lambda_n$, and

$$(2)\qquad \lim_{m\to\infty} p_k(m) = 0, \qquad 1 \leq k \leq n.$$

Under these assumptions it is natural to ask whether (1) has solutions y_1, $y_2, \dots, y_n$ which behave asymptotically in some sense like the solutions $x_r(m) = \lambda_r^m$ $(1 \leq r \leq n)$ of the constant coefficient equation

$$x(n+m) + a_1x(n+m-1) + \dots + a_nx(m) = 0.$$

If $\lambda_1, \lambda_2, \dots, \lambda_n$ have distinct moduli, then Poincaré's theorem [5] asserts that every nontrivial solution of (1) exhibits the asymptotic behavior

$$\lim_{m\to\infty} \frac{y(m+1)}{y(m)} = \lambda_r$$

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for some r in $\{1, 2, \ldots, n\}$ and Perron's theorem [4] asserts that (1) has solutions $y_1, y_2, \ldots, y_n$ such that

$$\lim_{m\to\infty} \frac{y_r(m+1)}{y_r(m)} = \lambda_r, \qquad 1 \le r \le n. \tag{3}$$

The conclusions of Poincaré's and Perron's theorems are weak, since (3) does not imply that $y_r(m) - \lambda_r^m$ becomes small (i.e., $o(\lambda_r^m)$) as $m \to \infty$. (We will use O and o in the usual way to indicate asymptotic behavior as $m \to \infty$.) To obtain this conclusion it is necessary to replace (2) with a stronger condition. For example, the following theorem is due to Evgrafov [2].

Theorem 1. *Suppose that the zeros $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $q(\lambda)$ are distinct and*

$$\sum_{m=0}^{\infty} |p_k(m)| < \infty, \qquad 1 \le k \le n.$$

Then (1) *has solutions $y_1, \ldots, y_n$ such that*

$$y_r(m) = \lambda_r^m(1 + o(1)), \qquad 1 \le r \le n. \tag{4}$$

The following theorem of Gelfond and Kubenskaya [3] provides an estimate of the $o(1)$ term in (4).

Theorem 2. *Suppose that $|\lambda_1| < |\lambda_2| < \cdots < |\lambda_n|$ and there is a nonincreasing sequence β such that*

$$\begin{gathered} |p_i(m)| \le \beta(m), \qquad m = 0, 1, \ldots, \quad 1 \le i \le n, \\ \lim_{m\to\infty} \frac{\beta(m+1)}{\beta(m)} = 1, \end{gathered} \tag{5}$$

and $\sum^{\infty} \beta(m) < \infty$. Let $\gamma(m) = \sum_{j=m}^{\infty} \beta(j)$. Then (1) *has solutions $y_1, \ldots, y_n$ such that*

$$y_r(m) = \lambda_r^m(1 + O(\gamma(m))), \qquad 1 \le r \le n. \tag{6}$$

Coffman [1] has shown that (5) can be weakened to

$$\liminf_{m\to\infty} \frac{\beta(m+1)}{\beta(m)} > \max_{1\le i<n} \left(\frac{|\lambda_i|}{|\lambda_{i+1}|}\right). \tag{7}$$

Theorems 1 and 2 do not apply if any of the series $\sum_{m=0}^{\infty} p_i(m)$ $(1 \le i \le n)$ converge conditionally. Moreover, even if these series converge absolutely, the estimate of the order of convergence in (6) may be too conservative, as our examples in §3 will illustrate.

The following theorem is our main result.

Theorem 3. *Suppose that the zeros $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $q(\lambda)$ are distinct, and let $0 < |\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_n|$. Suppose also that the series $\sum_{m=0}^{\infty} p_i(m)$ $(1 \le i \le n)$ converge* (*perhaps conditionally*) *and there are nonincreasing sequences ϕ and ψ such that $\lim_{m\to\infty} \phi(m) = \lim_{m\to\infty} \psi(m) = 0$, $\psi(m) = o(\phi(m))$,*

$$\sum_{j=m}^{\infty} p_i(j) = O(\phi(m)), \qquad 1 \le i \le n, \tag{8}$$

and

$$\sum_{j=m}^{\infty} |p_i(j)|\phi(n+j-i) = O(\psi(m)), \qquad 1 \le i \le n. \tag{9}$$

