PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 119, Number 3, November 1993

TAYLOR EXACTNESS AND KATO'S JUMP

ROBIN HARTE

(Communicated by Paul. S. Muhly)

ABSTRACT. The middle exactness condition of Joseph Taylor is related to the zero-jump condition of Tosio Kato, and some "commutative" Fredholm theory explored.

If $T: X \to Y$ and $S: Y \to Z$ are linear operators between complex spaces, we shall call the pair (S, T) *exact* iff

(0.1) $$S^{-1}(0) \subseteq T(X),$$

whether or not the chain condition

(0.2) $$ST = 0$$

is satisfied. For example if $T = 0$, this means that S is *one-to-one*; if $S = 0$, this means that T is *onto*. When S and T are bounded operators between normed spaces, we shall call the pair (S, T) *weakly exact* if

(0.3) $$S^{-1}(0) \subseteq \operatorname{cl} T(X)$$

and *split exact* if there are bounded $T': Y \to X$ and $S': Z \to Y$ for which

(0.4) $$S'S + TT' = I.$$

It is clear at once that

(0.5) $$(S, T) \text{ split exact} \Rightarrow (S, T) \text{ exact} \Rightarrow (S, T) \text{ weakly exact};$$

conversely, if S and T are both *regular* in the sense that there are bounded $T^\wedge: Y \to X$ and $S^\wedge: Z \to Y$ for which

(0.6) $$T = TT^\wedge T \quad \text{and} \quad S = SS^\wedge S$$

then there is the implication

(0.7) $$(S, T) \text{ weakly exact} \Rightarrow (S, T) \text{ split exact}.$$

Indeed if (0.3) and (0.6) both hold then [8, Theorem 10.3.3]

(0.8) $$(I - TT^\wedge)(I - S^\wedge S) = 0,$$

given two candidates for T' and S' to satisfy (0.4).

Received by the editors March 23, 1990 and, in revised form, March 5, 1992.
1991 *Mathematics Subject Classification*. Primary 47A53, 15A09.

Lemma 1. *If* $U\colon W \to X$, $T\colon X \to Y$, *and* $V\colon Y \to Z$ *are linear, there is the implication*

$$(V, TU) \text{ exact}, (T, U) \text{ exact} \Rightarrow (VT, U) \text{ exact} \tag{1.1}$$

and

$$(VT, U) \text{ exact}, (V, T) \text{ exact} \Rightarrow (V, TU) \text{ exact}. \tag{1.2}$$

If U, T, *and* V *are bounded, there is the implication*

$$(V, TU), (T, U) \text{ split exact} \Rightarrow (VT, U) \text{ split exact} \tag{1.3}$$

and

$$(VT, U), (V, T) \text{ split exact} \Rightarrow (V, TU) \text{ split exact}. \tag{1.4}$$

Proof. These are beefed up versions of parts of Theorem 10.9.2 and Theorem 10.9.4 of [8]; for example, if $V^{-1}(0) \subseteq TU(W)$ and $T^{-1}(0) \subseteq U(W)$ then

$$VTx = 0 \Rightarrow Tx \in V^{-1}(0) \subseteq TU(W) \Rightarrow x - Uw \in T^{-1}(0) \subseteq U(W). \quad \square$$

Lemma 1 does not extend to weak exactness; to violate the weak analogue of (1.2) take [6, Example 1] $U = 0$, T one-to-one dense but not onto, and $V^{-1}(0) = \mathbb{C}e$ with $e \in Y \setminus T(X)$.

