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EVERY NORMAL BAND WITH (REP) AND $(REP)^{op}$ IS AN AMALGAMATION BASE

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ABSTRACT. We shall prove that every normal band with the representation extension property and its dual is an amalgamation base in the class of all semigroups.

1. INTRODUCTION

A semigroup S is called an *amalgamation base* in the class of all semigroups (simply called an amalgamation base), if for any semigroups T_1, T_2 containing S as a subsemigroup the amalgam $[T_1, T_2; S]$ is embedded into a semigroup. A semigroup S has the *representation extension property* (denoted by (REP)) if for every embedding $S \to T$ of semigroups and every right S-set X, the canonical map: $X \to X \otimes T^1$ is injective (see [2, 6, 7]). The left–right dual of (REP) is denoted by $(REP)^{op}$. Hall [6] showed that any semigroup which is an amalgamation base always has (REP) and $(REP)^{op}$. The author [9] constructed an example of a monoid which has (REP) and $(REP)^{op}$ but is not an amalgamation base. However, such an example of regular semigroups is still unknown. In this direction, Bulman-Fleming and McDowell [4] determined the structure of normal bands with (REP) and $(REP)^{op}$ and, consequently, showed that every right (left) normal band with (REP) and $(REP)^{op}$ is left (right) absolutely flat (see [3]) and hence is an amalgamation base. The purpose of this paper is to prove the following stronger result.

Main Theorem. *A normal band has both (REP) and $(REP)^{op}$ if and only if it is an amalgamation base.*

Our method is to appeal the criterion for an amalgamation base given in [9], which is a modified version of Renshaw's Theorem [8, Theorem 6.11].

2. PRELIMINARIES

Throughout this paper, let S denote a semigroup and S^1 the semigroup with the adjoined identity 1 whether S has an identity or not. Let $\mathcal{J}$ $[\mathcal{L}, \mathcal{R}]$ denote *Green's $\mathcal{J}$- $[\mathcal{L}$-, $\mathcal{R}$-$]$ relation* on a semigroup. We often use the notation

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and conventions from Clifford and Preston's book [5] for semigroup theory. Let S-*Ens* (Ens-S, S-Ens-S) denote the category of all left S-sets (right S-sets, S-bisets). Let $X \in Ens\text{-}S$ and $Y \in S\text{-}Ens$. The *tensor product* over S of X and Y is denoted by $X \otimes_S Y$ (simply, $X \otimes Y$ if there is no confusion). Also, any element of $X \otimes Y$ is written in a form $x \otimes y$ $(x \in X,\ y \in Y)$. For brevity, $X \supset Y$ $(X, Y \in S\text{-}Ens$ $(Ens\text{-}S, S\text{-}Ens\text{-}S))$ means that Y is a left S- (right S-, S-bi) subset of X.

We will use the following results in the sequel.

Result 1 [9, Theorem 2.1]. *A semigroup S has (REP) if and only if, for each $M \in S\text{-}Ens$ with $M \supset S^1$ and each $X \in Ens\text{-}S$, the map: $X \to X \otimes M$ $(x \mapsto x \otimes 1)$ is injective.*

Result 2 [9, Theorem 2.2]. *A semigroup S is an amalgamation base if and only if for each $X \in Ens\text{-}S$, $Y \in S\text{-}Ens$, and $N \in S\text{-}Ens\text{-}S$ with $N \supset S^1$, the map: $X \otimes Y \to X \otimes N \otimes Y$ $(x \otimes y \to x \otimes 1 \otimes y)$ is injective.*

We recall that a *normal band* satisfies the identity $xyzx = xzyx$ (equivalently, $xyza = xzya$).

For a normal band S, let $S = \bigcup\{S_\lambda : \lambda \in \Lambda\}$ be the semilattice decomposition. In this case each S_λ is a $\mathscr{J}$-class of S. So by using the partial order $\geq$ on Λ, we define a quasi-order $\geq_{\mathscr{J}}$ on S by $s \geq_{\mathscr{J}} t$ $(s, t \in S)$ if and only if $\mathscr{J}_s \geq \mathscr{J}_t$. Then, for convenience, we sometimes write $t \leq_{\mathscr{J}} s$. Also, $s >_{\mathscr{J}} t$ means both $\mathscr{J}_s \geq \mathscr{J}_t$ and $\mathscr{J}_s \neq \mathscr{J}_t$. If necessary, we extend the quasi order $\geq_{\mathscr{J}}$ from S to S^1. Clearly, $1 >_{\mathscr{J}} s$ in S^1 for all $s \in S$.

