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MEASURABILITY PROPERTIES OF SETS OF VITALI'S TYPE

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ABSTRACT. Assume a group G acts on a set. Given a subgroup H of G, by an H-selector we mean a selector of the set of all orbits of H. We investigate measurability properties of H-selectors with respect to G-invariant measures.

Let us fix a set X and a group G acting on it. By μ we denote a G-invariant countably additive measure on X. The most common example of such a situation is an invariant measure on a group acting on itself by translations. Let H be a subgroup of G. By an *H-selector* (sometimes called a set of Vitali's type for H) we understand a set having exactly one point in common with each orbit of H. Measurability properties of selectors were first systematically studied by Cichoń, Kharazishvili, and Węglorz in [1].

Selectors are extremely useful in constructing sets nonmeasurable with respect to an invariant measure. The first example of a Lebesgue nonmeasurable set, due to Vitali [8], is just a Q-selector where Q is the group of rationals. Also for any finite invariant diffused measure on a group (acting on itself by translations) any H-selector for a countable subgroup H is nonmeasurable. In fact, in both cases above the constructed sets are nonmeasurable with respect to any invariant extension of a given measure. Kharazishvili in [3] and Erdős and Mauldin in [2] constructed a nonmeasurable set for any σ-finite invariant measure. Their example is the union of a family of H-selectors where H is a subgroup of cardinality ω_1. Strengthening the result from [2, 3] the author constructed in [6] sets nonmeasurable with respect to any invariant extension of a given σ-finite measure. These sets are subsets of H-selectors for an appropriately chosen countable group H.

In the present paper we take a closer look at measurability properties of selectors. Putting a freeness assumption on the action of G and assuming that G is uncountable we prove that for a σ-finite measure one can always find a countable group H such that no H-selector is measured by any invariant extension of the given measure. We show also that the situation for subgroups of full cardinality is just the opposite. Imposing a stronger freeness condition and

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assuming that the cardinality of G has uncountable cofinality we prove that any σ-finite ergodic measure admits an invariant extension which measures at least one H-selector for any subgroup H of full cardinality. This was conjectured for Lebesgue measure and for Borel uncountable subgroups of the reals by Cichoń.

Now we set some notation. We write $HY = \{hx: x \in Y, \ h \in H\}$ for $H \subset G$ and $Y \subset X$, and $hY = \{h\}Y$ for $h \in G$, $Y \subset X$. μ_* and μ^* denote the inner and outer measure, respectively. We say that the action of G is *μ-free* if $\mu^*(\{x \in X: hx = x\}) = 0$ for any $h \in G\backslash\{e\}$ ($e =$ the identity of G). Notice that the action of any subgroup of the group of all isometries of a Euclidean space is μ-free for any invariant extension of Lebesgue measure. The action of G is *free* if $\{x \in X: hx = x\} = \varnothing$ for any $h \in G\backslash\{e\}$. A measure $\overline{\mu}$ is called an *invariant extension* of μ if $\overline{\mu}$ is an invariant measure, each μ-measurable set is $\overline{\mu}$-measurable, and $\mu(Y) = \overline{\mu}(Y)$ for any μ-measurable set $Y \subset X$. μ is called *ergodic* if for any two measurable sets $A, B \subset X$ with $\mu(A) > 0$ and $\mu(B) > 0$ there is an $h \in G$ such that $\mu(A \cap hB) > 0$. Note that Lebesgue measure, or more generally Haar measure on a locally compact group, is ergodic. $|A|$ denotes the cardinality of A. N stands for the set of positive integers.

Our first theorem states that in the case of σ-finite measures one can always find a countable subgroup H such that H-selectors behave just like Q-selectors where Q is the group of the rationals on the real line.

Theorem 1. *Let G be uncountable, and let μ be σ-finite. Suppose G acts μ-freely on X. Then there exists a countable subgroup H of G such that each H-selector is nonmeasurable with respect to any invariant extension of μ.*

Proof. The terminology in this proof is from [6]. Since μ is σ-finite, one can construct by transfinite induction using [6, Lemma 3.3] a countable family of μ-measurable sets $\{A_n: n \in N\}$ such that $\mu(X\backslash \bigcup_{n=1}^{\infty} A_n) = 0$ and A_n is infinitely covered by some countable H_n, $n \in N$. Let H be the subgroup of G generated by $\bigcup_{n=1}^{\infty} H_n$. Let V be any H-selector, and let $\overline{\mu}$ be an invariant extension of μ for which V is measurable. Since $HV = X$ and H is countable, we have $\overline{\mu}(V) > 0$. But then $\overline{\mu}(V \cap A_n) > 0$ for some $n \in N$, which contradicts [6, Lemma 3.1]. $\square$

Notice that the group H in the above theorem may be very different from the group of the rationals. For example, if G is the group of all isometries of the n-dimensional Euclidean space and μ is a G-invariant extension of Lebesgue measure, then any countable infinite subgroup of G consisting of orthogonal linear transformations works. We can choose such a subgroup to be isomorphic to a free group with countably many generators $(n \geq 3)$ or to the infinite cyclic group $(n \geq 2)$.