Lemma 2. *If* $U\colon W \to X$ *and* $V\colon Y \to Z$ *are bounded and linear and* $T = TT^{\wedge}T\colon X \to Y$ *is regular, then*

$$V^{-1}(0) \subseteq T(X) \Rightarrow T^{\wedge}V^{-1}(0) \subseteq (VT)^{-1}(0) \tag{2.1}$$

and

$$T^{-1}(0) \subseteq U(W) \Rightarrow T^{\wedge}TU(W) \subseteq U(W). \tag{2.2}$$

Also

$$V'V + TT' = I \Rightarrow VTT^{\wedge} = V''V \tag{2.3}$$

and

$$T'T + UU' = I \Rightarrow T^{\wedge}TU = UU''. \tag{2.4}$$

Proof. The first part of this is essentially given by Mbekhta [10, Proposition 2.4]. To see (2.1) argue

$$Vy = 0 \Rightarrow VTT^{\wedge}y = VTT^{\wedge}Tx = VTx = Vy = 0.$$

For (2.3) take $V'' = VTT^{\wedge}V' + I - VV'$. $\square$

It is familar that the product of regular operators need not be regular [8, (7.3.6.17); 2, §2.8] and that regularity of the product need not imply regularity of the factors [8, (7.3.6.16); 2, §2.8].

Theorem 3. *If* $T\colon X \to Y$ *and* $S\colon Y \to Z$ *are bounded and linear and* (S, T) *is split exact, then*

$$ST \text{ regular} \Leftrightarrow S, T \text{ regular}. \tag{3.1}$$

Proof. If $ST = STUST$ and $S'S + TT' = I$ then

$$(I - TT')T(I - UST) = 0 = (I - STU)S(I - S'S).$$

Conversely, if $S = SS^\frown S$, $T = TT^\frown T$, and $S^{-1}(0) \subseteq \mathrm{cl}\, T(X)$, then (0.8) gives

$$STT^\frown S^\frown ST = S(TT^\frown + S^\frown S - I)T = ST. \quad \square$$

When $T\colon X \to X$ and $S\colon X \to X$ are complex linear operators on the same space X, we shall call the pair (S, T) *left nonsingular* if

$$S^{-1}(0) \cap T^{-1}(0) = \{0\}, \tag{3.2}$$

right nonsingular if

$$S(X) + T(X) = X, \tag{3.3}$$

and *middle nonsingular* if, in matrix notation,

$$(-S \quad T)^{-1}(0) \subseteq \begin{pmatrix} T \\ S \end{pmatrix}(X). \tag{3.4}$$

The last condition means of course that whenever $Sy = Tx$ there is z for which $y = Tz$ and $x = Sz$, and is a special case of (0.1). Each of these conditions is symmetric in S and T and is not restricted to pairs (S, T), which are *commutative* in the sense that

$$ST = TS. \tag{3.5}$$

Gonzalez [5, Proposition] has essentially shown

Theorem 4. *Necessary and sufficient for middle nonsingularity of* (S, T) *are the following three conditions*:

$$S^{-1}(0) \subseteq TS^{-1}(0), \tag{4.1}$$

$$T^{-1}(0) \subseteq ST^{-1}(0), \tag{4.2}$$

$$S(X) \cap T(X) \subseteq (ST)(TS - ST)^{-1}(0). \tag{4.3}$$

If (4.1) *and* (4.2) *hold then also*

$$(ST)^{-1}(0) + (TS)^{-1}(0) \subseteq S^{-1}(0) + T^{-1}(0). \tag{4.4}$$

Proof. Suppose first that middle nonsingularity (3.4) holds. Then

$$Sy = 0 \Rightarrow (-S \quad T)\begin{pmatrix} y \\ 0 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} T \\ S \end{pmatrix} x,$$

giving $y = Tx$ with $x \in S^{-1}(0)$; this proves (4.1) and similarly (4.2). Also

$$w = Tx = Sy \Rightarrow \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} T \\ S \end{pmatrix} z \Rightarrow w = STz = TSz,$$

giving (4.3). Conversely, if these conditions hold then, using first (4.3),

$$\begin{pmatrix} y \\ x \end{pmatrix} \in (-S \quad T)^{-1}(0) \Rightarrow Sy = Tx = STz = TSz,$$