Result 3 [4, Theorem 1]. *A normal band $S = \bigcup\{S_\lambda : \lambda \in \Lambda\}$ has (REP) and $(REP)^{\mathrm{op}}$ if and only if S has the following:*

(i) *$uau = vav$ for any $u, v, a \in S$ with $u\mathscr{J}v$, $u >_{\mathscr{J}} a$;*
(ii) *$|S_\lambda| \leq 2$ for each $\lambda \in \Lambda$; and*
(iii) *if $|S_\lambda| = 2$ $(\lambda \in \Lambda)$ then $\bigwedge S_\lambda$ does not exist with respect to the natural ordering $\geq$ of S.*

3. Proof of the main theorem

To prove the main theorem, it suffices to prove the "only if" part. In this section, we let S be a normal band with (REP) and $(REP)^{\mathrm{op}}$. Then we shall show first the preliminary lemmas.

Lemma 1. *Let S be as above, and $a, u, v \in S$. Let $X \in Ens\text{-}S$, $Y \in S\text{-}Ens$, $x, x' \in X$, and $y, y' \in Y$. Then:*

(i) *$xu = x'v$ implies $xuau = x'vav$; and*
(ii) *$uy = vy'$ implies $uauy = vavy'$.*

Proof. (i) If $uv >_{\mathscr{J}} uva$, then $vuauv = (vu)^2a(uv)^2 = uv(vuauv)vu$ (by Result 3(i)) $= uvavu$, so that $xuau = xuvau = x(uvavu) = x'v(uvavu) = x'v(vuauv) = x'(vuav) = x'vav$. If $uv\mathscr{J}uva$, then $xuau = xuvau = x(uvu) = xu$, and similarly $x'vav = x'v$. Hence (i) holds.

(ii) Similarly. □

Lemma 2 (cf. [1, Lemma 2]). *Let S, X, Y, x, and y be as above. Suppose that $x \otimes y = x' \otimes y'$ in $X \otimes_S Y$. Then there exist $s_1, \ldots, s_n, t_1, \ldots, t_n \in S^1$,*

$x_1, \dots, x_n \in X$, and $y_2, \dots, y_n \in Y$ such that

$$\begin{array}{rr} x = x_1 s_1, & s_1 y = t_1 y_2, \\ x_1 t_1 = x_2 s_2, & s_2 y_2 = t_2 y_3, \\ \vdots & \vdots \\ x_{n-1} t_{n-1} = x_n s_n, & s_n y_n = t_n y', \\ x_n t_n = x' & \end{array} \tag{1}$$

and

$$\begin{gathered} s_1 \geq_{\mathcal{J}} t_1 \geq_{\mathcal{J}} \cdots \geq_{\mathcal{J}} s_i \geq_{\mathcal{J}} t_i \leq_{\mathcal{J}} s_{i+1} \leq_{\mathcal{J}} t_{i+1} \leq_{\mathcal{J}} \cdots \leq_{\mathcal{J}} s_n \leq_{\mathcal{J}} t_n \\ (or\ s_1 \geq_{\mathcal{J}} t_1 \geq_{\mathcal{J}} \cdots \geq_{\mathcal{J}} s_i \leq_{\mathcal{J}} t_i \leq_{\mathcal{J}} s_{i+1} \leq_{\mathcal{J}} t_{i+1} \leq_{\mathcal{J}} \cdots \leq_{\mathcal{J}} s_n \leq_{\mathcal{J}} t_n) \end{gathered} \tag{2}$$

where $\geq_{\mathcal{J}}$ is the quasi order of S^1.

According to [1], *a set of equations* (1) *is called a scheme of length n over X and Y joining (x, y) to (x', y'). If a scheme satisfies* (2), *then we say that it is V-formed.*

Proof. By [1, Lemma 2], there exists a scheme (1) joining (x, y) to (x', y'). By appropriate substitution of s_i, t_i, we will show that (2) is satisfied. Let us assume in (1) that

$$s_i \in S \quad (1 < i \leq n), \qquad t_i \in S \quad (1 \leq i < n).$$

For if $s_i = 1 \quad (1 < i \leq n)$, then $s_{i-1} y_{i-1} = t_{i-1} t_i y_{i+1}$, $x_{i-1} t_{i-1} t_i = x_{i+1} s_{i+1}$; hence, the scheme gets shorter; similarly, if $t_i = 1 \quad (1 \leq i < n)$.

