PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 119, Number 2, October 1993

ON THE RIEMANNIAN GEOMETRY OF THE NILPOTENT GROUPS $H(p, r)$

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(Communicated by Jonathan M. Rosenberg)

ABSTRACT. We study some aspect of the left-invariant Riemannian geometry on a class of nilpotent Lie groups $H(p, r)$ that generalize the Heisenberg group H_{2p+1}. Let us prove that the groups of type H (or Kaplan's spaces) and the $H(p, r)$ groups have same common Riemannian properties but they are not the same spaces.

INTRODUCTION

The Heisenberg group H_{2p+1} with the left-invariant metric

$$ds^2 = (dx_i)^2 + \left(dz + \sum x_{2i-1}\, dx_{2i}\right)^2$$

is a typical model of a homogeneous Riemannian non-Euclidean structure.

The geometry of these metrics is strongly connected to contact geometry of the Pfaff equation

$$\omega = dz + \sum x_{2i-1} dx_{2i} = 0.$$

In fact, let $\mathscr{C}(\omega)$ be the group of contact transformations relative to ω (i.e., of the transformations preserving the codimension 1 distribution $\mathrm{Ker}(\omega)$). Then

$$\mathscr{C}(\omega) = \mathfrak{J}som(ds^2),$$

where $\mathfrak{J}som(ds^2)$ denotes the group of isometries of ds^2.

It is natural to study the Riemannian structures adapted to a generalized (i.e., of higher codimension) contact geometry.

Recall that in codimension 1, every contact equation is equivalent to $\omega = dz + \sum x_{2i-1} dx_{2i} = 0$. This is not true anymore in codimension greater than 1, where one has an infinity of models $[G_1]$.

In [GH] Haraguchi and the second author introduced a notion of *r-contact system* that seems to generalize in a remarkable way that of codimension 1 contact structure.

Received by the editors March 15, 1990.

1991 *Mathematics Subject Classification*. Primary 53C25, 53B20.

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Definition. Let S be a Pfaff system. S is called an *r-contact system* if the rank of S is r, its class is maximal, and S admits an integral foliation of dimension $(n-r)/(r+1)$ (which is the maximal dimension of such a foliation).

Theorem [H]. *Let M be an $(rp+r+p)$-dimensional manifold with an r-contact system S. Then S is locally equivalent to the system given by*

$$\omega_i = dz_i + \sum_{\alpha=1}^{p} y_i^\alpha \, dx_\alpha, \qquad i = 1, \dots, r.$$

The simplest examples of Lie groups admitting a left-invariant r-contact system are groups that generalize the Heisenberg group H_{2p+1}, denoted by $H(p, r)$ (see [GH]).

In this paper, we study some aspect of the Riemannian geometry of $H(p, r)$ equipped with a natural left-invariant metric whose isometries preserve the distribution associated to the r-contact system.

I. The groups $H(p, r)$

I.1. Lie algebra considerations.

Definition 1.1. A generalized Heisenberg group in the sense of Goze and Haraguchi [GH] is the product $H(p, r) = \mathscr{M}_{1p} \times \mathscr{M}_{pr} \times \mathscr{M}_{1r}$ of three Abelian topological groups of matrices of dimensions $1 \times p$, $p \times r$, and $1 \times r$ respectively, endowed with the multiplication

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + \tfrac{1}{2}(xy' - x'y)).$$

In [H] and [GH] Haraguchi and Goze proved the following result:

Proposition 1.2. (1) *The groups $H(p, r)$ are $(rp+r+p)$-dimensional, two-step nilpotent, connected, and simply connected.*

(2) *The center Z of $H(p, r)$ is r-dimensional and isomorphic with the Abelian topological group $\mathscr{M}_{1r}$.*

(3) *A group $H(p, r)$ admits discrete uniform subgroups. An attempt at classification can be found in* [H].

(4) *A group $H(p, r)$ admits an r-contact system* (*see Introduction*).