giving $y - Tz \in S^{-1}(0) \subseteq TS^{-1}(0)$ and $x - Sz \in T^{-1}(0) \subseteq ST^{-1}(0)$, so that there are u and v for which

$$y - Tz = Tu \quad \text{with } Su = 0 \qquad \text{and} \qquad x - Sz = Sv \quad \text{with } Tv = 0:$$

but now $\binom{T}{S}(z + u + v) = \binom{y}{x}$, as required by (3.4). Toward the last part we assume only (4.1) and claim

$$(ST)^{-1}(0) \subseteq S^{-1}(0) + T^{-1}(0), \tag{4.5}$$

for if $(ST)x = 0$ then $Tx \in TS^{-1}(0)$, giving $Tx = Tz$ with $Sz = 0$, and hence

$$x = (x - z) + z \in T^{-1}(0) + S^{-1}(0) . \quad \square$$

The conditions (4.3) and (4.4) are not together sufficient for either (4.1) or (4.2), even in the presence of commutivity. If for example

$$S = T = P = P^2 \neq I \tag{4.6}$$

is a nontrivial idempotent then both (4.3) and (4.4), and of course also (3.5), hold, while neither (4.1) nor (4.2) are satisfied. The conditions (4.1) and (4.2) are not together sufficient for (4.3): for example, take $S = T$ to be one-to-one with $T(X) \neq T^2(X)$. Specifically if $X = l_2$, we can take $S = T = U$, the forward shift with $(Ux)_{n+1} = x_n$ and $(Ux)_1 = 0$. Curto [4, pp. 71–72] has shown essentially that, in the presence of commutivity (3.5), middle nonsingularity (3.4) is equivalent to (4.1) together with

$$T^{-1}S(X) \subseteq S(X) , \tag{4.7}$$

and therefore also (4.2) together with

$$S^{-1}T(X) \subseteq T(X) . \tag{4.8}$$

"Duality" considerations then suggest that (4.7), (4.8), and (4.4) might together be equivalent to (3.4). This, however, fails without commutivity. If, for example, $X = l_2$, we can take $T = V$, the backward shift with $(Vx)_n = x_{n+1}$ and $S = W$ with $(Wx)_n = (1/n)x_n$ to satisfy both (4.7) and (4.8), and also (4.4), but not (3.4). Sufficient for the nonsingularity conditions (3.2)–(3.4) are the corresponding invertibility conditions: we call the pair (S , T) *left invertible* if there is another pair (S' , T') for which

$$S'S + T'T = I , \tag{4.9}$$

right invertible if there is another pair (S'' , T'') for which

$$SS'' + TT'' = I , \tag{4.10}$$

and *middle invertible* if there are pairs (S' , T') and (S'' , T'') for which, in matrix notation,

$$\begin{pmatrix} -S'' \\ T'' \end{pmatrix} (-S \quad T) + \begin{pmatrix} T \\ S \end{pmatrix} (T' \quad S') = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} . \tag{4.11}$$

In the context of pure linear algebra it is clear that "invertibility" and "nonsingularity" are equivalent, by the argument for (0.7); for bounded linear operators between normed spaces we require that the "left", "right", and "middle" inverses be made out of bounded operators. When the operators S and T commute and the space X is a Hilbert space then nonsingularity implies invertibility; for Banach spaces this question appears to be open still [7, pp. 73–74]. In general it is sufficient for left, right, and middle invertibility that (4.9) holds for a pair (S' , T') such that

(4.12) $\quad (S' , S), \ (S' , T), \ (T' , T), \ (T , S)$ are commutative.