Next, if t_i, s_{i+1} are incomparable with respect to $\geq_{\mathcal{J}}$, then one can insert new equations into the equations (1) as follows:

$$\begin{gathered} x_i t_i = x_{i+1}(s_{i+1} t_i s_{i+1}), \qquad (s_{i+1} t_i s_{i+1}) y_{i+1} = (s_{i+1} t_i s_{i+1}) y_{i+1}, \\ x_{i+1}(s_{i+1} t_i s_{i+1}) = x_{i+1} s_{i+1}, \qquad s_{i+1} y_{i+1} = t_{i+1} y_{i+2}. \end{gathered}$$

(If s_i, t_i are incomparable with respect to $\geq_{\mathcal{J}}$, then

$$s_i y_i = (t_i s_i t_i) y_{i+1}, \qquad x_i(t_i s_i t_i) = x_i(t_i s_i t_i), \qquad (t_i s_i t_i) y_{i+1} = t_i y_{i+1}.)$$

By repeating such insertions, we may assume any adjacent two elements of the sequence $s_1, t_1, \dots, s_n, t_n$ are $\mathcal{J}$-comparable. If scheme (1) is not V-formed, then several of the following four cases may occur. In each case, we will convert a part of the scheme into a V-formed scheme as follows.

Case 1. $s_i <_{\mathcal{J}} t_i \mathcal{J} \cdots \mathcal{J} t_{j-1} \mathcal{J} s_j >_{\mathcal{J}} t_j$. Then, by assumption, all $s_i, t_i, \dots, t_{j-1}, s_j, t_j$ are in S. Set

$$\begin{array}{ll} t'_k = t_k s_i t_k, & s'_{k+1} = s_{k+1} s_i s_{k+1}, \\ t''_k = t_k t_j t_k, & s''_{k+1} = s_{k+1} t_j s_{k+1}, \\ t^*_k = t_k s_i s_j t_j t_k, & s^*_{k+1} = s_{k+1} s_i s_j t_j s_{k+1} \qquad (i \leq k \leq j-1). \end{array}$$

By Result 3(i), we have

$$t'_k = s'_l, \quad t''_k = s''_l, \quad t^*_k = s^*_l \qquad (i \leq k < j,\ i < l \leq j).$$

From (1) we get

$$\begin{array}{ll} & s_i y_i (= s'_j y_j = s'_j s_j y_j = s'_j s_j t_j s_j y_j) = s^*_j y_j, \\ x_i s^*_j = x_i t^*_i, & t^*_i y_j (= t^*_i y_{i+1} = t_i t_j s_i t_i y_{i+1} = t_i t_j t_i y_{i+1}) = t''_i y_{i+1}, \\ x_i t''_i = x_j s''_j, & s''_j y_{i+1} (= s''_j y_j = s_j y_j) = t_j y_{j+1}, \end{array}$$

and $s_i \geq_{\mathscr{J}} s_i^* \mathscr{J} t_i^* \leq_{\mathscr{J}} t_i'' \leq_{\mathscr{J}} t_j$. This is a required scheme.

Case 2. $t_i <_{\mathscr{J}} s_{i+1} \mathscr{J} \cdots \mathscr{J} t_{j-1} \mathscr{J} s_j >_{\mathscr{J}} t_j$. Then by assumption, all t_i, $s_{i+1}, \ldots, t_{j-1}, s_j, t_j$ are in S.

