ON POINCARÉ TYPE INTEGRAL INEQUALITIES

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ABSTRACT. In this paper some new integral inequalities of the Poincaré type, on a region of rectangular dimensions, involving many functions in many variables are obtained. These in turn can be used to serve as generators of other integral inequalities.

1. INTRODUCTION

The multidimensional integral inequality [4, 7]

$$\lambda_0 \int_\Omega u^2 \, dx \le \int_\Omega |\bigtriangledown u|^2 \, dx \ , \tag{1}$$

where Ω is a bounded region in $\mathbb{R}^2$ or $\mathbb{R}^3$, $u \in C^1(\Omega)$, $u = 0$ on $\partial\Omega$, and λ_0 is the smallest eigenvalue of the problem

$$\begin{cases} \triangle u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \ , \end{cases}$$

is known as Poincaré's inequality. Since (1) plays a fundamental role in the theory as well as applications of partial differential equations, considerable attention and effort have been put on its generalizations to various aspects. For details one is referred to Beckenbach and Bellman [1], Hardy, Littlewood, and Pólya [3], Mitrinović [7], Nirenberg [8], and, more recently, Horgan [4–6], Pachpatte [9, 10], and Cheung [2]. It is the purpose of this paper to obtain some new generalizations of inequality (1) in the context involving many functions in many variables by methods which are rather elementary compared to the usual technique of considering certain eigenvalue problems.

2. NOTATION AND PRELIMINARIES

Throughout this paper, let $m \ge 2$ and $n \ge 1$ be any two fixed integers. Let α, β be indices running from 1 to m, and i, j from 1 to n. Let $\Omega = \prod_{i=1}^n [a_i, b_i] \subset \mathbb{R}^n$ be a fixed rectangular region. A general point in Ω will be denoted by $x = (x_1, \ldots, x_n)$, and the volume form will be denoted by $dx = dx_1 \cdots dx_n$. Let $C_0^1(\Omega)$ be the collection of all real-valued continuously

differentiable functions on Ω which vanish on the boundary $\partial\Omega$ of Ω. As usual, partial derivatives of $f \in C_0^1(\Omega)$ will be denoted by f_i.

Since all summations and products appearing in this paper will be either over α, β or i, j, and hence should be from 1 to m or from 1 to n, respectively, for the sake of simplicity we shall drop the terminals of the summation sign $\sum$ and the product sign $\prod$ unless confusion may occur. Furthermore, as usual, integrals of the form $\int_\Omega F(x)\,dx$ will be abbreviated by $\int_\Omega F$.

The following basic inequalities will be needed in the sequel.

Inequality A [7]. For any $p_\alpha > 0$ with $\sum_\alpha 1/p_\alpha = 1$, and for any $c_\alpha \geq 0$,

$$\prod_\alpha c_\alpha \leq \sum_\alpha \frac{1}{p_\alpha} c_\alpha^{p_\alpha} .$$

Inequality B [3]. For any $q_\alpha > 0$ and $c_\alpha \geq 0$,

$$\prod_\alpha c_\alpha^{q_\alpha} \leq \frac{1}{q} \sum_\alpha q_\alpha c_\alpha^q ,$$

where $q := \sum_\beta q_\beta$.

Inequality C [7]. For any $r_i \geq 0$ and $s \geq 0$,

$$\left(\sum_i r_i \right)^s \leq c(s,n) \sum_i r_i^s ,$$

where

$$c(s,n) = \begin{cases} n^{s-1} & \text{if } s > 1, \\ 1 & \text{if } 0 \leq s \leq 1 . \end{cases}$$

Inequality D (Generalization of Hölder's inequality). For any $p_\alpha > 0$ with $\sum_\alpha 1/p_\alpha = 1$ and any $g^\alpha \in \mathscr{L}^{p_\alpha}(\Omega)$, we have $\prod_\alpha g^\alpha \in \mathscr{L}^1(\Omega)$ and

$$\int_\Omega \prod_\alpha |g^\alpha| \leq \prod_\alpha \left(\int_\Omega |g^\alpha|^{p_\alpha} \right)^{1/p_\alpha} .$$

Inequality D is a simple generalization of the classical Hölder's inequality to the case involving more than two functions, which can be proved easily by induction.