Let r *be an integer in* $\{1, 2, \ldots, n\}$ *such that* $|\lambda_r| \neq |\lambda_s|$ *if* $r \neq s$. *If* $r > 1$, *suppose also that there is an integer* M_1 *and a number* ρ *such that*

$$1 < \rho < |\lambda_r/\lambda_{r-1}| \tag{10}$$

and $\rho^m \phi(m)$ *and* $\rho^m \psi(m)$ *are nondecreasing for* $m \ge M_1$. *Then* (1) *has a solution* y_r *such that*

$$y_r(m) = \lambda_r^m(1 + O(\phi(m))), \qquad m \to \infty. \tag{11}$$

The assumption concerning ρ is equivalent to the conditions

$$\liminf_{m\to\infty} \frac{\phi(m+1)}{\phi(m)} > \left|\frac{\lambda_{r-1}}{\lambda_r}\right| \quad \text{and} \quad \liminf_{m\to\infty} \frac{\psi(m+1)}{\psi(m)} > \left|\frac{\lambda_{r-1}}{\lambda_r}\right|,$$

which is related to Coffman's condition (7).

2. Proof of Theorem 3

We will prove Theorem 3 by means of a series of lemmas. For convenience we rewrite (1) as

$$y(n+m) + a_1 y(n+m-1) + \cdots + a_n y(m) = -\mathscr{L}y(m),$$

where

$$\mathscr{L}y(m) = p_1(m)y(n+m-1) + p_2(m)y(n+m-2) + \cdots + p_n(m)y(m).$$

By variation of parameters, y is a solution of (1) if and only if

$$y(m) = \sum_{k=1}^{n} \lambda_k^m u_k(m), \tag{12}$$

where

$$\sum_{k=1}^{n} \lambda_k^{m+i} \Delta u_k(m) = -\delta_{in}\mathscr{L}y(m), \qquad 1 \le i \le n.$$

Solving this system yields

$$\Delta u_k(m) = -A_k \lambda_k^{-m} \mathscr{L}y(m) \quad \text{with } A_k = 1/\lambda_k q'(\lambda_k). \tag{13}$$

Now let M be a positive integer which we will specify further below. For now we assume that if $r > 1$ then $\rho^m\phi(m)$ and $\rho^m\psi(m)$ are nondecreasing for $m \ge M$ for some ρ satisfying (10). From (12) and (13), if the sequence y_r satisfies

$$\begin{aligned} y_r(m) = \lambda_r^m &- \sum_{k=1}^{r-1} A_k \sum_{j=M}^{m-1} \lambda_k^{m-j} \mathscr{L}y_r(j) \\ &+ \sum_{k=r}^{n} A_k \sum_{j=m}^{\infty} \lambda_k^{m-j} \mathscr{L}y_r(j), \qquad m \ge M \end{aligned} \tag{14}$$

(where the first sum is vacuous if $r = 1$), then y_r satisfies (1) for $m \ge M$.

It is convenient to rewrite (14) in terms of the sequence v_r defined by

$$v_r(m) = \lambda_r^{-m} y_r(m) - 1. \tag{15}$$

Then

$$\mathscr{L} y_r(j) = P_r(j) + \mathscr{M}_r v_r(j), \tag{16}$$

where

$$P_r(j) = \sum_{i=1}^{n} \lambda_r^{n+j-i} p_i(j), \tag{17}$$

and

$$\mathscr{M}_r u(j) = \sum_{i=1}^{n} \lambda_r^{n+j-i} p_i(j) u(n+j-i) \tag{18}$$

for any sequence u. From (14)–(16),

$$\begin{aligned} v_r(m) = F_r(m) &- \sum_{k=1}^{r-1} A_k \left(\frac{\lambda_k}{\lambda_r}\right)^m \sum_{j=M}^{m-1} \lambda_k^{-j} \mathscr{M}_r v_r(j) \\ &+ \sum_{k=r}^{n} A_k \left(\frac{\lambda_k}{\lambda_r}\right)^m \sum_{j=m}^{\infty} \lambda_k^{-j} \mathscr{M}_r v_r(j), \end{aligned} \tag{19}$$

where

$$F_r(m) = -\sum_{k=1}^{r-1} A_k \left(\frac{\lambda_k}{\lambda_r}\right)^m \sum_{j=M}^{m-1} \lambda_k^{-j} P_r(j) + \sum_{k=r}^{n} A_k \left(\frac{\lambda_k}{\lambda_r}\right)^m \sum_{j=m}^{\infty} \lambda_k^{-j} P_r(j). \tag{20}$$