Set

$$\begin{aligned} s_k' &= s_k t_i s_k, & t_k' &= t_k t_i t_k, \\ s_k'' &= s_k t_j s_k, & t_k'' &= t_k t_j t_k, \\ s_k^* &= s_k t_i s_j t_j s_k, & t_k^* &= t_k t_i s_j t_j t_k \qquad (i+1 \leq k \leq j). \end{aligned}$$

By Result 3(i), we have

$$s_k' = t_l', \quad s_k'' = t_l'', \quad s_k^* = t_l^* \quad (i+1 \leq k \leq j,\ i+1 \leq l < j).$$

From (1) we get

$$x_i t_i (= x_{i+1} s_{i+1}') = x_j s_j', \qquad s_j' y_{i+1} (= s_j' y_j = s_j' s_j y_j = s_j' s_j t_j s_j y_j) = s_j^* y_j,$$
$$x_j s_j^* (= x_{i+1} s_{i+1}^* = x_{i+1} s_{i+1} t_i s_j t_j s_{i+1} = x_{i+1} s_{i+1} s_j t_j s_{i+1} = x_{i+1} s_{i+1} t_j s_{i+1}) = x_j s_j'',$$
$$s_j'' y_j = t_j y_{j+1}$$

and $t_i \geq_{\mathscr{J}} s_j' \geq_{\mathscr{J}} s_i^* \leq_{\mathscr{J}} s_j'' \leq_{\mathscr{J}} t_j$. We are done.

Case 3. $s_i <_{\mathscr{J}} t_i \mathscr{J} \cdots \mathscr{J} t_{j-1} >_{\mathscr{J}} s_j$. By reversely ordering the equations (1), it is just Case 2.

Case 4. $t_i <_{\mathscr{J}} s_{i+1} \mathscr{J} \cdots \mathscr{J} t_{j-1} >_{\mathscr{J}} s_j$. In a way similar to the above, this is Case 1.

Notice that the subband of S^1 generated by all the s_i, t_i in (1) is finite (of course, it has finitely many $\mathscr{J}$-classes) and it contains all the elements $s_i', t_i', s_i'', t_i'', s_i^*, t_i^*$ occurring in the substitutions above. Thus by finitely repeating those substitutions of parts of the scheme by V-formed one, scheme (1) becomes V-formed. □

Lemma 3. *Let S, X, Y be as above and $x, x' \in XS$ and $y, y' \in SY$.*

(i) *If $x \otimes y = x' \otimes y'$ in $X \otimes_S Y$, then $xa \otimes y = x'a \otimes y'$ in $X \otimes_S Y$ for all $a \in S$.*

(ii) *If $xs \otimes y = x' \otimes y'$, $x \otimes y = x't \otimes y'$ in $X \otimes_S Y$ for some $s, t \in S$, then $x \otimes y = x' \otimes y'$.*

Proof. (i) By Lemma 2, there exist $x_1, \ldots, x_n \in X$, $y_2, \ldots, y_n \in Y$, $s_1, \ldots, s_n$, and $t_1, \ldots, t_n \in S^1$ such that

$$\begin{aligned} x &= x_1 s_1, & s_1 y &= t_1 y_2, \\ x_1 t_1 &= x_2 s_2, & s_2 y_2 &= t_2 y_3, \\ &\vdots & &\vdots \\ x_{n-1} t_{n-1} &= x_n s_n, & s_n y_n &= t_n y', \\ x_n t_n &= x'. \end{aligned} \tag{3}$$

Here we may assume that all s_i, t_i belong to S. For, if $s_1 = 1$, then s_1, t_1 can be replaced by s, st_1, respectively, where s is any element of S with $xs = x$. Also if $t_n = 1$, then t_n can be also replaced by some element of S. Further if $s_i = 1$ $(2 \leq i)$, then as seen in the proof of Lemma 2 the scheme gets shorter; similarly, if $t_i = 1$ $(1 \leq i < n)$.

Note next that $efy = efey$ $(efy' = efey')$ for all $e, f \in S$. For, by assumption, we can write $y = hy$ $(h \in S)$ and by normality of S, $efy = ef(hy) = (eefh)y = (efeh)y = efey$. (Similarly, $efy' = efey'$.)

Thus by using Lemma 1 and the note above, we get

$$xa = x_1(s_1a), \qquad (s_1a)y(= (s_1as_1)y) = (t_2at_2)y_2$$
$$((s_nas_n)y_n(= (t_nat_n)y') = (t_na)y', \qquad x_n(t_na) = x'a).$$

So, by Lemma 1, we get a scheme joining (xa, y) to $(x'a, y')$ by replacing s_i, t_i by s_ias_i, t_iat_i respectively. Then (i) holds.