3. MAIN RESULTS

Let $M = \max\{b_i - a_i : i = 1, \ldots, n\}$.

Theorem 1. *For any $f^\alpha \in C_0^1(\Omega)$ and any real numbers $p_\alpha \geq 2$ satisfying $\sum_\alpha 1/p_\alpha = 1$,*

$$\int_\Omega \prod_\alpha |f^\alpha| \leq \frac{1}{n} \sum_\alpha \frac{1}{p_\alpha} \left(\frac{M}{2} \right)^{p_\alpha} \int_\Omega |\nabla f^\alpha|^{p_\alpha} .$$

Theorem 2. *For any $f^\alpha \in C_0^1(\Omega)$ and any real numbers $q_\alpha > 0$ with $q := \sum_\alpha q_\alpha \geq 2$,*

$$\int_\Omega \prod_\alpha |f^\alpha|^{q_\alpha} \leq \frac{1}{n} \left(\frac{M}{2} \right)^q \sum_\alpha \frac{q_\alpha}{q} \int_\Omega |\nabla f^\alpha|^q .$$

Note that these results generalize the existing Poincaré type integral inequalities [6, 7, 9, 10] in the literature. To prove these, we first observe the following

Lemma 1. *If $f \in C_0^1(\Omega)$, then, for any $t \in \Omega$,*

$$|f(t)| \leq \frac{1}{2n} \sum_i \int_{a_i}^{b_i} \left| f_i(t_1, \ldots, t_{i-1}, u_i, t_{i+1}, \ldots, t_n) \right| du_i .$$

Proof. For each i we have

$$f(t) = \int_{a_i}^{t_i} f_i(t_1, \ldots, t_{i-1}, u_i, t_{i+1}, \ldots, t_n) du_i$$

and also

$$f(t) = - \int_{t_i}^{b_i} f_i(t_1, \ldots, t_{i-1}, u_i, t_{i+1}, \ldots, t_n) du_i .$$

Taking their absolute values and summing with respect to i, we then have

$$2n|f(t)| \leq \sum_i \int_{a_i}^{b_i} \left| f_i(t_1, \ldots, t_{i-1}, u_i, t_{i+1}, \ldots, t_n) \right| du_i$$

and thus the assertion follows.   Q.E.D.

Proof of Theorem 1. By Inequality A, Lemma 1, and Inequality C, we have

$$\prod_\alpha |f^\alpha(t)| \leq \sum_\alpha \frac{1}{p_\alpha} |f^\alpha(t)|^{p_\alpha} \leq \sum_\alpha \frac{1}{p_\alpha} \left[\frac{1}{2n} \sum_i \int_{a_i}^{b_i} |f_i^\alpha| du_i \right]^{p_\alpha}$$

$$\leq \sum_\alpha \frac{1}{p_\alpha} \left(\frac{1}{2n} \right)^{p_\alpha} \left[c(p_\alpha, n) \sum_i \left(\int_{a_i}^{b_i} |f_i^\alpha| du_i \right)^{p_\alpha} \right] .$$

Since $p_\alpha > 1$, we have $c(p_\alpha, n) = n^{p_\alpha - 1}$ for all α and so, by Hölder's inequality,

$$\prod_\alpha |f^\alpha(t)| \leq \sum_\alpha \frac{1}{np_\alpha} \left(\frac{1}{2} \right)^{p_\alpha} \sum_i \left(\int_{a_i}^{b_i} |f_i^\alpha| du_i \right)^{p_\alpha}$$

$$\leq \sum_\alpha \frac{1}{np_\alpha} \left(\frac{1}{2} \right)^{p_\alpha} \sum_i \left[(b_i - a_i)^{(p_\alpha - 1)/p_\alpha} \left(\int_{a_i}^{b_i} |f_i^\alpha|^{p_\alpha} du_i \right)^{1/p_\alpha} \right]^{p_\alpha}$$

$$\leq \sum_\alpha \frac{1}{np_\alpha} \left(\frac{1}{2} \right)^{p_\alpha} M^{p_\alpha - 1} \sum_i \int_{a_i}^{b_i} |f_i^\alpha|^{p_\alpha} du_i .$$