We shall use on $H(p, r)$ the global left-invariant coframe

$$(1.1) \qquad \begin{cases} \vartheta^\alpha = dx_\alpha, \ \vartheta^{(\alpha, i)} = dy_i^\alpha, & i = 1, \dots, r; \\ \vartheta^i = dz_i + \tfrac{1}{2}(y_i^\alpha \, dx_\alpha - x_\alpha \, dy_i^\alpha), & \alpha = 1, \dots, p. \end{cases}$$

The frame of left-invariant vector fields dual to the 1-forms (1.1) is

$$E_\alpha = \frac{\partial}{\partial x_\alpha} - \frac{1}{2} y_i^\alpha \frac{\partial}{\partial z_i}, \quad E_{(\alpha, i)} = \frac{\partial}{\partial y_i^\alpha} + \frac{1}{2} x_\alpha \frac{\partial}{\partial z_i}, \quad E_i = \frac{\partial}{\partial z_i}.$$

The Maurer-Cartan equations for the Lie algebra $\mathfrak{h}(p, r)$ are given by

$$\begin{cases} d\vartheta^\alpha = 0, \\ d\vartheta^{(\alpha, i)} = 0, \\ d\vartheta^i = -\vartheta^\alpha \wedge \vartheta^{(\alpha, i)}. \end{cases}$$

Remarks 1.3. From the Maurer-Cartan equations it follows at once that:

(1) the group $H(p, r)$ is isomorphic to the Heisenberg group H_{2p+1} if and only if $\dim Z = r = 1$;

(2) the derived group $H'(p, r)$ is an r-dimensional group and coincides with the center Z of $H(p, r)$;

(3) the Lie algebra $\mathfrak{h}(p, r)$ of $H(p, r)$ is the direct sum of three Abelian subalgebras $\mathfrak{h}(p, r) = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathscr{Z}$, where $\mathscr{Z}$ is the center of $\mathfrak{h}(p, r)$ (the Lie algebra of the center Z of $H(p, r)$).

I.2. **A Riemannian structure for** $H(p, r)$. We shall consider on the group $H(p, r)$ the left-invariant metric tensor g given by

$$g = \sum (\vartheta^\alpha)^2 + \sum (\vartheta^{(\alpha, i)})^2 + \sum (\vartheta^i)^2. \tag{1.2}$$

Therefore, with respect to the metric g, the vector fields $\{E_A\}$ form an orthonormal frame.

To obtain the Levi-Civita connection we compute the connection 1-forms (by means of $d\vartheta^A = \vartheta^B \wedge \vartheta^A_B$)

$$\vartheta^i_\alpha = -\tfrac{1}{2}\vartheta^{(\alpha, i)}, \qquad \vartheta^i_{(\alpha, i)} = \tfrac{1}{2}\vartheta^\alpha, \qquad \vartheta^\alpha_{(\alpha, i)} = \tfrac{1}{2}\vartheta^i.$$

The curvature forms ($\Omega^A_B = d\vartheta^A_B + \vartheta^C_B \wedge \vartheta^A_C$) are

$$\begin{aligned}
\Omega^i_j &= \tfrac{1}{4}\vartheta^{(\alpha, j)} \wedge \vartheta^{(\alpha, i)}, & \Omega^\alpha_\beta &= \tfrac{1}{4}\vartheta^{(\alpha, i)} \wedge \vartheta^{(\beta, i)},\\
\Omega^{(\alpha, i)}_{(\alpha, j)} &= \tfrac{1}{4}\vartheta^j \wedge \vartheta^i, & \Omega^{(\alpha, i)}_{(\beta, i)} &= \tfrac{1}{4}\vartheta^\beta \wedge \vartheta^\alpha,\\
\Omega^{(\alpha, i)}_{(\beta, j)} &= 0, & \Omega^i_{(\alpha, j)} &= \tfrac{1}{4}\vartheta^j \wedge \vartheta^{(\alpha, i)},\\
\Omega^i_{(\alpha, i)} &= \tfrac{1}{4}\vartheta^i \wedge \vartheta^{(\alpha, i)}, & \Omega^i_\alpha &= \tfrac{1}{4}\vartheta^i \wedge \vartheta^\alpha,\\
\Omega^\alpha_{(\alpha, i)} &= -\tfrac{3}{4}\vartheta^\alpha \wedge \vartheta^{(\alpha, i)}, & \Omega^\beta_{(\alpha, i)} &= -\tfrac{1}{4}\vartheta^\beta \wedge \vartheta^{(\alpha, i)}.
\end{aligned}$$