(ii) This is an immediate consequence of (i). □

Remarks. 1. Lemma 3(i) is false without assumption that $x, x' \in XS$ and $y, y' \in SY$. For instance, let S be a left zero semigroup. Then $1 \otimes a = a \otimes a$ in $S^1 \otimes S$, but $b \otimes a \neq aba \otimes a$.

2. Given a scheme (3) of length n joining (x, y) to (x', y') (not necessarily, $x, x' \in XS$), it is shown, in the proofs of Lemmas 2 and 3, that it is possible to assume all the s_i, t_i except possibly s_1, t_n belong to S and that s_1 is in S if $x \in XS$ (t_n is in S if $x' \in XS$). Under these assumptions, if $x \in X - XS$ and $y \in Y - SY$, then $s_1 = t_1 = 1$ and $n = 1$; that is, $x = x'$ and $y = y'$. Otherwise, one can find $x'' \in XS$ and $y'' \in SY$ such that $x \otimes y = x'' \otimes y''$.

The proof of the "only if" part of the main theorem. We will appeal to Result 2. Let S be a normal band with (REP) and $(REP)^{op}$. Suppose

$$x \otimes (1 \otimes y) = x' \otimes (1 \otimes y') \quad \text{in } X \otimes (W \otimes Y) \tag{4}$$

where $x, x' \in X$, $y, y' \in Y$, $S^1 \subset W$, $X \in Ens\text{-}S$, $W \in S\text{-}Ens\text{-}S$, and $Y \in S\text{-}Ens$. Then we shall show that

$$x \otimes y = x' \otimes y' \quad \text{in } X \otimes Y. \tag{5}$$

By the remarks after Lemma 3, we may assume that $x, x' \in XS$ and $y, y' \in SY$.

Here we may assume that W has the following property:

$$aws \in S,\ a\mathscr{R}b,\ \text{and } a >_{\mathscr{J}} s\ (a, b, s \in S,\ w \in W) \quad \text{implies} \quad bws = bsbws \in S. \tag{6}$$

Proof of (6). Let ξ be the congruence on W generated by the relation $(bws, bsbws)$ and $\xi|_S$ the restriction to S of ξ. Then we shall show that $\xi|_S$ is an identity relation on S. For our purpose, it suffices to show that

$$ubwsv = u'bwsv'\ (u, u', v, v' \in S) \quad \text{implies} \quad ubsbwsv = u'bsbwsv'. \tag{7}$$

If $a = b$, then, by assumption, $bws \in S$ and so, by normality of S, $bws = b^2ws^3 = b(sbws)s$. Hence (7) holds. Then we can assume that $a \neq b$. By Result 3(ii), $\mathscr{J}_a = \mathscr{R}_a$. If $u, u' \geq_{\mathscr{J}} b$, then $ub = b$, $u'b = b$ and, hence,

$$(ub)sbwsv = bsbwsv = bs(ub)wsv = bs(u'bwsv') = u'bsbwv'$$

as required. If $u \not\geq_{\mathscr{J}} b$ (or $u' \not\geq_{\mathscr{J}} b$), then, by Result 3(i),

$$bub = b(bub)b = a(bub)a = aua$$

so that, by assumption,

$$ubwsv = (ubub)wsv = u(aua)wsv \in S.$$

Then, by normality of S,

$$\begin{aligned} ubwsv &= ub(bub)wsv = ub(aua)w(ssv) = ubs(aua)wsv \\ &= ubs(bub)wsv = ubsbwsv. \end{aligned}$$

Then $u'bwsv' \in S$. Similarly, $u'bwsv' = u'bsbwsv'$. In any case, (7) holds. Therefore, $\xi|_S$ is an identity relation on S. So S can be naturally embedded in W/ξ. Hence $bws = bsbws = (asa)ws \in S$, which proves (6).

Hereafter, by Result 1, we may identify $y \in Y$ with $1 \otimes y \in W \otimes Y$.

By Lemma 2, we obtain a V-formed scheme of length n over X and $W \otimes Y$ joining $(x, 1 \otimes y)$ to $(x', 1 \otimes y')$ as follows:

$$\begin{aligned} x &= x_1 a_1, & a_1(1 \otimes y) &= b_1(w_2 \otimes y_2), \\ x_1 b_1 &= x_2 a_2, & a_2(w_2 \otimes y_2) &= b_2(w_3 \otimes y_3), \\ &\vdots & &\vdots \\ x_{n-1} b_{n-1} &= x_n a_n, & a_n(w_n \otimes y_n) &= b_n(1 \otimes y'), \\ x_n b_n &= x' \end{aligned} \tag{8}$$

where $x_i \in X$, $w_i \in W$, $y_i \in Y$, and $a_i, b_i \in S^1$.

