A CLASS OF RIEZS SETS

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Abstract. Let $G$ be a metrizable compact abelian group. A subset $\Lambda$ in the
dual group is said to be ergodic if every $f \in L^\infty(G)$ whose spectrum lies in
a translate of $\Lambda$ has a unique invariant mean. It is shown that such a set is a
Riesz set.

Introduction

It is well known that Rosenthal sets are Riesz sets [1, 4, 3]. A natural class of
sets is lying between them; it is connected with the possibility for an element of
$L^\infty(G)$ to have a unique invariant mean. A subset $\Lambda$ of a discrete abelian group
is said to be ergodic if every $f \in L^\infty(G)$ whose spectrum lies in a translate of
$\Lambda$ has a unique invariant mean. Katznelson [8] proved that $N$, which is a
Riesz set, is not an ergodic set. A few years ago, Françoise Lust-Piquard [6]
showed that $\mathbb{P} \cap (5\mathbb{Z} + 3)$, $\mathbb{P}$ being the set of primes, is an ergodic set which is
not a Rosenthal set. Of course, each Rosenthal set is ergodic since continuous
functions have a unique invariant mean. In view of these results, the interplay
between ergodic and Riesz sets arises naturally. We prove in this paper that
every ergodic set is a Riesz set.

Definitions and notation

Throughout the paper $G$ will be a metrizable compact abelian group and
$\Gamma = \hat{G}$ will be its countable discrete dual group.

As usual $\mathcal{M}_\Lambda$, $L^1_\Lambda$, $L^\infty_\Lambda$, $L^\infty(G)$, $L^1(G)$, $C(G)$, $L^\infty(G)$ respectively whose spectrum (i.e., the support
of their Fourier transform) lies in $\Lambda \subseteq \Gamma$.

A subset $\Lambda \subseteq \Gamma$ is said to be a Riesz set if each measure $\mu \in \mathcal{M}_\Lambda$ is absolutely
continuous (w.r.t. the Haar measure); in short, $\mathcal{M}_\Lambda = L^1_\Lambda$; it is said to be a
Rosenthal set if each $f \in L^\infty_\Lambda$ is represented by a continuous function; in short,
$L^\infty_\Lambda = C_\Lambda$.

An invariant mean on $L^\infty$ is a translation-invariant continuous linear form
on $L^\infty$ such that $\|M\| = M(1) = 1$. An element $f \in L^\infty$ is said to be ergodic
if all the invariant means take the same value on it; it is said to be totally ergodic.
if $\gamma f$ is ergodic for every $\gamma \in \Gamma$. For such an $f$, we have $M(\gamma f) = \hat{f}(-\gamma)$ for each invariant mean $M$.

**Definition.** $\Lambda \subseteq \Gamma$ is said to be an ergodic set if every $f \in L^\infty_\Lambda$ is totally ergodic.

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**Ergodic and Riesz sets**

The main result is

**Theorem 1.** Let $G$ be a metrizable compact abelian group. Every ergodic set in $\Gamma = \widehat{G}$ is a Riesz set.

**Proof.** For every $\mu \in \mathcal{M}_\Lambda$, we write $\mu = \mu_a + \mu_s$, its Lebesgue-Radon-Nikodým decomposition. For every $f \in L^\infty$, we have $\mu * f \in L^\infty_\Lambda$ and so $\mu * f$ is totally ergodic. Since $\mu_a * f$ is totally ergodic, as a continuous function, we obtain that $\mu_s * f$ is totally ergodic. The following proposition allows us to conclude that $\mu_s = 0$.

**Proposition 2.** Let $G$ be a metrizable compact abelian group. For every singular measure $\sigma \neq 0$, there is an $f \in L^\infty$ such that $\sigma * f$ is not totally ergodic.

**Proof.** First, let $\mu$ be a singular measure such that $\theta = \mu(G) \neq 0$.

Since the measure $\nu = |\mu|$ is singular, there is a Borel set $B$ such that $m(B) = 0$ ($m$ being the Haar measure on $G$) and such that $\nu(B) = \|\nu\|$. It will be convenient to take $B$ symmetric.

Let $D$ be a countable dense subgroup of $G$. We set $A = B + D$; since $D$ is countable, we have $m(A) = 0$, and since $B \subseteq A$, we have $\nu(A) = \|\nu\|$.

Let now $\Omega$ be an open set of $G$ containing $A$ and such that $m(\Omega) \leq 1/4$.

We are going to show that $\mu * 1_\Omega$ is not ergodic. For this, we shall use the fact that an element $f \in L^\infty$ is ergodic if and only if there is a constant function in the norm-closed convex hull of the orbit of $f$ by $G$ [5, 8].

Let $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$ and $x_1, \ldots, x_n \in G$ be given. For each $i \in \{1, \ldots, n\}$, the function $\varphi_i$ defined by

$$
\varphi_i(x) = (1_{(\Omega,x_i)} * \nu)(x) = (1_{\Omega} * \nu)_{x_i}(x) = \nu(\Omega + x - x_i)
$$

is l.s.c.; therefore the set

$$
U_i = \{x \in G : \nu(\Omega + x - x_i) > \|\nu\| - |\theta|/4\}
$$

is open. Moreover, it is dense since it contains $D + x_i$. Then, $U = \bigcap_{i=1}^n U_i$ is also a dense open set, so $m(U) > 0$.