Therefore,

$$\int_\Omega \prod_\alpha |f^\alpha(t)| dt \le \sum_\alpha \frac{1}{np_\alpha} \left(\frac{1}{2}\right)^{p_\alpha} M^{p_\alpha - 1} \sum_i \int_\Omega \int_{a_i}^{b_i} |f_i^\alpha|^{p_\alpha} du_i \, dt$$

$$= \sum_\alpha \frac{1}{np_\alpha} \left(\frac{1}{2}\right)^{p_\alpha} M^{p_\alpha - 1} \sum_i \int_{a_i}^{b_i} \int_\Omega |f_i^\alpha|^{p_\alpha} dt \, du_i$$

$$= \sum_\alpha \frac{1}{np_\alpha} \left(\frac{1}{2}\right)^{p_\alpha} M^{p_\alpha - 1} \sum_i \left(\int_\Omega |f_i^\alpha|^{p_\alpha} dt\right)(b_i - a_i)$$

$$\le \sum_\alpha \frac{1}{np_\alpha} \left(\frac{M}{2}\right)^{p_\alpha} \int_\Omega \sum_i |f_i^\alpha|^{p_\alpha} dt$$

$$= \sum_\alpha \frac{1}{np_\alpha} \left(\frac{M}{2}\right)^{p_\alpha} \int_\Omega \left[\left(\sum_i |f_i^\alpha|^{p_\alpha}\right)^{2/p_\alpha}\right]^{p_\alpha/2} dt$$

$$\le \sum_\alpha \frac{1}{np_\alpha} \left(\frac{M}{2}\right)^{p_\alpha} \int_\Omega \left[c\left(\frac{2}{p_\alpha}, n\right) \sum_i (|f_i^\alpha|^{p_\alpha})^{2/p_\alpha}\right]^{p_\alpha/2} dt$$

by Inequality C. Since $p_\alpha \ge 2$, $c(\frac{2}{p_\alpha}, n) = 1$ for all α and so

$$\int_\Omega \prod_\alpha |f^\alpha(t)| dt \le \sum_\alpha \frac{1}{np_\alpha} \left(\frac{M}{2}\right)^{p_\alpha} \int_\Omega \left(\sum_i |f_i^\alpha|^2\right)^{p_\alpha/2} dt$$

$$= \frac{1}{n} \sum_\alpha \frac{1}{p_\alpha} \left(\frac{M}{2}\right)^{p_\alpha} \int_\Omega |\nabla f|^{p_\alpha} dt \, . \quad \text{Q.E.D.}$$

Proof of Theorem 2. By Inequality B, Lemma 1, and Inequality C, we have

$$\prod_\alpha |f^\alpha(t)|^{q_\alpha} \le \frac{1}{q} \sum_\alpha q_\alpha |f^\alpha(t)|^q \le \frac{1}{q} \sum_\alpha q_\alpha \left[\frac{1}{2n} \sum_i \int_{a_i}^{b_i} |f_i^\alpha| du_i\right]^q$$

$$\le \frac{1}{q} \sum_\alpha q_\alpha \left(\frac{1}{2n}\right)^q \left[c(q, n) \sum_i \left(\int_{a_i}^{b_i} |f_i^\alpha| du_i\right)^q\right] \, .$$