We are going to prove (5) by induction on the length n of scheme (8).

By the remarks after Lemma 3, we may assume, in (8),

all the a_i, b_i belong to S.

If $n = 1$, then, obviously, $x \otimes y = x' \otimes y'$. Assuming that (4) implies (5) when $n \leq m$, we proceed to the case where $n = m + 1$. First we may assume

$$a_1 \mathscr{R} b_1 \mathscr{R} a_2 \quad \text{and} \quad b_1 \neq a_2. \tag{9}$$

Proof of (9). If $b_1 >_{\mathscr{J}} a_1$, then we obtain the ascending chain

$$a_1 <_{\mathscr{J}} b_1 \leq_{\mathscr{J}} a_2 \leq_{\mathscr{J}} \cdots \leq_{\mathscr{J}} a_n \leq_{\mathscr{J}} b_n$$

since scheme (8) is V-formed. In this case, regarding scheme (8) as joining (x', y') to (x, y), we can assume that $a_1 \geq_{\mathscr{J}} b_1$.

Next, if $a_1 >_{\mathscr{J}} b_1$, then $a_1(1 \otimes y) = b_1(w_2 \otimes y_2) = (a_1 b_1 a_1)(1 \otimes y)$. Consequently,

$$\begin{aligned} x &= x_1 a_1, & a_1(1 \otimes y) &= (a_1 b_1 a_1)(1 \otimes y), \\ x_1(a_1 b_1 a_1) &= x(a_1 b_1 a_1), \end{aligned}$$

so that

$$x \otimes y = x(a_1 b_1 a_1) \otimes y,$$

while

$$x(a_1 b_1 a_1) = x_1(a_1 b_1 a_1), \qquad (a_1 b_1 a_1)(1 \otimes y) = b_2(w_2 \otimes y_2).$$

Therefore, we may assume that $a_1 \mathscr{J} b_1$.

If $b_1 >_{\mathscr{J}} a_2$, then

$$x = x_1 a_1, \qquad a_1(1 \otimes y)(= b_1(w_2 \otimes y_2)) = b_1 a_1 (1 \otimes y),$$
$$x_1(b_1 a_1)(= (x_2 a_2)(b_1 a_1) = (x_1 b_1 a_2)(b_1 a_1)$$
$$= x_1(a_1 a_2 a_1) a_1 \quad \text{[by Result 3(i)]}) = x(a_1 a_2 a_1).$$

On the other hand,

$$x(a_1 a_2 a_1) = x_1(a_1 a_2 a_1), \qquad (a_1 a_2 a_1)(1 \otimes y) = (b_1 a_2 b_1)(w_2 \otimes y_2),$$
$$x_1(b_1 a_2 b_1) = x_2 a_2.$$

Hence, we may assume that $a_2 \geq_{\mathscr{J}} b_1$.

If $a_2 >_{\mathscr{J}} b_1$, then, since scheme (8) is V-formed,

$$a_1 <_{\mathscr{J}} b_1 \leq_{\mathscr{J}} a_2 \leq_{\mathscr{J}} \cdots \leq_{\mathscr{J}} a_n \leq_{\mathscr{J}} b_n.$$

In this case, as shown above, we can reduce to the case that $b_n \mathscr{J} a_n \mathscr{J} b_{n-1}$. By renumbering reversely the equations (8), we may assume that $a_1 \mathscr{J} b_1 \mathscr{J} a_2$.

If $\mathscr{J}_{a_1} = \mathscr{L}_{a_1}$, then we can replace $w_2 \otimes y_2$ by $1 \otimes y$ in (8) and scheme (8) gets shorter. On the other hand, if $\mathscr{J}_{a_1} = \mathscr{R}_{a_1}$ and $b_1 = a_2$, then $x = x a_1 = x_1(b_1 a_1) = (x_2 a_2) a_1 = x_2 a_1$. So we can remove x_1, $w_2 \otimes y_2$ from scheme (8). Hence (9) may be assumed.