Moreover, for each $x \in U$, we have

$$
\forall i \in \{1, \ldots, n\} \quad \nu(\Omega + x - x_i) > \|\nu\| - |\theta|/4.
$$

We have then, for each $x \in U$,

$$
\left| \sum_{i=1}^n \lambda_i \mu(\Omega + x - x_i) - \mu(\Omega) \right| = \left| \sum_{i=1}^n \lambda_i [\mu(\Omega + x - x_i) - \mu(\Omega)] \right| 
\leq \sum_{i=1}^n \lambda_i \nu((\Omega + x - x_i)\Delta \Omega) < \frac{|\theta|}{4}
$$
and so, since $\mu(\Omega) = \mu(G) = \theta$,

$$\left| \sum_{i=1}^{n} \lambda_i (\Omega + x - x_i) \right| > \frac{3|\theta|}{4}.$$  

Therefore,

$$\left\| \sum_{i=1}^{n} \lambda_i (1_{\Omega} * \mu) \right\|_{\infty} > \frac{3|\theta|}{4}.$$  

Hence, there is no constant function in the norm-closed convex hull of $\{(1_{\Omega} * \mu)_{x} ; x \in G\}$, since if there is one, namely, $C \Omega$, we do have by the above calculations $C \geq 3|\theta|/4$; but, on the other hand, we also have

$$C = \int_{\Omega} (1_{\Omega} * \mu) \, dm = \mu(G)m(\Omega),$$  

and so $|C| \leq |\theta|/4$.

This proves that $1_{\Omega} * \mu$ is not ergodic.

To finish the proof, let $\gamma \in \Gamma$ be such that $\hat{\sigma}(\gamma) \neq 0$. Setting $\mu = \gamma\sigma$, we have $\mu(G) \neq 0$, and then the above proof shows that

$$\gamma \sigma \ast (\gamma 1_{\Omega}) = (\gamma \sigma) \ast 1_{\Omega}$$  

is not ergodic; therefore, $\sigma \ast (\gamma 1_{\Omega})$ is not totally ergodic. □

**Corollary 3.** Let $\Gamma$ be a countable (infinite) discrete abelian group. The set $\mathcal{E}rg(\Gamma)$ of all the ergodic sets in $\Gamma$ is a coanalytic subset of $\mathcal{P}(\Gamma)$ which is not a Borel subset.

**Proof.** From [2, Theorem 2] and Theorem 1 above, $\mathcal{E}rg(\Gamma)$ is not an analytic subset of $\mathcal{P}(\Gamma)$. It is then enough to show that it is coanalytic.

An element $f \in L^\infty(G)$ is ergodic if and only if for every $k \geq 1$ there is an integer $n$ and elements $x_1, \ldots, x_n$ in the group $G$ such that

$$(*) \quad \left\| \frac{1}{n} \sum_{i=1}^{n} f_{x_i} - \hat{f}(0)1 \right\|_{\infty} \leq \frac{1}{k};$$  

hence, the set $E(G)$ of all the totally ergodic elements of $L^\infty(G)$ can be written as

$$E(G) = \bigcap_{\gamma \in \Gamma} \bigcap_{k \geq 1} \bigcup_{n \geq 1} \bigcup_{M \geq 1} F_{k, n, M},$$  

where

$$F_{k, n, M} = \{ f \in L^\infty(G) ; \| f \|_{\infty} \leq M \text{ and } \exists x_1, \ldots, x_n \in G : (*) \}.$$  

$F_{k, n, M}$ is $w^*$-compact, as the projection of the compact

$$\{(x_1, \ldots, x_n, f) \in G^n \times (B_{L^\infty(G)}(0, M), w^*) ; \ (\ast) \}.$$  

This shows that $E(G)$ is a $K_{G\delta}$. To finish the proof, it suffices to check that $\mathcal{P}(\Gamma) \setminus \mathcal{E}rg(\Gamma)$ is the projection of the $G_{\delta\sigma}$:

$$\{(\Lambda, f) \in \mathcal{P}(\Gamma) \times (L^\infty(G), w^*) ; \hat{f} = 0 \text{ on } \Lambda^c \text{ and } f \in L^\infty(G) \setminus E(G)\}$$  

and so is analytic. □
Remark. As was recalled in the introduction, every Rosenthal set is a Riesz set. More precisely, we have the following implications:

\[(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v)\]

between the properties:

(i) \(A\) is a Rosenthal set;
(ii) the space \(L^1/L^1_{\Gamma\setminus\{A\}}\) contains no isomorphic copy of \(l_1\) (since its dual space \(L_{\infty}^A\) has the Radon-Nikodým Property);
(iii) every element of \(L_{\infty}^A\) is represented by a Riemann-integrable function (by \([7, \text{Corollary IV.4}]\));
(iv) \(A\) is an ergodic set (since every Riemann-integrable function is totally ergodic; see, for instance, \([8]\) or \([9]\));
(v) \(A\) is a Riesz set (by Theorem 1).

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REFERENCES