Hence, the Ricci tensor ρ is given by

$$\begin{cases}
\rho(E_i, E_j) = \frac{p}{2}\delta^i_j, & i, j = 1, \dots, r;\\
\rho(E_\alpha, E_\beta) = -\frac{r}{2}\delta^\alpha_\beta, & \alpha, \beta = 1, \dots, p;\\
\rho(E_{(\alpha, i)}, E_{(\beta, j)}) = -\frac{1}{2}\delta^i_j\delta^\alpha_\beta. &
\end{cases} \tag{1.3}$$

As a consequence [J], the Riemannian space $(H(p, r), g)$ is not an Einstein space (i.e., $\rho_{AB} \neq K g_{AB}$, $A, B = 1, \dots, rp + r + p$).

Finally we compute the scalar curvature, which is $\tau = \sum \rho_{AB} = -\frac{1}{2}rp$.

I.3. **Geodesics and Killing vector fields on** $(H(p, r), g)$. Further, let $J: \mathscr{Z} \to \mathrm{End}(\mathfrak{h}_1 \oplus \mathfrak{h}_2)$ be the linear map defined by

$$g(J(a)X, Y) = g([X, Y], a),$$

where $a \in \mathscr{Z}$ and $X, Y \in \mathfrak{h}_1 \oplus \mathfrak{h}_2$. It is easy to see that the endomorphism $J(a)$ satisfies the following conditions:

$$\begin{aligned}
&\|J(a)X\| = \|X\|\,\|a\|, \qquad a \in \mathscr{Z},\ X \in \mathfrak{h}_1 \oplus \mathfrak{h}_2;\\
&J(a)^2 = -\|a\|^2 I; \qquad g(J(a)X, Y) + g(J(a)Y, X) = 0.
\end{aligned} \tag{1.4}$$

Using polarization we obtain from (1.4)

$$g(J(a)X, J(b)X) = g(a, b)\|X\|^2,$$
$$g(J(a)X, J(a)Y) = \|a\|^2 g(X, Y)$$

for all $X, Y \in \mathfrak{h}_1 \oplus \mathfrak{h}_2$ and $a, b \in \mathscr{Z}$.

The geodesics of $(H(p, r), g)$ are the solutions of Euler-Lagrange equations for the Lagrangian

$$L = \tfrac{1}{2}[(\dot{x}_\alpha)^2 + (\dot{y}_i^\alpha)^2 + (\dot{z}_i + \tfrac{1}{2}y_i^\alpha \dot{x}_\alpha - \tfrac{1}{2}x_\alpha \dot{y}_i^\alpha)^2]$$

associated to the metric (1.2). These equations are

$$\begin{aligned} \ddot{x}_\alpha &= -\dot{y}_i^\alpha(\dot{z}_i + \tfrac{1}{2}y_i^\alpha \dot{x}_\alpha - \tfrac{1}{2}x_\alpha \dot{y}_i^\alpha), \\ \ddot{y}_i^\alpha &= \dot{x}_i^\alpha(\dot{z}_i + \tfrac{1}{2}y_i^\alpha \dot{x}_\alpha - \tfrac{1}{2}x_\alpha \dot{y}_i^\alpha), \\ \tfrac{d}{dt}(\dot{z}_i &+ \tfrac{1}{2}y_i^\alpha \dot{x}_\alpha - \tfrac{1}{2}x_\alpha \dot{y}_i^\alpha) = 0. \end{aligned} \tag{1.5}$$

The last equation implies at once $\dot{z}_i + \frac{1}{2}y_i^\alpha \dot{x}_\alpha - \frac{1}{2}x_\alpha \dot{y}_i^\alpha = \text{const}$. We restrict our attention to the geodesics $\gamma(t) = (x(t), y(t), z(t))$, starting at identity with the velocity vector $\dot{\gamma}(0) = (\lambda, \mu, \nu)$, i.e., satisfying the initial condition $x(0) = y(0) = z(0) = 0$, and $\dot{x}(0) = \lambda$, $\dot{y}(0) = \mu$, $\dot{z}(0) = \nu$. Then the last of (1.5) becomes

$$\dot{z}_i(t) + \tfrac{1}{2}y_i^\alpha(t)\dot{x}_\alpha(t) - \tfrac{1}{2}x_\alpha(t)\dot{y}_i^\alpha(t) = \nu_i \quad \text{for all } t,$$

and the first two equations in (1.5) reduce to

$$\begin{cases} \ddot{x}_\alpha = -\nu_i \dot{y}_i^\alpha, \\ \ddot{y}_i^\alpha = \nu_i \dot{x}_\alpha. \end{cases}$$