Case 1. There exists some $2 \leq i < n$ such that all a_k, b_k $(1 \leq k \leq i)$ belong to $\mathscr{J}_{a_1}$ but $a_1 >_{\mathscr{J}} a_{i+1}$. Since $\mathscr{J}_{a_1} = \mathscr{R}_{a_1}$, by multiplying the equations (8) on the left by a_1 from the right, we get

$$x = x a_1 = x_1 a_1 = x_2 a_1 = \cdots = x_i a_1 = x_{i+1} a_{i+1} a_1,$$

so that $x = x_i(a_1 a_{i+1} a_1)$, while, by Result 3(i), $a_k a_{i+1} a_k = b_k a_{i+1} b_k$ for all $1 \leq k \leq i$. So from (8) we obtain a scheme of length $\leq m$ joining $(x, 1 \otimes y)$ to $(x', 1 \otimes y')$ as follows:

$$\begin{aligned} x &= x_i(a_1 a_{i+1} a_1), & (a_1 a_{i+1} a_1)(1 \otimes y) &= (b_i a_{i+1} b_i)(w_{i+1} \otimes y_{i+1}), \\ x_i(b_i a_{i+1} b_i) &= x_{i+1} a_{i+1}, & a_{i+1}(w_{i+1} \otimes y_{i+1}) &= b_{i+1}(w_{i+2} \otimes y_{i+2}), \\ &\vdots & &\vdots \\ x_{n-1} b_{n-1} &= x_n a_n, & a_n(w_n \otimes y_n) &= b_n(1 \otimes y'), \\ x_n b_n &= x'. \end{aligned}$$

By the inductive assumption, $x \otimes y = x' \otimes y'$.

Case 2. There exists some $1 \leq i < n$ such that all a_k, b_k $(1 \leq k \leq i)$ belong to $\mathscr{J}_{a_1}$ but $a_1 \mathscr{J} a_{i+1} >_{\mathscr{J}} b_{i+1}$. By applying Lemma 1(ii) to (8) we have

$$(a_1 b_{i+1} a_1) y (= (b_1 b_{i+1} b_1)(w_2 \otimes y_2) = \cdots = (b_i b_{i+1} b_i)(w_{i+1} \otimes y_{i+1})$$
$$= (a_{i+1} b_{i+1} a_{i+1})(w_{i+1} \otimes y_{i+1}) = a_{i+1}(w_{i+1} \otimes y_{i+1})) = b_{i+1}(w_{i+2} \otimes y_{i+2}).$$

Also, $x(a_1 b_{i+1} a_1) = x_{i+1}(a_1 b_{i+1} a_1)$. Then there exists a scheme of length $< n$ over X and $W \otimes Y$ joining $(x(a_1 b_{i+1} a_1), y)$ to (x', y'). Consequently, it follows from the inductive assumption that $x(a_1 b_{i+1} a_1) \otimes y = x' \otimes y'$. We have to prove that $x \otimes y = x \otimes (a_1 b_{i+1} a_1) y$. Since $a_{i+1}(w_{i+1} \otimes y_{i+1}) = a_1(a_1 b_{i+1} a_1 y_{i+1})$, $x_{i+1} a_1 = x$, this case can be reduced to the case for all a_j, b_j $(1 \leq j \leq n)$. So we proceed to the next case.

Case 3. All a_i, b_i $(1 \le i \le n)$ belong to $\mathscr{R}_{a_1}$. Then

$$x' = x = xs \quad \text{for all } s \in \mathscr{R}_{a_1}. \tag{10}$$

From Lemma 2, it follows that for each $1 \le i \le n$, there exists a V-formed scheme of length n_i over W and Y joining (a_iw_i, y_i) to (b_iw_{i+1}, y_{i+1}) as follows:

$$\begin{aligned} a_iw_i &= w_{i1}s_{i1}, & s_{i1}y_i &= t_{i1}y_{i2}, \\ w_{i1}t_{i1} &= w_{i2}s_{i2}, & s_{i2}y_{i2} &= t_{i2}y_{i3}, \\ &\vdots & &\vdots \\ w_{i\,n_i-1}t_{i\,n_i-1} &= w_{in_i}s_{in_i}, & s_{in_i}y_{in_i} &= t_{in_i}y_{i+1}, \\ w_{in_i}t_{in_i} &= b_iw_{i+1} \end{aligned} \tag{11}$$

where $w_i(w_1 = 1,\ w_{n+1} = 1)$, $w_{i1}, \dots, w_{in_i} \in X$, $y_i(y_1 = y,\ y_{n+1} = y')$, $y_{i2}, \dots, y_{in} \in Y$, $s_{i1}, \dots, s_{in_i}$, and $t_{i1}, \dots, t_{in_i} \in S^1$.

