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TOWERS ARE UNIVERSALLY MEASURE ZERO AND ALWAYS OF FIRST CATEGORY

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ABSTRACT. We improve a few known results about universally measure zero and always of first category sets. Our main tool is the fact that any tower with respect to a Borel relation is such a set.

INTRODUCTION

Let ω denote the set of all natural numbers, and let ω_1 denote the first uncountable cardinal. $P(X)$ denotes the set of all subsets of X and $X \times Y$ denotes the Cartesian product of X and Y. Ordinal numbers are denoted by $k, m, n, \alpha, \beta, \lambda$, where the first three symbols are reserved for natural numbers. An inequality $\alpha < \beta$ means that the ordinal $\alpha + 1$ is a subset of β. We use the definition of ordinal numbers such that $\alpha = \{\beta : \beta < \alpha\}$; in particular, $n = \{0, 1, 2, \dots, n-1\}$.

The *natural topology* is the topology on the set $P(\omega)$ of all subsets of natural numbers for which the sets

$$U_X^n = \{Y \subseteq \omega : Y \cap n = X \cap n\},$$

where $X \subseteq \omega$ and $n \in \omega$, constitute a base of open sets. Let us note that sets U_X^n are closed.

A family of subsets of natural numbers is *a universally measure zero set* if for every countably additive nonzero and finite measure which is defined on its Borel subsets, with respect to the topology inherited from the natural topology, there is a one point set with positive measure. In the literature one can find other equivalent definitions of universally measure zero sets; see Sierpiński and Szpilrajn [11], Laver [7], or Miller [9].

There are a few known constructions of uncountable universally measure zero sets; see Miller [9]. The first ones are due to Hausdorff [5] and Sierpiński and Szpilrajn [11]. Those constructions give universally measure zero sets of cardinality ω_1. Other constructions, which produce universally measure zero sets with cardinality greater than ω_1 in suitable models for the ZFC axioms of set theory, were given by Grzegorek [4] and Cichoń [1].

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We describe a construction of uncountable universally measure zero sets, namely, uncountable towers with respect to a Borel relation. This gives results which seem to be new. We obtain a new proof of the fact that an (ω_1, ω_1^*) gap forms a universally measure zero set; compare Hausdorff [5] and Laver [7]. We obtain universally measure zero sets which can have cardinality greater than those from Grzegorek's paper [4]. We reprove Cichoń's results about so-called Banach numbers; compare Cichoń [1].

A family of subsets of the natural numbers is an always of first category set if every dense-in-itself subfamily of it is a first category set in itself; compare Kuratowski [6, p. 516] or Miller [9].

In Lusin [8] it was observed that a certain construction of uncountable sets produces always of first category sets; compare Kuratowski [6, p. 519]. In this note we observe that any tower with respect to a Borel relation is simultaneously a universally measure zero set and an always of first category set.

The main lemma

Assume that for every natural number n there is given a relation $<_n$ on subsets of n. For subsets X and Y of the natural numbers we write $Y < X$ if for almost all n there holds $Y \cap n <_n X \cap n$.

We define a relation $<^n$ on subsets of natural numbers as

$$<^n = \{(Y, X): Y \cap n <_n X \cap n\}.$$

A standard quantifier-counting argument, similar to §1C of [10], gives that any relation $<^n$ is a closed-open set in the product $P(\omega) \times P(\omega)$. Thus, $<$ is an F_σ set in $P(\omega) \times P(\omega)$.

A family T of subsets of the natural numbers is a *tower* (compare Dordal [3]) if it is well ordered by the relation $<$, i.e., $T = \{T_\alpha : \alpha < \lambda\}$ and if $\alpha < \beta$, then $T_\alpha < T_\beta$ and there does not hold $T_\beta < T_\alpha$.

The following lemma needs only the assumption that a relation $<$ is a Borel subset of $P(\omega) \times P(\omega)$. The proof of it is a modification of essentially already known results; see [10, §5A.10]. Our main improvement is that we use a suitable definition of a universally measurable zero set (this is from Sierpiński and Szpilrajn [11]).