Then the equations of these geodesics are, if $\nu \neq 0$,

$$\begin{cases} x(t) = \dfrac{1-\cos(\|\nu\|t)}{\|\nu\|^2}J(\nu)\mu + \dfrac{\lambda}{\|\nu\|}\sin(\|\nu\|t), \\ y(t) = \dfrac{1-\cos(\|\nu\|t)}{\|\nu\|^2}J(\nu)\lambda - \dfrac{H}{\|\nu\|}\sin(\|\nu\|t) + (H+\mu)t, \\ z(t) = \dfrac{1-\cos(\|\nu\|t)}{\|\nu\|}L - \dfrac{N}{\|\nu\|}\sin(\|\nu\|t) + (1 + K\nu - 2M\cos(|\nu\|t))t, \end{cases}$$

where

$$H_{(\alpha, i)} = \frac{\mu_{(\alpha, j)}\nu_j\nu_i}{\|\nu\|^2}; \quad L_i = H_{(\alpha, i)}\lambda_\alpha + \mu_{(\alpha, i)}\lambda_\alpha; \quad M_i = \frac{H_{(\alpha, i)}\mu_{(\alpha, j)}\nu_j}{\|\nu\|^2};$$
$$K = \frac{\|\lambda\|^2 + \|\mu\|^2 + 2\|J(\nu)\mu\|^2}{\|\nu\|^2}; \qquad N = M + \frac{Lt}{2} + K\nu;$$

and

$$\begin{cases} x(t) = \lambda t, \\ y(t) = \mu t, \qquad \text{if } \nu = 0, \\ z(t) = 0. \end{cases}$$

Now recall that $J(\nu)^2 = -\|\nu\|^2 \,\mathrm{Id}$. Then if $H + \mu = 0$, except when $\nu = 0$, for a geodesic γ starting at the identity with initial vector (λ, μ, ν), one obtains the expression

$$(1.6)\qquad \begin{aligned} X(t) &= \frac{(I - e^{tJ(\nu)})}{\|\nu\|^2} J(\nu)\xi, \\ Z(t) &= \left(t + \frac{1}{2}\frac{\|\xi\|^2}{\|\nu\|^2}\left(t - \frac{\sin\|\xi\|t}{\|\xi\|}\right)\right)\nu, \end{aligned}$$

where $X(t) = (x(t), y(t))$, $Z(t) = z(t)$, and $\xi = \lambda + \mu$.

Proposition 1.4. *Let* $\mathfrak{I}_g$ *be the group of isometries of* $(H(p, r), g)$.

(i) *If* $r > 1$ *then* $\dim \mathfrak{I}_g = rp + p + r + p(p-1)/2 + r(r-1)/2$.

(ii) *If* $r = 1$ *then* $\dim \mathfrak{I}_g = (p+1)^2$.

Proof. It is sufficient to compute the Killing vector fields. For $r > 1$ a basis of Killing vector fields on $(H(p, r), g)$ has been found in [R]. It is

$$\frac{\partial}{\partial x_\alpha} + y_i^\alpha \frac{\partial}{\partial z_i}, \qquad \frac{\partial}{\partial y_i^\alpha} - x_\alpha \frac{\partial}{\partial z_i}, \qquad \frac{\partial}{\partial z_i},$$
$$y_i^\beta \frac{\partial}{\partial y_i^\alpha} - y_i^\alpha \frac{\partial}{\partial y_i^\beta} - x_\beta \frac{\partial}{\partial x_\alpha} + x_\alpha \frac{\partial}{\partial x_\beta},$$
$$y_i^\alpha \frac{\partial}{\partial y_j^\alpha} - y_j^\alpha \frac{\partial}{\partial y_i^\alpha} - z_j \frac{\partial}{\partial z_i} + z_i \frac{\partial}{\partial z_j}.$$

For $r = 1$ a basis is

$$\frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial z},$$
$$y_i \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial x_i},$$
$$x_i \frac{\partial}{\partial y_j} - x_j \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial x_i}.$$

II. Natural reductivity

II.1. **Definition of a Riemannian homogeneous naturally reductive space.** Let (M, g) be a connected n-dimensional Riemannian homogeneous manifold. Further let $M = G/K$, where G is a group of isometries for M and K is the isotropy subgroup at a point p of M.