Set $s'_{ij} = s_{ij}a_1s_{ij}$ and $t'_{ij} = t_{ij}a_1t_{ij}$.

Subcase 3.1. There exist some of all the s'_{ij}, t'_{ij}, which are under a_1 with respect to $\ge_{\mathscr{J}}$. Then we shall show that there exist $u, v \in S$ such that $a_1 > u$, $a_1 > v$ and $x \otimes y = xu \otimes y$, $x' \otimes y = x'v \otimes y'$. Suppose first that all s'_{pq}, t'_{pq} $(2 \le i,\ 1 \le p \le i-1,\ 1 \le q \le n_p)$, s'_{iq}, t_{iq} $(1 \le q \le r-1)$ belong to $\mathscr{R}_{a_1}$, but $a_1 >_{\mathscr{J}} s'_{ir}$.

Set $u = a_1s'_{ir}a_1$. Since $eu = u$ for all $e \in S$ with $e \ge_{\mathscr{J}} a_1$, it follows from (11) that

$$a_pw_pu = b_pw_{p+1}u \quad (1 \le p \le i-1), \qquad a_iw_iu = w_{ir-1}u.$$

By applying (6) to the equations just above, we obtain $u = w_{ir-1}u$, so that $u = w_{ir-1}s_{ir-1}u = w_{ir}s'_{r1}u$. By Result 3(iii), u is not the greatest lower bound of $\mathscr{R}_{a_1}$, since by Result 3(ii) and (9) $|\mathscr{R}_{a_1}| = 2$. So there exists $u' \in S$ such that u' is a lower bound of $\mathscr{R}_{a_1}$ but $u' \ne uu'$. In the same way as above, $u' = u'a_1 = w_{ir}s'_{ir}u'$. Hence $u' = uu'$, which is a contradiction. Thus it must hold that all s'_{pq}, t'_{pq} $(2 \le i,\ 1 \le p \le i-1,\ 1 \le q \le n_p)$, s'_{iq}, t'_{iq} $(1 \le q \le r-1)$, and s'_{ir} belong to $\mathscr{R}_{a_1}$, but $a_1 >_{\mathscr{J}} t'_{ir}$.

Then, by Result 3(i), $a_kt'_{ij}a_k = b_kt'_{ij}b_k$ $(1 \le k \le n)$, say a^*. From equations (11) on the right, we have $a^*y = a^*y_{k+1}$ $(1 \le k \le i)$, which, together with (10), yields $x \otimes y = x \otimes a^*y$. By the same way as above, we can find $b^* \in S$ satisfying that $b_n > b^*$ and $x' \otimes y' = x' \otimes b^*y'$, as required.

Moreover, by multiplying the right side of (8) by a^*, b^*, respectively that $a^*y = a^*y'$ and $b^*y = b^*y'$. Hence, $x \otimes y = x' \otimes a^*y'$ and $x' \otimes y' = x \otimes b^*y$. By Lemma 3(ii), we conclude that $x \otimes y = x' \otimes y'$.

Subcase 3.2. All the s'_{ij}, t'_{ij} belong to $\mathscr{R}_{a_1}$. By applying Lemma 1 to (11), we obtain schemes joining (x, y_i) to (x, y_{i+1}) as follows:

$$\begin{aligned} x &= xs'_{i1}, & s'_{i1}y_i &= t'_{i1}y_{i2}, \\ xt'_{i1} &= xs'_{i2}, & s'_{i2}y_{i2} &= t'_{i2}y_{i3}, \\ &\vdots & &\vdots \\ xt'_{i\,n_i-1} &= xs'_{in_i}, & s'_{in_i}y_{in_i} &= t'_{in_i}y_{i+1}, \\ xt'_{in_i} &= x. \end{aligned} \tag{12}$$

From (10) and (12), it follows that $x \otimes y = x' \otimes y'$. This completes the proof of the main theorem.

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