Now we turn our attention to selectors of subgroups of higher cardinality. We need some new notions. By an *ideal* on a set Y we understand a family of subsets of Y not containing Y, closed under taking subsets and finite unions. If I is an ideal, then a family of subsets of Y is called *disjoint modulo I* if the intersection of any two of its members is in I. Define $\mathrm{sat}(I) = \min\{\kappa:$ if $\mathscr{F}$ is a disjoint modulo I family of subsets of Y then $|\mathscr{F}| < \kappa\}$ and $\mathrm{add}(I) = \min\{|\mathscr{F}|: \mathscr{F} \subset I$ and $\bigcup \mathscr{F} \notin I\}$. As usual I is called a σ-ideal if $\mathrm{add}(I) > \omega$. Two ideals I_1, I_2 on Y are called *coherent* if $A_1 \cup A_2 \neq Y$ for any $A_1 \in I_1$ and $A_2 \in I_2$. If I_1 and I_2 are coherent, we denote by $[I_1, I_2]$ the ideal generated by I_1 and I_2. Clearly $\operatorname{add}([I_1, I_2]) \geq \min(\operatorname{add}(I_1), \operatorname{add}(I_2))$. An ideal I on X is called *invariant* if for any $A \in I$ and $h \in G$ we have $hA \in I$. For any cardinal number λ and any set Y let $[Y]^{<\lambda}$ (resp. $[Y]^{\lambda}$, $[Y]^{\leq\kappa}$) denote $\{A \subset Y : |A| < \lambda\}$ (resp. $\{A \subset Y : |A| = \lambda\}$, $\{A \subset Y : |A| \leq \lambda\}$). We identify ordinal numbers with the sets of their predecessors. For a cardinal number λ let $\operatorname{cf}(\lambda) = \min\{\kappa : \kappa$ is an ordinal and $\exists f : \kappa \to \lambda\ \lambda = \bigcup_{\alpha<\kappa} f(\alpha)\}$. A cardinal number κ is called *regular* if $\operatorname{cf}(\kappa) = \kappa$. For any cardinal λ, $\operatorname{cf}(\lambda)$ is a regular cardinal.

Lemma 1. *Let I be an ideal on Y, and let κ be a regular cardinal with $\kappa \leq \operatorname{add}(I)$ and $\kappa < \operatorname{sat}(I)$. Then there exists an ideal J such that:*

(i) $J \supset I$;
(ii) $\operatorname{add}(J) \geq \kappa$;
(iii) *for each $A \notin I$ there is $B \in J \setminus I$ with $B \subset A$.*

Proof. (The presented proof follows a suggestion of Blass which substantially simplifies the author's original argument.) Since $\kappa < \operatorname{sat}(I)$, we can find a maximal disjoint modulo I family of cardinality $\geq \kappa$. Denote this family $\mathscr{B}$. Let

$$J = \left\{B \subset Y : \exists C \in I\ \exists \mathscr{S} \subset [\mathscr{B}]^{<\kappa}\ B \subset C \cup \bigcup \mathscr{S}\right\}.$$

Obviously J fulfills (i). Since κ is regular and $\kappa \leq \operatorname{add}(I)$, J fulfills (ii). If $A \notin I$, then by maximality of $\mathscr{B}$ there exists $B \in \mathscr{B}$ with $A \cap B \notin I$. Clearly $A \cap B \in J$, so (iii) is satisfied, too. □

Notice that by Ulam's theorem if $\operatorname{add}(I)$ is a successor cardinal, then $\operatorname{add}(I) < \operatorname{sat}(I)$. In this case (ii) means simply $\operatorname{add}(J) \geq \operatorname{add}(I)$. Nevertheless in general the condition $\kappa < \operatorname{sat}(I)$ cannot be dropped. For if I and J are as in the above lemma we have $\operatorname{add}(J) < \operatorname{sat}(I)$ because applying (iii) and (i) one can construct $\operatorname{add}(J)$ pairwise disjoint sets outside of I.

In the sequel we will use only the following corollary of Lemma 1. This corollary can also be inferred from a much deeper result of Węglorz [9]. The author decided to present the direct proof here because of its simplicity.

Corollary 1. *Let κ be a cardinal, and let I be an ideal on Y. Then there exists an ideal J such that:*

(i) $J \supset I$;
(ii) $\operatorname{add}(J) \geq \operatorname{add}(I)$;
(iii) $\forall A \in [Y]^{\kappa}\ \exists B \in J \cap [Y]^{\kappa}\ B \subset A$.