We denote by $\mathfrak{g}$ (respectively $\mathfrak{k}$) the Lie algebra of G (respectively K). Then the space $M = G/K$ is called *naturally reductive* if there exists a vector subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}; \qquad [\mathfrak{m}, \mathfrak{k}] \subseteq \mathfrak{m};$$
$$\langle [X, Y]_\mathfrak{m}, Z\rangle + \langle [X, Z]_\mathfrak{m}, X\rangle = 0, \quad X, Y, Z \in \mathfrak{m},$$

where $\langle\ ,\ \rangle$ denotes the inner product induced on $\mathfrak{m}$ from g by identification of $\mathfrak{m}$ with T_pM. Further $[\ ,\ ]_\mathfrak{m}$ is the projection of $[\ ,\ ]$ on $\mathfrak{m}$.

In [TV] Tricerri and Vanhecke proved the following result:

Theorem 2.1. *Let (M, g) be a connected, simply connected, and complete Riemannian manifold. Then (M, g) is a naturally reductive homogeneous space if and only if there exists a tensor field T of type $(1, 2)$ such that*

$$\text{(AS)} \qquad \begin{cases} \text{(i)} \quad g(T_X Y, Z) + g(Y, T_X Y) = 0, \\ \text{(ii)} \quad (\nabla_X R)(Y, Z) = [T_X, R_{YZ}] - R_{T_X YZ} - R_{YT_X Z}, \\ \text{(iii)} \quad (\nabla_X T)(Y) = [T_X, T_Y] - T_{T_X Y}, \end{cases}$$

and

$$T_X Y + T_Y X = 0, \qquad X, Y, Z \in \mathfrak{X}(M),$$

where ∇ denotes the Levi-Civita connection and R is the Riemannian curvature tensor.

II.2. **Nonnatural reductivity of the $(H(p, r), g)$ groups.**

Theorem 2.2. *The homogeneous manifold $(H(p, r), g)$ is naturally reductive if and only if $H(p, r)$ is a Heisenberg group (i.e., $r = 1$).*

Proof. Suppose that $(H(p, r), g)$ is a naturally reductive homogeneous space. Then there exists a tensor field T of type $(1, 2)$ satisfying the conditions (AS) and such that

$$T_X Y + T_Y X = 0.$$

Let ρ denote the Ricci tensor of the manifold $(H(p, r), g)$. By contraction, from (AS)(ii) we have

$$(\nabla_X \rho)(Y, Z) = -\rho(T_X Y, Z) - \rho(Y, T_X Z).$$

Since T_X is a skew-symmetric operator, the previous condition gives

$$\mathfrak{S}_{X, Y, Z} (\nabla_X \rho)(Y, Z) = 0. \tag{2.1}$$

On the other hand, from (1.3) one has

$$\begin{aligned} (\nabla_{E_i} \rho)(E_\alpha, E_{(\alpha, i)}) &= -(p + r)/4, \\ (\nabla_{E_\alpha} \rho)(E_i, E_{(\alpha, i)}) &= (p + 1)/4, \\ (\nabla_{E_{(\alpha, i)}} \rho)(E_\alpha, E_i) &= (1 - r)/4. \end{aligned}$$

Now we combine these relations with condition (2.1) and obtain $r = 1$. Conversely, if $r = 1$, the group $H(p, 1)$ is isomorphic to the Heisenberg group H_{2p+1}. In this case we know that (H_{2p+1}, g) is a naturally reductive space for every left-invariant metric g [GP].

