

## BOUNDARY BEHAVIOR OF HOLOMORPHIC FUNCTIONS OF $A_{q,s}^p$

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**ABSTRACT.** In this paper we prove that the Sobolov spaces  $A_{q,s}^p(D)$  on bounded strongly pseudoconvex domains  $D$  are continuously contained in  $\text{BMOA}(\partial D)$  for  $0 < p < \infty$ ,  $q \geq 0$ , and  $s = (n + q)/p$ .

### 1. INTRODUCTION

Let  $D$  be a bounded strongly pseudoconvex domain in  $C^n$  with smooth boundary  $\partial D$ . Let  $\delta(z)$  be the distance from  $z$  to  $\partial D$ , and let  $dV_q = C_q \delta(z)^{q-1} dV$  for each  $q > 0$ , where  $dV$  is the  $C^n = R^{2n}$  volume element and  $C_q$  is chosen so that  $dV_q$  is a probability measure on  $D$ . As  $q \rightarrow 0^+$ , these measures (as measures on  $\bar{D}$ ) converge to the normalized surface measure on  $\partial D$ , which is denoted by  $dV_0$ . We use  $L_q^p$  for  $L^p(dV_q)$  and  $\|\cdot\|_{p,q}$  for the  $L_q^p$  norm. The space of all holomorphic functions on  $D$  satisfying

$$\|f\|_{p,q,s} = \left( \sum_{|\alpha| \leq s} \left\| \frac{\partial^\alpha f}{\partial z^\alpha} \right\|_{p,q}^p \right)^{1/p} < +\infty$$

is denoted by  $A_{q,s}^p$  for  $0 < p < \infty$ ,  $q \geq 0$ , and  $s$  a nonnegative integer. For noninteger values of  $s > 0$ ,  $A_{q,s}^p$  and  $\|\cdot\|_{p,q,s}$  can be defined by interpolation; see [1] for the details. Let  $\text{BMOA}$  be the space of all holomorphic functions in  $D$  whose boundary values are in  $\text{BMO}(\partial D)$  (see [9, pp. 235, 253]). The  $\text{BMO}$  norm is denoted by  $\|\cdot\|_{\text{BMO}}$  (strictly speaking,  $\|\cdot\|_{\text{BMO}}$  is a norm on functions modulo constants). Let  $\mathcal{B}$  be the space of all Bloch functions defined in [6] and  $\mathcal{B}_0$  be the little Bloch space of all holomorphic functions satisfying

$$(1) \quad \sup_{\substack{\xi \in T_z(D) \\ |\xi|=1}} |f_*(z) \cdot \xi| / F_K^D(z, \xi) \rightarrow 0 \quad \text{as } \delta(z) \rightarrow 0,$$

where  $f_*$  and  $F_K^D$  are as defined in [6]. For  $\alpha > 0$  let  $\Lambda_\alpha$  be the Lipschitz space of order  $\alpha$  [4].

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In [3] Graham proved that  $Rf(z) \in A_{0,0}^p(B)$  implies  $f \in A_{0,0}^{np/(n-p)}(B)$  if  $0 < p < n$  and  $f \in \Lambda_{1-n/p}$  if  $p > n$ , where  $B = \{z \in \mathbb{C}^n : |z| < 1\}$  and  $Rf(z) = \sum_{j=1}^n z_j \partial f / \partial z_j$ . Employing analysis on the Heisenberg group, Krantz also obtained these results, and, furthermore, he proved that  $Rf(z) \in A_{0,0}^p(B)$  implies  $f \in \text{BMOA}$ ; see [5]. Later on, Beatrous and Burbea proved that, for  $0 < p < \infty$  and  $q \geq 0$ ,  $A_{q,s}^p(B) \subset A_{q,0}^{p(n+q)/((n+q)-ps)}(B)$  if  $0 \leq s < (n+q)/p$ ,  $A_{q,s}^p \subset \text{BOMA} \cap \mathcal{B}_0$  if  $s = (n+q)/p$ , and  $A_{q,s}^p \subset \Lambda_{s-(n+q)/p}$  if  $s > (n+q)/p$  (see [2, Theorem 2.7]). In [1] Beatrous proved a theorem which implies that on strongly pseudoconvex domains the inclusion  $A_{q,s}^p \subset A_{q,0}^{p(n+q)/((n+q)-ps)}$  is continuous if  $s < (n+q)/p$ . In this paper we prove

**Theorem 1.** *Let  $D$  be a bounded strongly pseudoconvex domain with smooth boundary, and let  $0 < p < \infty$ ,  $q \geq 0$ . If  $s = (n+q)/p$  then  $A_{q,s}^p \subset \text{BMOA} \cap \mathcal{B}_0$  and the inclusion is continuous with respect to the BMO norm in  $\text{BMO} \cap \mathcal{B}_0$ .*