Lemma. *Each tower is a universally measure zero and an always of first category set.*

Proof. Let $T = \{T_\alpha : \alpha < \lambda\}$ be a tower. Suppose, to the contrary, that there exists a countably additive, nonzero, and finite measure defined on Borel subsets of T such that one-point sets have measure zero. We can assume that λ is the least ordinal for which the assumptions above are fulfilled; if necessary, one can take a suitable ordinal less than λ.

Let us consider the Borel subset $E = < \bigcap (T \times T)$ of the product $T \times T$. It is measurable with respect to product measure. Each horizontal section $E \cap (T \times \{T_\beta\})$ is contained in

$$\{T_\alpha : \alpha < \beta\} \times \{T_\beta\} \cup \{(T_\beta, T_\beta)\}.$$

For $\beta < \lambda$ the sets $\{T_\alpha : \alpha < \beta\}$ have measure zero. Therefore, the set E has measure zero because of Fubini's theorem.

On the other hand each vertical section $E \cap (\{T_\beta\} \times T)$ contains

$$\{T_\beta\} \times \{T_\alpha : \beta < \alpha\}.$$

For $\beta < \lambda$, sets $\{T_\alpha : \beta < \alpha\}$ have full measure and, in particular, have positive measure. Therefore, the set E has positive measure because of Fubini's theorem. We have a contradiction, which implies that T has to be a universally measure zero set.

The proof that each tower is an always of first category set is similar. One has to use the Kuratowski-Ulam theorem (see Kuratowski [6, p. 249]) instead of Fubini's theorem. □

Applications

Let us consider the following relation on subsets of the natural numbers: $X <^* Y$ if the set X is almost contained in the set Y. It can be defined in the form of our schema from the previous part. Namely, one can write $X <^{n+1} Y$ if $n \in X$ implies $n \in Y$. Hausdorff [5] gave a construction of uncountable universally measure zero sets using this relation. First he constructed an (ω_1, ω_1^*) gap, i.e., two families $\{A_\alpha : \alpha < \omega_1\}$ and $\{\beta_\alpha : \alpha < \omega_1\}$ of subsets of natural numbers linearly ordered in the manner

$$A_0 <^* A_1 <^* \cdots <^* A_\alpha <^* \cdots <^* B_\alpha <^* \cdots <^* B_1 <^* B_0$$

and such that there does not exist a set C for which there hold all the inequalities $A_\alpha <^* C <^* B_\alpha$. Then he showed that any (ω_1, ω_1^*) gap forms a universally measure zero set. His proof essentially uses properties of (ω_1, ω_1^*) gaps; compare Laver [7]. The last result can be deduced from our theorem. Any (ω_1, ω_1^*) gap is the union of countably many towers with respect to relation $<^*$ and its reverse (one has to use the observation that for each X there are only countably many Y which satisfy $X <^* Y <^* X$).

In Grzegorek [4] there was given a construction of universally measure zero sets of cardinality equal to the least cadinality of nonmeasurable sets; we denote it $\mathrm{non}(L)$. One can produce a relation, via our schema from the previous part, for which there exist towers of cardinality b, and in some model for ZFC there can exist towers of cardinality greater than b; see Dordal [3]. The cardinal b is taken as in Fremlin [2], where it was noted that one cannot prove $b \leq \mathrm{non}(L)$ using ZFC axioms only. So our theorem gives a stronger result than Grzegorek's in some models for ZFC.

A cardinal number λ is called a Banach number (see Chicoń [1]) if there are a universally measure zero set of cardinality λ and a nonmeasurable subset of reals of cardinality λ. In any model for ZFC with $b > \mathrm{non}(L)$ there are at least two Banach numbers, namely, b and $\mathrm{non}(L)$. Thus we have reproved the result of Cichoń [1]: it is consistent with ZFC that there exist many Banach numbers.

One can define a relation as needed above as follows: $X <_* Y$ if X is finite or if for almost all n there holds $x_n < y_n$, where $x_0 < x_1 < \cdots$ is the increasing sequence of all members of X and $y_0 < y_1 < \cdots$ is the increasing sequence of all members of Y. Thus b is the least cardinality of unbounded towers with respect to the relation $<_*$ restricted to the set of all infinite subsets of natural numbers. The relation $<_*$ can be obtained in the form of the schema from the previous part, when one puts $X <^{2n} Y$ if for every x_k greater than n there holds $x_k < y_k$ and for odd n we let $X <^n Y$ hold for each X and Y.

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