The reader may suspect that there is an analogue for Theorem 4 with "invertibility" in place of "nonsingularity": the author has been unable to find it. The invertible analogues of the conditions (4.1) and (4.2), and of (4.7) and (4.8), are not hard to find—each consists of either a column or a row from (4.11): the reader is invited to think up invertible analogues for (4.3) and (4.4). Theorem 4 should also have an analogue for "weak exactness": thus, (3.2) is equivalent to the implication

$$SU = TU = 0 \Rightarrow U = 0, \tag{4.13}$$

the weakly exact analogue of (3.3) is

$$VS = VT = 0 \Rightarrow V = 0, \tag{4.14}$$

and the weakly exact analogue of (3.4) is

$$(-S \quad T)\begin{pmatrix} -U' \\ U \end{pmatrix} = (V \quad V')\begin{pmatrix} T \\ S \end{pmatrix} = 0 \Rightarrow (V \quad V')\begin{pmatrix} -U' \\ U \end{pmatrix} = 0. \tag{4.15}$$

It is not hard, starting from the "invertible" versions of (4.1) and (4.2), and of (4.7) and (4.8), to write down corresponding weak versions of these four conditions.

Definition 5. If $T: X \to X$ is linear then its hyperrange and hyperkernel are the subspaces

$$T^\infty(X) = \bigcap_{n=1}^\infty T^n(X) \tag{5.1}$$

and

$$T^{-\infty}(0) = \bigcup_{n=1}^\infty T^{-n}(0). \tag{5.2}$$

When T is continuous on a normed space X neither of these need be closed. If we write

$$\mathrm{comm}(T) = \{S \in BL(X, X) : ST = TS\} \tag{5.3}$$

for the *commutant* of T and

$$\mathrm{comm}^{-1}(T) = \mathrm{comm}(T) \cap BL^{-1}(X, X) \tag{5.4}$$

for the *invertible commutant* of T, then we can collect the following

Lemma 6. *If $T \in BL(X, X)$ is arbitrary then*

$$T^{-1}T^{-\infty}(0) \subseteq T^{-\infty}(0) \tag{6.1}$$

and

$$T \text{ essentially one-to-one } \Rightarrow T^\infty(X) \subseteq TT^\infty(X). \tag{6.2}$$

If $S \in \mathrm{comm}(T)$ then

$$ST^{-\infty}(0) \subseteq T^{-\infty}(0) \quad \text{and} \quad ST^\infty(X) \subseteq T^\infty(X). \tag{6.3}$$

If $S \in \mathrm{comm}^{-1}(T)$ then

$$(T - S)^{-1}(0) \subseteq T^\infty(X) \quad \text{and} \quad T^{-\infty}(0) \subseteq (T - S)(X). \tag{6.4}$$

Proof. This is [8, Theorem 7.8.3]. □

We shall call the operator $T: X \to X$ *self-exact* if the pair (T, T) satisfies (0.1):

$$T^{-1}(0) \subseteq T(X), \tag{6.5}$$

n-exact if (T, T^n) satisfies (0.1):

$$T^{-1}(0) \subseteq T^n(X), \tag{6.6}$$

and *hyperexact* if

$$T^{-1}(0) \subseteq T^\infty(X). \tag{6.7}$$

There are various equivalent forms of these conditions:

Theorem 7. *If $T: X \to X$ is linear and $n \in \mathbb{N}$ with $m + k = n + 1$ then*

$$T^{-1}(0) \subseteq T^n(X) \Leftrightarrow T^{-k}(0) \subseteq T^m(X) \Leftrightarrow T^{-n}(0) \subseteq T(X) \tag{7.1}$$

and

$$T^{-1}(0) \subseteq T^\infty(X) \Leftrightarrow T^{-\infty}(X) \subseteq T^\infty(X) \Leftrightarrow T^{-\infty}(0) \subseteq T(X). \tag{7.2}$$

If $T = TT^\wedge T$ is regular with $T^{-1}(0) \subseteq T^\infty(X)$ then

$$T^\wedge T^\infty(X) \subseteq T^\infty(X) \quad \text{and} \quad T^\wedge T^{-\infty}(0) \subseteq T^{-\infty}(0). \tag{7.3}$$