Since $q > 1$, we have $c(q, n) = n^{q-1}$. By Hölder's inequality,

$$\prod_\alpha |f^\alpha(t)|^{q_\alpha} \le \frac{1}{nq} \left(\frac{1}{2}\right)^q \sum_{\alpha, i} q_\alpha \left(\int_{a_i}^{b_i} |f_i^\alpha| du_i\right)^q$$

$$\le \frac{1}{nq} \left(\frac{1}{2}\right)^q \sum_{\alpha, i} q_\alpha \left[(b_i - a_i)^{(q-1)/q} \left(\int_{a_i}^{b_i} |f_i^\alpha|^q du_i\right)^{1/q}\right]^q$$

$$\le \frac{1}{nq} \left(\frac{1}{2}\right)^q M^{q-1} \sum_{\alpha, i} q_\alpha \int_{a_i}^{b_i} |f_i^\alpha|^q du_i \, .$$

Therefore,

$$\int \prod_\alpha |f^\alpha(t)|^{q_\alpha} dt \le \frac{1}{nq} \left(\frac{1}{2}\right)^q M^{q-1} \sum_{\alpha,\, i} q_\alpha \int_\Omega \int_{a_i}^{b_i} |f_i^\alpha|^q du_i \, dt$$

$$= \frac{1}{nq} \left(\frac{1}{2}\right)^q M^{q-1} \sum_{\alpha,\, i} q_\alpha \int_{a_i}^{b_i} \int_\Omega |f_i^\alpha|^q dt \, du$$

$$= \frac{1}{nq} \left(\frac{1}{2}\right)^q M^{q-1} \sum_{\alpha,\, i} q_\alpha (b_i - a_i) \int_\Omega |f_i^\alpha|^q dt$$

$$\le \frac{1}{nq} \left(\frac{M}{2}\right)^q \sum_\alpha q_\alpha \int_\Omega \sum_i |f_i^\alpha|^q dt$$

$$= \frac{1}{nq} \left(\frac{M}{2}\right)^q \sum_\alpha q_\alpha \int_\Omega \left[\left(\sum_i |f_i^\alpha|^q\right)^{2/q}\right]^{q/2} dt$$

$$\le \frac{1}{nq} \left(\frac{M}{2}\right)^q \sum_\alpha q_\alpha \int_\Omega \left[c\left(\frac{2}{q}, n\right) \sum_i (|f_i^\alpha|^2)^{2/q}\right]^{q/2} dt$$

by Inequality C. Since $q \ge 2$, we have $c(\frac{2}{q}, n) = 1$ and so

$$\int_\Omega \prod_\alpha |f^\alpha(t)| \, dt \le \frac{1}{nq} \left(\frac{M}{2}\right)^q \sum_\alpha q_\alpha \int_\Omega \left(\sum_i |f_i^\alpha|^2\right)^{q/2} dt$$

$$= \frac{1}{n} \left(\frac{M}{2}\right)^q \sum_\alpha \frac{q_\alpha}{q} \int_\Omega |\nabla f^\alpha|^q dt \,. \quad \text{Q.E.D.}$$

Corollary 1. *If $f^\alpha \in C_0^1(\Omega)$, then*

$$\int_\Omega \prod_\alpha |f^\alpha| \le \frac{1}{nm} \sum_\alpha \left(\frac{M}{2}\right)^m \int_\Omega |\nabla f^\alpha|^m \,.$$

Proof. This follows from Theorem 1 by setting $p_\alpha = m$ for all α or from Theorem 2 by setting $q_\alpha = 1$ for all α. Q.E.D.

Corollary 2. *If $f \in C_0^1(\Omega)$, then*

$$\int_\Omega |f|^m \le \frac{1}{n} \left(\frac{M}{2}\right)^m \int_\Omega |\nabla f|^m \,.$$

Proof. This follows from Corollary 1 by setting $f^\alpha = f$ for all α. Q.E.D.

Remark. Notice that when $m = n = 2$ and Ω is a square, Corollary 2 is sharper than the result in [7] in the sense that our constant here is $M^2/8$, comparing to $7M^2/12$ there.