Proof. If $[Y]^{<\kappa}$ is not contained in I, take $A \in [Y]^{<\kappa} \setminus I$ and define $J = \{B \subset Y : B \cap A \in I\}$. Then clearly (i), (ii), and (iii) are fulfilled. Assume that $[Y]^{<\kappa} \subset I$. If $\operatorname{add}(I) > \kappa$ or $\operatorname{sat}(I) \leq \kappa$, put $J = I$. Then (i) and (ii) are obviously satisfied. When $\operatorname{add}(I) > \kappa$, we have $[Y]^{\kappa} \subset I$ as $[Y]^{<\kappa} \subset I$ and (iii) is fulfilled. When $\operatorname{sat}(I) \leq \kappa$, (iii) is again true since each set from $[Y]^{\kappa}$ can be divided into κ many pairwise disjoint sets from $[Y]^{\kappa}$. If $\operatorname{sat}(I) > \kappa \geq \operatorname{add}(I)$, notice that $\operatorname{add}(I)$ is a regular cardinal and apply Lemma 1 ($\operatorname{add}(I)$ playing the role of the κ in the lemma). As for (iii), by Lemma 1(iii) each set from $[Y]^{\kappa} \setminus I$ contains a set from $(J \setminus I) \cap [Y]^{\leq\kappa}$ and we have $J \cap [Y]^{\kappa} \supset (J \setminus I) \cap [Y]^{\leq\kappa}$ since $[Y]^{<\kappa} \subset I$. □

Now we prove a lemma concerning extensions of invariant ideals. Our method of construction owes much to ideas of Kakutani and Oxtoby [5] and Hulanicki [4].

Lemma 2. *Assume G is uncountable and acts freely on X. Let I be an invariant ideal on X. Then there exists an invariant ideal J such that*:

(i) $J \supset I$;
(ii) $\mathrm{add}(J) \geq \min(\mathrm{add}(I), \mathrm{cf}(|G|))$;
(iii) *J contains an H-selector for each subgroup H of G with $|H| = |G|$.*

Proof. Let W be a G-selector. Put $\lambda = |G|$ and $\kappa = \mathrm{cf}(\lambda)$. Let $\{G_\alpha : \alpha < \kappa\}$ be a family of subgroups of G such that $G_\alpha \subset G_\beta$ for $\alpha < \beta < \kappa$, $|G_\alpha| < \lambda$, and $\bigcup_{\alpha<\kappa} G_\alpha = G$. For convenience we assume also that $G_\alpha \setminus \bigcup_{\xi<\alpha} G_\xi \neq \varnothing$. Let $X_\alpha = (G_\alpha \setminus \bigcup_{\xi<\alpha} G_\xi)W$. We define an ideal on κ as

$$I' = \left\{ D \subset \kappa : \bigcup_{\alpha \in D} X_\alpha \in I \right\}.$$

First we show that I' is coherent with $[\kappa]^{<\kappa}$. Take $D \in [\kappa]^{<\kappa}$. Since κ is regular, we can find $\beta < \kappa$, which is greater than all elements of D. Take $h \in G_\beta \setminus \bigcup_{\xi<\beta} G_\xi$. Since G acts freely and W is a G-selector, $h(\bigcup_{\alpha\in D} X_\alpha) \cap \bigcup_{\alpha\in D} X_\alpha = \varnothing$, i.e., $h(\bigcup_{\alpha\in\kappa\setminus D} X_\alpha) \cup \bigcup_{\alpha\in\kappa\setminus D} X_\alpha = X$. Thus $\kappa\setminus D \notin I'$ as I is invariant.

Put $\overline{I} = [I', [\kappa]^{<\kappa}]$. Then $\mathrm{add}(\overline{I}) \geq \min(\mathrm{add}(I), \kappa)$. Let $\overline{J}$ be an ideal on κ extending $\overline{I}$ whose existence is guaranteed by Corollary 1. Let

$$J' = \left\{ A \subset X : \exists D \in \overline{J}\ A \subset \bigcup_{\alpha\in D} X_\alpha \right\}.$$

J' is invariant. Let $h \in G$. Then $h \in G_\beta$ for some $\beta < \kappa$. It is enough to check that $hA \in J'$ for A of the form $\bigcup_{\alpha\in D} X_\alpha$ for some $D \in \overline{J}$. But then $hA \setminus A \subset \bigcup_{\alpha<\beta} X_\alpha \in J'$ since $\{\alpha : \alpha < \beta\} \in \overline{J}$. Notice that J' and I are coherent. Otherwise there are $A_1 \in I$, $A_2 \in J'$ such that $A_1 \cup A_2 = X$. Then there is $D \in \overline{J}$ such that $A_2 \subset \bigcup_{\alpha\in D} X_\alpha$. Thus $\bigcup_{\alpha\in\kappa\setminus D} X_\alpha \subset A_1$ whence $\kappa\setminus D \in \overline{I}$. But $\overline{I} \subset \overline{J}$ and thus D, $\kappa\setminus D \in \overline{J}$, a contradiction.