## 2. PROOF OF THE THEOREM

If  $f \in A_{q,s}^p$ , we take three numbers  $p_2 > \max(n, p)$ ,  $q_2 \geq 0$ , and  $s_2 \geq 0$  such that

$$(2) \quad (n+q_2)/p_2 - (n+q)/p = s_2 - s.$$

Using Theorem 1.5(iii) of [1] for  $M = D$  we have  $f \in A_{q_2,s_2}^{p_2}$  and  $\|f\|_{p_2,q_2,s_2} \leq C\|f\|_{p,q,s}$ . Observe that  $s_2 = (n+q_2)/p_2$  since  $s = (n+q)/p$ . Applying Theorem 1.2 in [1] we obtain  $A_{q_2,s_2}^{p_2} = A_{p_2-n,1}^{p_2}$ , and the norms are equivalent. Therefore, it is sufficient to prove that  $f \in A_{p-n,1}^p$  ( $p > n$ ) implies  $f \in \text{BMOA} \cap \mathcal{B}_0$  and  $\|f\|_{\text{BMO}} \leq C\|f\|_{p,p-n,1}$ .

For  $z \in D$  near  $\partial D$ , let  $\Delta(z)$  be a polydisc centered at  $z$  with radius  $c_1\delta(z)$  in the complex normal direction and radius  $c_1\delta(z)^{1/2}$  in  $n-1$  orthogonal complex tangential directions ( $c_1$  is fixed and small enough). Since  $D$  has smooth boundary, there exists a constant  $c_2$  such that

$$\delta(z)/c_2 \leq \delta(\xi) \leq c_2\delta(z) \quad \text{for } \xi \in \Delta(z).$$

From the plurisubharmonicity of  $|\nabla f|^p$  we have

$$\begin{aligned} |\nabla f|^p(z)\delta(z)^p &\leq C\delta(z)^{p-(n+1)} \int_{\Delta(z)} |\nabla f|^p(\xi) dV(\xi) \\ &\leq C \int_{\Delta(z)} |\nabla f|^p(\xi)\delta(\xi)^{p-(n+1)} dV(\xi) \\ &\leq C \int_{\delta(\xi) \leq c_2\delta(z)} |\nabla f|^p(\xi) dV_{p-n}(\xi) \rightarrow 0 \quad \text{as } \delta(z) \rightarrow 0. \end{aligned}$$

This means that for any  $\varepsilon > 0$  there exists  $\delta_0 > 0$  such that

$$(3) \quad |\nabla f|(z)\delta(z) < \varepsilon \quad \text{for } \delta(z) < \delta_0.$$

By the argument of Lemma 4.8 in [8], we can prove that, for any complex

tangential direction  $\mu$  at  $z$ , the complex tangential derivative  $\nabla_\mu f$  satisfies

$$(4) \quad |\nabla_\mu f|(z)\delta(z)^{1/2} \leq C\varepsilon \quad \text{for } \delta(z) \leq \delta_0/4.$$

By a proof similar to that on p. 152 of [6] we know that (3) and (4) imply (1). Hence,  $f \in \mathcal{B}_0$ .

Now we are going to prove  $f \in \text{BMOA}$ . For  $f \in A_{p-n,1}^p$ , using Corollary 2.3 in [1] we have

$$f(z) = \int_D [f(\zeta)K_0(z, \zeta) + \mathcal{D}f(\zeta)K_1(z, \zeta)] dV_{p-n+1}(\zeta),$$

where  $\mathcal{D}f = \langle df, \partial\rho \rangle$  (see [1, p. 93]) and  $K_0$  and  $K_1$  are kernels of type  $p$ . Let  $D_\varepsilon = \{z \in D : \rho(z) \leq -\varepsilon\}$  and  $d\sigma_\varepsilon$  be the Hausdorff measure of  $(2n-1)$  dimensions on  $\partial D_\varepsilon$ , where  $\rho(z)$  is the defining function of  $D$  and  $\varepsilon > 0$ . From Theorem 2.4 in [1] we obtain

$$\begin{aligned} & \int_{\partial D_\varepsilon} |f(z)| d\sigma_\varepsilon(z) \\ & \leq \int_D (|f(\zeta)| + |\mathcal{D}f(\zeta)|) dV_{p-n+1}(\zeta) \int_{\partial D_\varepsilon} (|K_0(z, \zeta)| + |K_1(z, \zeta)|) d\sigma_\varepsilon(z) \\ & \leq C \int_D (|f(\zeta)| + |\mathcal{D}f(\zeta)|) \delta(\zeta)^{n-p} dV_{p-n+1}(\zeta) \\ & = C \int_D (|f(\zeta)| + |\mathcal{D}f(\zeta)|) dV(\zeta) \\ & \leq C \left\{ \int_D (|f|^p(\zeta) + |\mathcal{D}f(\zeta)|^p) \delta(\zeta)^{p-n-1} dV(\zeta) \right\}^{1/p} \\ & \quad \times \left\{ \int_D \delta(\zeta)^{p'(n/p-1/p')} dV(\zeta) \right\}^{1/p'} \\ & \leq C \|f\|_{p,p-n,1} \left\{ \int_D \delta(\zeta)^{p'n/p-1} dV(\zeta) \right\}^{1/p'} \leq C \|f\|_{p,p-n,1}, \end{aligned}$$