If $S \in \mathrm{comm}^{-1}(T)$ then

$$(T - S)^{-\infty}(0) \subseteq T^\infty(X) \quad \text{and} \quad T^{-\infty}(0) \subseteq (T - S)^\infty(X) \tag{7.4}$$

and

$$T^{-\infty}(0) \cap (T - S)^{-\infty}(0) = \{0\}, \tag{7.5}$$

and for each $m, n \in \mathbb{N}$

$$T^m(X) + (T - S)^n(X) = X. \tag{7.6}$$

Proof. The first half of this comes from Lemmas 1 and 2, taking U and V to be powers of T. For (7.4)–(7.6) factorize $(T^m - S^m)^n$ in two ways to see that $((T - S)^n, T^m)$ satisfies (4.9)–(4.11) for each m and n:

$$S^{mn} - r_{m,n}(T, S)T^m = (T - S)^n q_m(T, S)^n \tag{7.7}$$

for certain polynomials q_m and $r_{m,n}$. □

It is clear from (7.7) that (7.5) remains valid if $S \in \mathrm{comm}(T)$ is one-to-one, and (7.6) if S is onto. We cannot replace m and n by ∞ in (7.6): for a counterexample take $T = U$ to be the forward shift on $X = l_2$ and $S = I$. Our condition (6.7) is apparently weaker than, but actually the same as, the "perfection" of Saphar [14, Definition 2], in which the hyperrange is replaced by a possibly smaller transfinite version [14, Definition 1]. The reason both definitions in fact agree is because the condition (6.7) also implies the right-hand side of (6.2). Mbekhta has noticed this: if $y \in T^\infty(X)$ then $y = Tx_0 = T^{n+1}x_n$ for each $n \in \mathbb{N}$, giving

$$x_0 - T^n x_n \in T^{-1}(0) \subseteq T^\infty(X) \tag{7.8}$$

and, hence, $x_0 \in T^n(X)$ for each n. The same condition forces the restriction of T to the hyperkernel to be onto: remembering (6.1)

$$y \in T^{-\infty}(0) \subseteq T(X) \Rightarrow y = Tx \quad \text{with } x \in T^{-1}T^{-\infty}(0) \subseteq T^{-\infty}(0). \tag{7.9}$$

Definition 8. Call $T \in BL(X, X)$ *hyperregular* if it is regular and hyperexact. We shall say that T is *consortedly regular* if there are sequences (S_n) in $\mathrm{comm}^{-1}(T)$ and $(T_n^\wedge)$ in $BL(X, X)$ for which

$$(8.1)\qquad \|S_n\| + \|T_n^\wedge - T^\wedge\| \to 0 \quad \text{and} \quad T - S_n = (T - S_n)T_n^\wedge(T - S_n),$$

and *holomorphically regular* if there is $\delta > 0$ and a holomorphic mapping $T_z^\wedge : \{|z| < \delta\} \to BL(X, X)$ for which

$$(8.2)\qquad T - \lambda I = (T - \lambda I)T_\lambda^\wedge(T - \lambda I) \quad \text{for each } |\lambda| < \delta.$$

Mbekhta [10, Théorème 2.6] has essentially proved

Theorem 9. *If X is complete and $T \in BL(X, X)$ then*

(9.1) *T consortedly regular $\Rightarrow$ T hyperregular $\Rightarrow$ T holomorphically regular.*