If v_r satisfies (19), then the sequence y_r defined by

$$y_r(m) = (1 + v_r(m)) \lambda_r^m \tag{21}$$

satisfies (1). This motivates us to look for v_r as a fixed point (sequence) of the transformation

$$v = F_r + \mathscr{T}_r u, \tag{22}$$

where

$$\begin{aligned} \mathscr{T}_r u(m) = &- \sum_{k=1}^{r-1} A_k \left(\frac{\lambda_k}{\lambda_r}\right)^m \sum_{j=M}^{m-1} \lambda_k^{-j} \mathscr{M}_r u(j) \\ &+ \sum_{k=r}^{n} A_k \left(\frac{\lambda_k}{\lambda_r}\right)^m \sum_{j=m}^{\infty} \lambda_k^{-j} \mathscr{M}_r u(j). \end{aligned} \tag{23}$$

Now let $M \geq M_1$ and let $\mathscr{B}$ be the Banach space of sequences $u = \{u(m)\}_{m=M}^{\infty}$ such that $u(m) = O(\phi(m))$, with norm

$$\|u\| = \sup_{m \geq M} \{|u(m)|/\phi(m)\}. \tag{24}$$

We will show that (22) is a contraction mapping of $\mathscr{B}$ into itself if M is sufficiently large.

Lemma 1. *Suppose that the series* $\sum^{\infty} w(j)$ *converges, and let* σ *be a nonincreasing sequence such that*

$$\sigma(m) \geq \sup_{\nu \geq m} |W(\nu)|, \quad \text{where } W(m) = \sum_{j=m}^{\infty} w(j).$$

Let γ *be a complex constant.*

(a) *If* $|\gamma| < 1$ *then*

$$\left| \sum_{j=m}^{\infty} \gamma^j w(j) \right| \leq K_1 |\gamma|^m \sigma(m), \tag{25}$$

where K_1 *depends only on* γ.

(b) *If* $|\gamma| > 1$ *and there is a number* ρ *such that* $1 < \rho < |\gamma|$ *and* $\rho^m \sigma(m)$ *is nondecreasing for* $m \geq M$, *then*

$$\left| \sum_{j=M}^{m-1} \gamma^j w(j) \right| \leq K_2 |\gamma|^m \sigma(m), \qquad m \geq M+1, \tag{26}$$

where K_2 *is a constant which depends only on* γ *and* ρ.

Proof. (a) Summation by parts yields

$$\sum_{j=m}^{N} \gamma^j w(j) = -\gamma^N W(N+1) + \gamma^m W(m) + \sum_{j=m+1}^{N} (\gamma^j - \gamma^{j-1}) W(j), \qquad m < N.$$

Letting $N \to \infty$ and applying routine estimates yields (25) with

$$K_1 = 1 + \frac{|1-\gamma|}{1-|\gamma|}.$$

(b) Summation by parts yields

$$\sum_{j=M}^{m-1} \gamma^j w(j) = \gamma^M W(M) - \gamma^{m-1} W(m) + \left(1 - \frac{1}{\gamma}\right) \sum_{j=M+1}^{m-1} \gamma^j W(j);$$

therefore,

$$\left| \gamma^{-m} \sum_{j=M}^{m-1} \gamma^j w(j) \right| \leq \frac{|\gamma|^{M-m}}{\rho^M} (\rho^M \sigma(M)) + \frac{1}{|\gamma|} \sigma(m) + \left| 1 - \frac{1}{\gamma} \right| \sum_{j=M+1}^{m-1} \frac{|\gamma|^{j-m}}{\rho^j} (\rho^j \sigma(j)),$$

and the monotonicity of $\rho^m \sigma(m)$ implies (26) with

$$K_2 = 1 + \frac{1}{|\gamma|} + \frac{\rho |\gamma - 1|}{|\gamma|(|\gamma| - \rho)}.$$

Lemma 2. *The sequence* F_r *defined by* (20) *for* $m \geq M$ *is in* $\mathscr{B}$.