Let $J = [J', I]$. Clearly J is invariant and $J \supset I$. Since $\mathrm{add}(J') \geq \mathrm{add}(\overline{J}) \geq \mathrm{add}(\overline{I}) \geq \min(\mathrm{add}(I), \kappa)$, we have $\mathrm{add}(J) \geq \min(\mathrm{add}(I), \kappa)$. Thus (i) and (ii) are fulfilled. Now we show (iii). Let H be a subgroup of G with $|H| = \lambda$. Put $D = \{\alpha < \kappa : H \cap G_\alpha \setminus \bigcup_{\xi<\alpha} G_\xi \neq \varnothing\}$. Then $D \in [\kappa]^\kappa$, so there is $D' \in \overline{J} \cap [\kappa]^\kappa$ with $D' \subset D$. Let $B = \bigcup_{\alpha\in D'} X_\alpha \in J$. Pick $x \in X$. There exist $y \in W$ and $h \in G$ with $x = hy$. We can also find $\beta \in D'$ such that $h \in G_\alpha$ for some $\alpha < \beta$. Then $Hh \cap (G_\beta \setminus \bigcup_{\xi<\beta} G_\xi) = Hh \cap (G_\beta \setminus \bigcup_{\xi<\beta} G_\xi)h \neq \varnothing$ whence $H\{x\} \cap X_\beta = Hh\{y\} \cap (G_\beta \setminus \bigcup_{\xi<\beta} G_\xi)W \neq \varnothing$. Thus $H\{x\} \cap B \neq \varnothing$ for any $x \in X$. Now we can choose an H-selector inside B. $\square$

The following lemma is essentially due to Szpilrajn [7, §2].

Lemma 3 (Szpilrajn). *Let μ be an invariant measure on X, and let J be an invariant σ-ideal on X such that $\mu_*(A) = 0$ for $A \in J$. Then there exists an* *invariant extension of* μ *defined on the* σ*-algebra generated by the* σ*-algebra of* μ*-measurable sets and* J.

The next theorem shows that under certain assumptions a subgroup of full cardinality with properties like those in Theorem 1 cannot be constructed.

Theorem 2. *Assume* $\mathrm{cf}(|G|) > \omega$. *Suppose also that* G *acts freely on* X. *Let* μ *be* σ*-finite and ergodic. Then there exists an invariant extension* $\overline{\mu}$ *of* μ *such that for each subgroup* H *of* G *with* $|H| = |G|$ *there is a* $\overline{\mu}$*-measurable* H*-selector.*

Proof. Consider I_μ the invariant σ-ideal of μ-measure 0 sets. Let J be an ideal extending I_μ produced in Lemma 2. As $\mathrm{add}(J) \geq \min(\mathrm{add}(I_\mu), \mathrm{cf}(|G|)) > \omega$, J is a σ-ideal. Now we show that the assumption of Lemma 3 is fulfilled. Suppose $\mu_*(A) > 0$ for some $A \in J$. As J is closed under taking subsets, we can assume that A is μ-measurable and $\mu(A) > 0$. Using the σ-finiteness and the ergodicity of μ we find a countable set $K \subset G$ with $\mu(X \backslash KA) = 0$, i.e., $X \backslash KA \in I_\mu \subset J$. As $KA \in J$ we get a contradiction. Thus $A \notin J$. Now Lemma 3 yields an invariant extension $\overline{\mu}$ of μ for which all sets from J are measurable. In particular, for each subgroup H of cardinality $|G|$ there is a $\overline{\mu}$-measurable H-selector. □

Since $\mathrm{cf}(2^\omega) > \omega$, Theorem 2 gives the following corollary. (The "if" direction of the second part of the corollary can be shown by the same argument as in the standard proof that any Q-selector, where Q denotes the rationals, is not Lebesgue measurable.)

Corollary 2. *There exists a translation invariant extension of Lebesgue measure on the real line which measures at least one* H*-selector for each group of translations* H *with* $|H| = 2^\omega$. *In particular, assuming the Continuum Hypothesis, each* H*-selector of a subgroup* H *of the reals is nonmeasurable with respect to any invariant extension of Lebesgue measure if and only if* H *is countable and dense.*

In the context of Theorems 1 and 2 the following question seems to be interesting. Let G act freely on X, and let μ be invariant, σ-finite, and ergodic. Does there exist an invariant extension of μ which measures at least one H-selector for each uncountable subgroup H of G? The author does not know the answer even for Lebesgue measure on the real line (without assuming the Continuum Hypothesis of course).

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