III. Geodesic symmetries

Let (M, g) be a smooth n-dimensional Riemannian manifold. For every point m in M, consider a neighborhood U_m of m such that for every point $p \in U_m$ there exists a unit vector $\xi \in T_m M$ and a real number r such that $p = \exp_m(r\xi)$. Then the *local geodesic symmetry centered at* m is the diffeomorphism $\mathfrak{s}_m : U_m \to U_m$ defined by $\mathfrak{s}_m(\exp(r\xi)) = \exp(-r\xi)$.

Definition 3.1 [DN]. A Riemannian manifold is a *D'Atri space* if every local geodesic symmetry is volume-preserving (up to sign).

Locally symmetric manifolds are the simplest D'Atri spaces, but there are also a lot of nonsymmetric examples. In particular, all naturally reductive homogeneous Riemannian manifolds are D'Atri spaces. The converse is false. In

[K] Kaplan gives a family of examples of nonnaturally reductive D'Atri spaces. These spaces are connected and simply connected nilpotent Lie groups whose Lie algebras $\mathfrak{n}$ split as $\mathfrak{n} = V \oplus Z$, where Z is the center of $\mathfrak{n}$, and satisfy

$$g(X, Y) = 0 \quad \text{for all } X \in V, \, Y \in Z; \qquad \|\mathrm{ad}_X^*(Y)\| = \|X_V\| + \|Y_Z\|,$$

where * denotes the adjoint relative to g and X_v (resp. Y_z) is the projection of X on V (resp. Z). These spaces are called *of type* H.

It is well known that if an analytic Riemannian manifold is a D'Atri space then it satisfies the Ledger conditions of odd order. Now, the first Ledger condition is equivalent to (2.1). The Riemannian manifold $(H(p, r), g)$ is not a D'Atri space but let D be the distribution given by

$$D = \{(\lambda, \mu, \nu) \in \mathfrak{h}(p, r) | \mu_{(\alpha, j)}\nu_j\nu_i + \|\nu\|^2\mu_{(\alpha, i)} = 0\};$$

then

Theorem 3.2. *Let* X *be a vector in* $\mathfrak{h}(p, r)\backslash D$. *Then the local geodesic symmetry with respect to the geodesic through* e *and determined by* X *is volume-preserving* (*up to sign*).

Proof. It follows the same lines as Kaplan's proof.

It is sufficient to consider the geodesic symmetry $\mathfrak{s}$ at the identity e of $H(p, r)$ along the geodesic γ such that $\dot\gamma \in \mathfrak{h}(p, r)\backslash D$. Let $\exp_H \colon \mathfrak{h}(p, r) \to H(p, r)$ denote the Lie exponential maps. Let U be a normal neighborhood of e. For X in $\exp^{-1} U$, let $F(X) \in \mathfrak{h}(p, r)\backslash D$ denote the tangent vector at e of the geodesic joining e to $\exp_H X$, and put $\Sigma = F^{-1} \circ (-F)$. Then the geodesic symmetry $\mathfrak{s}$ maps $\exp_H X$ to $\exp_H \Sigma X$.

The geodesic symmetry $\mathfrak{s}$ can also be computed easily. Indeed, if we put $F(X) = (a_1, a_2)$ from (1.6) we have

$$X = \frac{(I - e^{J(a_2)})}{\|a_2\|} J(a_2)a_1, \qquad Z = \left(1 + \frac{1}{2}\frac{\|a_1\|}{\|a_2\|^2}\right)\left(1 - \frac{\sin\|a_2\|}{\|a_2\|}\right)a_2,$$

and therefore

$$\sum(X) = -e^{-J(a_2)}X, \qquad \sum(Z) = -Z.$$

By using conditions (1.4) one gets

$$\|X\|^2 = 2\frac{\|a_1\|^2}{\|a_2\|^2}(1 - \cos\|a_2\|),$$

$$\|Z\| = \left(1 + \frac{1}{2}\frac{\|a_1\|^2}{\|a_2\|^2}\left(1 - \frac{\sin\|a_2\|}{\|a_2\|}\right)\right)\|a_2\|.$$

For small $\|X\|$, $\|Z\|$, these equations determine $\|a_1\|$, $\|a_2\|$ uniquely, so we can write

$$\mathfrak{s}(\exp_H X + Z) = \exp_H\left(\sum(X + Z)\right) = \exp_H(-e^{\beta(\|X\|, \|Z\|)J(\|Z\|)}X - Z)$$

for some function β depending only on the length of X and Z.