Proof. If T is consortedly regular then, using (6.4), there is inclusion $T^{-k}(0) \subseteq (T - S_n)(X)$ for arbitrary k and n, where S_n satisfies (8.1); hence if $T^k x = 0$ then $x = (T - S_n)T_n^\wedge x$, giving

$$(I - TT^\wedge)x = ((T - S_n)T_n^\wedge - TT^\wedge)x \to 0 \quad \text{as } n \to \infty,$$

and hence $x = TT^\wedge x \in T(X)$. This gives, without completeness, the first implication of (9.1). Conversely, suppose $T = TT^\wedge T$ is hyperregular and $S \in \mathrm{comm}(T)$ with $\|S\|\,\|T^\wedge\| < 1$. Using (6.3) and (7.3) and expanding $(I - T^\wedge S)^{-1}$ in the geometric series gives

$$S(I - T^\wedge S)^{-1}T^{-1}(0) \subseteq \mathrm{cl}\, T^{-\infty}(0) \subseteq \mathrm{cl}\, T(X)$$

and hence

$$(I - TT^\wedge)S(I - T^\wedge S)^{-1}(I - T^\wedge T) = 0,$$

which by (3.8.4.3) from the proof of [11, Theorem 3.8.4] says

$$(9.2)\qquad T - S = (T - S)(I - T^\wedge S)^{-1}T^\wedge(T - S).$$

Specializing to scalar $S = \lambda I$ gives the second implication of (9.1). □

The derivation of (9.2) is based on Caradus [3]; cf. also [12, Theorem 3.9] of Nashed. If we observe

$$(9.3)\qquad T^\wedge(T - S) + (I - T^\wedge T) = I - T^\wedge S,$$

that $I - T^\wedge S$ sends the null space of $T - S$ into the null space of T, then we can see why if T is Fredholm and $I - T^\wedge S$ is one-to-one then $\dim(T - S)^{-1}(0) \le \dim T^{-1}(0)$ [8, Theorem 6.4.5]. Conversely if $T = TT^\wedge T$ is hyperregular and $S \in \mathrm{comm}^{-1}(T)$ has small enough norm,

$$(9.4)\qquad \begin{gathered}(T - S)^\wedge T + I - (T - S)^\wedge(T - S) = I + (T - S)^\wedge S \\ \text{with } (T - S)^\wedge = (I - T^\wedge S)^{-1}T^\wedge,\end{gathered}$$

furnishing an invertible operator which sends the null space of T into the null space of $T - S$. In the Fredholm case this is the Kato *zero jump* condition [1, 15, 13].

Theorem 9 says (cf. [14, Théorème 3] that the hyperregular operators form a "commutatively open" subset of $BL(X, X)$ and hence that a certain kind of "spectrum" is closed in $\mathbb{C}$. We may also observe that the topological boundary of the usual spectrum is contained in this "hyperregular spectrum":

(9.5) $$\{T \in \mathrm{cl}_{\mathrm{comm}} BL^{-1}(X, X): T \text{ hyperregular}\} \subseteq BL^{-1}(X, X).$$

We are claiming that if hyperregular T is the limit of a sequence $T - S_n$ of invertible operators that commute with T then T must also be invertible. It follows from (9.2) that if $I - T^\frown S$ and $T - S$ are both invertible then so is $T^\frown$; since the argument extends to $T^\frown T T^\frown$, this also makes T invertible.

The spectral mapping theorem for polynomials extends to the "hyperregular spectrum":

Theorem 10. *If $ST = TS$ then*

(10.1) $$ST \textit{ self-exact} \Rightarrow S, T \textit{ self-exact}$$

and

(10.2) $$ST \textit{ hyperregular} \Rightarrow S, T \textit{ hyperregular}.$$

If $ST = TS$ and (S, T) is middle exact then

(10.3) $$S, T \textit{ self-exact} \Rightarrow ST \textit{ self-exact}$$

and

(10.4) $$S, T \textit{ hyperregular} \Rightarrow ST \textit{ hyperregular}.$$

Proof. The first part is an extension of Mbekhta [11, Lemme 4.15]: if $(ST)^{-1}(0) \subseteq (ST)(X)$ then

$$T^{-1}(0) \subseteq (ST)^{-1}(0) \subseteq (ST)(X) = (TS)(X) \subseteq T(X),$$

and similarly for S and powers T^n and S^m. This gives (10.1) and most of (10.2); for the regularity of S and T observe that if $ST = STUST$ and $(ST)^{-1}(0) \subseteq (ST)X$ then $(I - STU)(I - UST) = 0$ giving (since $ST = TS$)

$$TSU - TSU^2TS + UTS = I;$$

now apply (3.1). Conversely, for (10.3), use (4.1)–(4.4):

$$(ST)^{-1}(0) \subseteq S^{-1}(0) + T^{-1}(0) \subseteq S(X) \cap T(X) \subseteq (ST)(X).$$