Proof. We apply Lemma 1 with $w(j) = p_i(j)$ and $\gamma = \lambda_r / \lambda_k$. From (17),

$$\sum_{j=m}^{\infty} \lambda_k^{-j} P_r(j) = \sum_{i=1}^{n} \lambda_r^{n-i} \sum_{j=m}^{\infty} \left(\frac{\lambda_r}{\lambda_k} \right)^j p_i(j).$$

Since $|\lambda_r/\lambda_k| < 1$ for $r < k \leq n$, we can infer from (8) and Lemma 1(a) that

$$\left|\sum_{j=m}^{\infty} \lambda_k^{-j} P_r(j)\right| \leq A\left(\frac{\lambda_r}{\lambda_k}\right)^m \phi(m), \qquad m \geq M,\ r \leq k \leq n,$$

for some constant A. Therefore, the second sum in (20) is $O(\phi(m))$. Similarly, Lemma 1(b) implies that the first sum in (20) is $O(\phi(m))$. Therefore, $F_r \in \mathscr{B}$.

In the following lemma let

$$\zeta(M) = \sup_{m \geq M} [\psi(m)/\phi(m)]. \tag{27}$$

Since $\psi(m) = o(\phi(m))$ by assumption, $\zeta(M)$ is well defined and

$$\lim_{M \to \infty} \zeta(M) = 0. \tag{28}$$

Lemma 3. *If $u \in \mathscr{B}$ then $\mathscr{T}_r u \in \mathscr{B}$ and*

$$\|\mathscr{T}_r u\| \leq J\zeta(M)\|u\|, \tag{29}$$

where J is independent of u and M.

Proof. From (18) and (24),

$$\begin{aligned} &\left|\sum_{j=m}^{\infty} \lambda_k^{-j} \mathscr{M}_r u(j)\right| \\ &\quad \leq \|u\| \sum_{i=1}^{n} |\lambda_r|^{n-i} \sum_{j=m}^{\infty} \left|\frac{\lambda_r}{\lambda_k}\right|^j |p_i(j)|\phi(n+j-i), \qquad r \leq k \leq n, \end{aligned} \tag{30}$$

and

$$\begin{aligned} &\left|\sum_{j=M}^{m-1} \lambda_k^{-j} \mathscr{M}_r u(j)\right| \\ &\quad \leq \|u\| \sum_{i=1}^{n} |\lambda_r|^{n-i} \sum_{j=M}^{m-1} \left|\frac{\lambda_r}{\lambda_k}\right|^j |p_i(j)|\phi(n+j-i), \qquad 1 \leq k \leq r-1. \end{aligned} \tag{31}$$

Now (9) and (30) imply that

$$\left|\sum_{j=m}^{\infty} \lambda_k^{-j} \mathscr{M}_r u(j)\right| \leq \alpha\|u\| \left|\frac{\lambda_r}{\lambda_k}\right|^m \psi(m), \qquad r \leq k \leq n,$$

where α is independent of u and M. By applying Lemma 1(b) with $w(j) = |p_i(j)|\phi(n+j-1)$ and $\gamma = \lambda_r/\lambda_k$, we see from (31) that

$$\left|\sum_{j=M}^{m-1} \lambda_k^{-j} \mathscr{M}_r u(j)\right| \leq \beta\|u\| \left|\frac{\lambda_r}{\lambda_k}\right|^m \psi(m), \qquad 1 \leq k \leq r-1,$$

where β is independent of u and M. From (23) and the last two inequalities we see that $|\mathscr{T}_r u(m)| \leq J\psi(m)\|u\|$ $(m \geq M)$ for some J independent of u and m. This together with (24) and (27) implies (29).