where  $1/p + 1/p' = 1$ . Hence,  $f$  is in the Hardy space  $A_{0,0}^1 = H^1$ . Then we know from [7] that

$$\int_{\partial D} |f(\zeta + \lambda\nu(\zeta)) - f(\zeta)| dV_0(\zeta) \rightarrow 0 \quad \text{as } \lambda \rightarrow 0^+,$$

where  $\nu(\zeta)$  is the inward unit normal of  $\partial D$  and  $f(\zeta)$  is the admissible limit of  $f$  at  $\zeta \in \partial D$ . Therefore,  $f(\zeta)$  must be of analytic type (for the definition see [9]). Additionally, we claim that  $(|f(z)| + |\nabla f(z)|) dV(z)$  is a Carleson measure on  $D$ . In fact, for any Carleson window  $\tilde{B}_t(\zeta_0)$  as in [9],

$$\tilde{B}_t(\zeta_0) = \{B_t(\zeta_0) + \lambda\nu(\zeta_0) : \lambda \in (0, t)\},$$

where  $B_t(\zeta_0)$  is the nonisotropic ball of  $\partial D$  at  $\zeta_0$  with radius  $t$ , we have

$$\begin{aligned}
 & \int_{\tilde{B}_t(\zeta_0)} (|f(z)| + |\nabla f(z)|) dV(z) \\
 & \leq \left\{ \int_{\tilde{B}_t(\zeta_0)} (|f(z)| + |\nabla f(z)|)^p \delta(z)^{p-(n+1)} dV(z) \right\}^{1/p} \\
 (5) \quad & \times \left\{ \int_{\tilde{B}_t(\zeta_0)} \delta(z)^{p((n+1)/p-1)/(p-1)} dV(z) \right\}^{(p-1)/p} \\
 & \leq C \|f\|_{p,p-n,1} \left\{ \int_{\tilde{B}_t(\zeta_0)} \delta(z)^{(n+1-p)/(p-1)} dV(z) \right\}^{(p-1)/p} \\
 & \leq C \|f\|_{p,p-n,1} \left\{ t^n \int_0^t t^{(n+1-p)/(p-1)} dt \right\}^{(p-1)/p} \\
 & = C \|f\|_{p,p-n,1} (t^n \cdot t^{n/(p-1)})^{(p-1)/p} = C \|f\|_{p,p-n,1} t^n.
 \end{aligned}$$

Therefore, we know from Theorem 2.1.3(ii) of [9] that  $f \in \text{BMO}$ . From the proof on p. 253 of [9] and (5), we obtain

$$(6) \quad \|f\|_{\text{BMO}} \leq C \|f\|_{p,p-n,1}.$$

This completes the proof.

*Remark 1.* As mentioned by Varopoulos in [9], Theorem 2.1.3(ii) of [9] is still valid for strongly pseudoconvex domains, but for simplicity's sake it was stated there only in the case  $D = \{z \in C^n : |z| < 1\}$ .

*Remark 2.* We can also prove

$$A_{q,s}^p \subset \Lambda_{s-(n+q)/p} \quad \text{for } s > (n+q)/p.$$

In order to prove this assertion, we take  $k > s$  a fixed integer. From [1] we have  $A_{q,s}^p = A_{q+p(k-s),k}^p$ . For  $f \in A_{q+p(k-s),k}^p$  and  $|\alpha| \leq k$

$$\begin{aligned}
 \left| \frac{\partial^\alpha f}{\partial z^\alpha}(z) \right|^p & \leq C \delta(z)^{-(n+1)-(q+p(k-s)-1)} \int_{\Delta(z)} \left| \frac{\partial^\alpha f}{\partial z^\alpha}(\zeta) \right|^p \delta(\zeta)^{q+p(k-s)-1} dV(\zeta) \\
 & \leq C \|f\|_{p,q+p(k-s),k}^p \cdot \delta(z)^{p((s-(n+q)/p)-k)}.
 \end{aligned}$$

Hence,

$$(7) \quad \max_{|\alpha| \leq k} \left| \frac{\partial^\alpha f}{\partial z^\alpha}(z) \right| \leq C \|f\|_{p,q+p(k-s),k} \cdot \delta(z)^{(s-(n+q)/p)-k}.$$

Applying Theorem 8.8.6 of [4], we have  $f \in \Lambda_{s-(n+q)/p}$ .

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