This gives (10.3) and most of (10.4); the regularity of ST is (3.1) again. □

The middle exactness condition in the second part is unnecessarily strong; it misses the rather easy

(10.5) $$T \text{ hyperregular} \Rightarrow T^n \text{ hyperregular}.$$

Three applications of Lemma 1 show that, if $ST = TS$,

(10.6) $$(S^2, T^2), (T^2, S^2) \text{ exact}$$

is sufficient for (10.3). By (1.1) (S^2, T^2) and (T, T) exact imply (S^2T, T) exact, and (S, S) exact imply (ST^2, S) and, hence, (ST, S) exact; then (1.2) says that (S^2T, T) and (ST, S) exact imply (ST, ST) exact.

REFERENCES

1. C. Apostol, *The reduced minimum modulus*, Michigan Math. J. **32** (1985), 279–294.
2. S. R. Caradus, *Operator theory of the pseudo-inverse*, Queen's Papers in Pure and Appl. Math., vol. 38, Queen's Univ., Kingston, Ontario, 1978.
3. ——, *Perturbation theory for generalized Fredholm operators*. II, Proc. Amer. Math. Soc. **62** (1977), 72–76.
4. R. E. Curto, *Applications of several complex variables to multiparameter spectral theory*, Surveys of Some Recent Results in Operator Theory. II, Pitman Res. Notes Math. Ser., vol. 192, Longman Sci. Tech, Harlow, 1988, pp. 25–90.
5. M. Gonzales, *Null spaces and ranges of polynomials of operators*, Publ. Mat. Univ. Barcelona **32** (1988), 167–170.
6. M. Gonzalez and R. E. Harte, *The death of an index theorem*, Proc. Amer. Math. Soc. **108** (1990), 151–156.
7. R. E. Harte, *Invertibility, singularity and Joseph L. Taylor*, Proc. Royal Irish Acad. Sect. A **81** (1981), 71–79.
8. ——, *Invertibility and singularity for bounded linear operators*, Dekker, New York, 1988.
9. T. Kato, *Perturbation theory for nullity, deficiency and other quantities for linear operators*, J. Analyse Math. **6** (1958), 261–322.
10. M. Mbekhta, *Généralization de la décomposition de Kato aux opérateurs paranormeaux et spectraux*, Glasgow Math. J. **29** (1987), 159–175.
11. ——, *Résolvant généralisé et théorie spectrale*, J. Operator Theory **21** (1989), 69–105.
12. M. Z. Nashed, *Perturbations and approximations for generalized inverses and linear operator equations*, Generalized Inverses and Applications (M. Z. Nashed, ed.), Academic Press, New York, 1976, pp. 325–396.
13. M. O Séarcoid and T. T. West, *Continuity of the generalized kernel and range of semi-Fredholm operators*, Math. Proc. Cambridge Philos. Soc. **105** (1989), 513–522.
14. P. Saphar, *Contribution a l'étude des applications linéaires dans un espace de Banach*, Bull. Soc. Math. France **92** (1964), 363–384.
15. T. T. West, *A Riesz-Schauder theorem for semi-Fredholm operators*, Proc. Royal Irish Acad. Sect. A **87** (1987), 137–146.

DEPARTMENT OF PURE MATHEMATICS, QUEEN'S UNIVERSITY, BELFAST BT7 1NN, UNITED KINGDOM
E-mail address: r.harte@v2.qub.ac.uk