We can now complete the proof of Theorem 1. Lemmas 2 and 3 and (22) imply that $\mathscr{T}_r$ maps $\mathscr{B}$ into itself. If u_1 and u_2 are in $\mathscr{B}$, then Lemma 3 implies that

$$\|\mathscr{T}_r u_1 - \mathscr{T}_r u_2\| \le J\zeta(M)\|u_1 - u_2\|.$$

Because of (28), we can choose M so large that $\zeta(M) < 1/J$; then the mapping defined by (22) is a contraction mapping of $\mathscr{B}$ and its fixed point v_r satisfies (19) for $m \ge M$. Therefore, y_r as defined by (21) satisfies (1) for $m \ge M$ and has the asymptotic behavior (11).

3. A remark and examples

Remark 1. Since v_r is the fixed point of (22), we have $v_r = F_r + \mathscr{T} v_r$, where $F_r = O(\phi)$ and $\mathscr{T} v_r = O(\psi)$. Since $\psi(m) = o(\phi(m))$, the asymptotic formula (11) can be written more precisely as

$$y_r(m) = \lambda_r^m(1 + F_r(m) + O(\psi(m))),$$

where F_r, which is $O(\phi(m))$, is the known sequence defined by (20).

In the following examples we consider the difference equation

$$y(m+2) + (a_1 + \varepsilon(m)/m)y(m+1) = a_2 y(m) = 0, \tag{32}$$

where

$$\lambda^2 + a_1\lambda + a_2 = (\lambda - \lambda_1)(\lambda - \lambda_2),$$

with $0 < |\lambda_1| < |\lambda_2|$.

Example 1. Let $\varepsilon(m) = 1/m$. Applying Theorem 2 with $\beta(m) = 1/m^2$ and $\gamma(m) = \sum_{j=m}^{\infty} 1/j^2 = O(1/m)$ shows that (32) has solutions y_1 and y_2 such that

$$\begin{aligned} y_1(m) &= \lambda_1^m(1 + O(1/m)), \\ y_2(m) &= \lambda_2^m(1 + O(1/m)). \end{aligned} \tag{33}$$

However, Theorem 3 and Remark 1 yield sharper estimates

$$\begin{aligned} y_1(m) &= \lambda_1^m(1 + F_1(m) + O(1/m^2)), \\ y_2(m) &= \lambda_2^m(1 + F_2(m) + O(1/m^2)), \end{aligned}$$

where Lemma 2 implies that the known sequences $F_1(m)$ and $F_2(m)$ are $O(1/m)$.

Example 2. Let $\varepsilon(m) = (-1)^m/m$. Then Theorem 2 implies only that (32) has solutions satisfying (33). However, now (8) and (9) hold with $\phi(m) = 1/m^2$ and $\psi(m) = 1/m^3$, respectively, so Theorem 3 and Remark 1 yield the sharper estimates

$$\begin{aligned} y_1(m) &= \lambda_1^m(1 + F_1(m) + O(1/m^3)), \\ y_2(m) &= \lambda_2^m(1 + F_2(m) + O(1/m^3)), \end{aligned}$$

where Lemma 2 implies that $F_1(m)$ and $F_2(m)$ are $O(1/m^2)$.

More generally, let $\varepsilon(m) = (-1)^m\delta(m)$, where $\delta(m)$ is nonincreasing,

$$\lim_{m\to\infty} \delta(m) = 0, \quad \text{and} \quad \liminf_{m\to\infty} \frac{\delta(m+1)}{\delta(m)} > \left|\frac{\lambda_1}{\lambda_2}\right|^{1/2}.$$

Then Theorem 3 and Remark 1 imply that (32) has solutions such that

$$y_1(m) = \lambda_1^m(1 + F_1(m) + O(\delta^2(m)/m)),$$
$$y_2(m) = \lambda_2^m(1 + F_2(m) + O(\delta^2(m)/m)),$$

where Lemma 2 implies that $F_1(m)$ and $F_2(m)$ are $O(\delta(m)/m)$. However, Theorem 2 does not apply unless $\sum^\infty \delta(j)/j < \infty$, in which case it yields the weaker estimates

$$y_1(m) = \lambda_1^m(1 + O(\gamma(m))),$$
$$y_2(m) = \lambda_2^m(1 + O(\gamma(m))),$$

where $\gamma(m) = \sum_{j=m}^\infty \delta(j)/j$.

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