Corollary 3. *If $f^\alpha \in C_0^1(\Omega)$ and $p_\alpha \ge 2$ are real numbers satisfying $\sum_\alpha 1/p_\alpha = 1$, then*

$$\int_\Omega \sum_\beta \prod_{\alpha \ne \beta} |f^\alpha \nabla f^\beta| \le \left(\frac{M}{2}\right)^{m-1} \left[\sum_\alpha \left(\frac{1}{n}\right)^{1-1/p_\alpha}\right] \left[\sum_\alpha \frac{1}{p_\alpha} \int_\Omega |\nabla f^\alpha|^{p_\alpha}\right].$$

Proof. By Inequality D and Corollary 2, we have

$$\int_\Omega \sum_\beta \prod_{\alpha \neq \beta} |f^\alpha \bigtriangledown f^\beta|$$

$$\leq \sum_\beta \left[\prod_{\alpha \neq \beta} \left(\int_\Omega |f^\alpha|^{p_\alpha} \right)^{1/p_\alpha} \left(\int_\Omega |\bigtriangledown f^\beta|^{p_\beta} \right)^{1/p_\beta} \right]$$

$$\leq \sum_\beta \left[\prod_{\alpha \neq \beta} \left(\frac{1}{n} \left(\frac{M}{2} \right)^{p_\alpha} \int_\Omega |\bigtriangledown f|^{p_\alpha} \right)^{1/p_\alpha} \left(\int_\Omega |\bigtriangledown f^\beta|^{p_\beta} \right)^{1/p_\beta} \right]$$

$$= \left[\sum_\beta \left(\frac{1}{n} \right)^{1-1/p_\beta} \left(\frac{M}{2} \right)^{m-1} \right] \prod_\alpha \left(\int_\Omega |\bigtriangledown f^\alpha|^{p_\alpha} \right)^{1/p_\alpha}.$$

Therefore, by Inequality B, we arrive at

$$\int_\Omega \sum_\beta \prod_{\alpha \neq \beta} |f^\alpha \bigtriangledown f^\beta|$$

$$\leq \left[\sum_\beta \left(\frac{1}{n} \right)^{1-1/p_\beta} \left(\frac{M}{2} \right)^{m-1} \right] \cdot \frac{1}{\sum (1/p_\beta)} \sum_\alpha \frac{1}{p_\alpha} \left(\int_\Omega |\bigtriangledown f^\alpha|^{p_\alpha} \right)^{\sum (1/p_\beta)}$$

$$= \left(\frac{M}{2} \right)^{m-1} \left[\sum_\alpha \left(\frac{1}{n} \right)^{1-1/p_\alpha} \right] \left[\sum_\alpha \frac{1}{p_\alpha} \int_\Omega |\bigtriangledown f^\alpha|^{p_\alpha} \right]. \quad \text{Q.E.D.}$$

Corollary 4. *If* $f^\alpha \in C_0^1(\Omega)$, *then*

$$\int_\Omega \sum_\beta \prod_{\alpha \neq \beta} |f^\alpha \bigtriangledown f^\beta| \leq \left(\frac{M}{2} \right)^{m-1} \left(\frac{1}{n} \right)^{1-1/m} \sum_\alpha \int_\Omega |\bigtriangledown f^\alpha|^m.$$

Proof. This follows immediately from Corollary 3 by setting $p_\alpha = m$ for all α. Q.E.D.

Remark. Further interesting integral inequalities of the Poincaré-type could easily be generated from the results above. For instance, by taking $m = 2$ in Corollary 4, we get

$$\int_\Omega |f \bigtriangledown g + g \bigtriangledown f| \leq \frac{M}{2\sqrt{n}} \left(\int_\Omega |\bigtriangledown f|^2 + \int_\Omega |\bigtriangledown g|^2 \right),$$

and by putting $f = g$ in the last expression, we have

$$\int_\Omega |f \bigtriangledown f| \leq \frac{M}{2\sqrt{n}} \int_\Omega |\bigtriangledown f|^2.$$

These integral inequalities are in general of great interest and have been proven to be very useful in the study of quantitative as well as qualitative properties of solutions of many differential and integral equations.

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