Although $\mathfrak{s}$ is not an isometry, it acts isometrically on the spheres of $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ and $\mathcal{Z}$ centered at 0. Therefore, it preserves the Euclidean Lebesgue measure in $\mathfrak{h}(p, r)$. The Riemannian volume element defines a Haar measure on $H(p, r)$. The Haar measure of a nilpotent group is the exponential of a Lebesgue measure. Finally the geodesic symmetry $\mathfrak{s}$ is volume-preserving.

IV. Lie algebras of type H and Lie algebras $\mathfrak{h}(p, r)$

The groups of type H and the $H(p, r)$ groups have some common Riemannian properties. Let us prove that they are not the same spaces.

Let H be a group of type H. Its Lie algebra $\mathfrak{h}$ is called an *algebra of type* H. In [G_2] Goze proved the following result:

Proposition 4.1. *Let G be a connected, simply connected n-dimensional two-step nilpotent group with r-dimensional center Z. Then there exists on G a Pfaffian system S of rank r and class n.*

Let $\mathfrak{h}(r)$ be an algebra of type H, where r is the dimension of its center. Then

(1) The Pfaffian system S defines on $\mathfrak{h}(r)$ an *r-contact structure* (an early generalization of contact structure introduced in [L]), i.e.,

$$(d\alpha)^{(n-r)/2} = 0 \quad \operatorname{mod} S \; \forall \alpha \in S.$$

The Pfaffian system S defines on $\mathfrak{h}(p, r)$ an r-contact system.

(2) $\mathfrak{h}(r)$ is a model for a Lie algebra with an r-contact structure. $\mathfrak{h}(p, r)$ is a model for a Lie algebra with an r-contact system.

(3) Every Lie algebra $\mathfrak{h}(r)$, with $\dim \mathfrak{h}(r) = rp + r + p$, can be contracted onto $\mathfrak{h}(p, r)$.

(4) The Engel invariant $c(s)$ for a Pfaffian system (S) satisfies the Gardner inequalities

$$\frac{n-r}{r-1} \leq c(S) \leq \frac{n-r}{2}.$$

For r-contact systems we have $c(S) = (n-r)L/(r-1)$; but an r-contact structure satisfies $c(S) = (n-r)/2$.

From this follows the

Proposition [G_2]. *The Lie algebra $\mathfrak{h}(r)$ is an $\mathfrak{h}(p, r)$ algebra if and only if it is a Heisenberg algebra.*

References

[DN] J. E. D'Atri and H. K. Nickerson, *Geodesic symmetries in spaces with special curvature tensor*, J. Differential Geom. **9** (1974), 251–262.

[G_1] M. Goze, *Étude locale des systèmes de Pfaff*, Publ. IRMA, Strasbourg, 1989.

[G_2] ——, *Systèmes de Pfaff associés aux algèbres de Lie de type H*, Rend. Sem. Mat. Univ. Politec. Torino **46** (1988), 91–110.

[GH] M. Goze and Y. Haraguchi, *Sur les r-systèmes de contact*, C. R. Acad. Sci. Paris Sér. I Math. **294** (1982), 95.

[GP] M. Goze and P. Piu, *Classification des métriques invariantes à gauche sur le groupe de Heisenberg*, Rend. Circ. Mat. Palermo **39** (1990), 299–306.

[H] Y. Haraguchi, *Sur une généralisation des structurers de contact*, Thèse, Mulhouse, 1981.

[J] G. R. Jensen, *Homogeneous Einstein spaces of dimension four*, J. Differential Geom. **3** (1969), 309–349.

[K] A. Kaplan, *On the geometry of groups of Heisenberg type*, Bull. London Math. Soc. **15** (1983), 35–42.

[R] W. Reyes, *A note about Killing vector fields on* $H(p, r)$, Rend. Sem. Mat. Univ. Cagliari **55** (1985), 25–29.

[TV] F. Tricerri and L. Vanhecke, *Homogeneous structures on Riemannian manifolds*, London Math. Soc. Lecture Note Ser., vol 83., Cambridge Univ. Press, New York, 1983.

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