UNIVALENT LOGHARMONIC RING MAPPINGS

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(Communicated by Clifford J. Earle, Jr.)

Abstract. Univalent logharmonic ring mappings are characterized in terms of univalent starlike mappings. An existence and uniqueness theorem is also given.

1. Introduction

Let $D$ be a domain of $\mathbb{C}$ and denote by $H(D)$ the set of all analytic functions on $D$. A logharmonic mapping is a solution of the nonlinear elliptic partial differential equation

$$
\overline{f_z} = (a f / f_z)f_z,
$$

where the second dilatation function $a$ is in $H(D)$ and $|a(z)| < 1$ for all $z \in D$. If $f$ does not vanish in $D$, then $f$ is of the form

$$
f = HG,
$$

where $H$ and $G$ are locally analytic (possibly multivalued) functions on $D$. On the other hand, if $f$ vanishes at $z_0$ but is not identically zero, then $f$ admits the local representation

$$
f(z) = (z-z_0)^m|z-z_0|^{2\beta}h(z)g(z)
$$

where $m \in \mathbb{N}$, $\text{Re} \beta > -m/2$, and $h$ and $g$ are analytic in a neighbourhood of $z_0$ (see, e.g., [1]).

Univalent logharmonic mappings defined on the unit disk $U$ have been studied in [1, 2].

In this paper we investigate the family $\mathcal{A}$ of all univalent logharmonic mappings which map an annulus $A(r, 1) = \{ z : r < |z| < 1 \}$, $0 < r < 1$, onto an annulus $A(R, 1)$ for some $R \in [0, 1)$ and which satisfy the condition

$$
\frac{1}{2\pi} \int_{|z|=\rho} d \arg f(\rho e^{it}) = 1 \quad \text{for all } \rho \in (r, 1).
$$

The last condition says that the outer boundary corresponds to the outer boundary. We call an element $f \in \mathcal{A}$ a univalent logharmonic ring mapping.

If $a \equiv 0$, then $R = r$ and $f(z) = e^{ia}z$, $\alpha \in R$, are the only mappings in $\mathcal{A}$. In the case of univalent harmonic mappings from $A(r, 1)$ onto $A(R, 1)$ it is possible that $R = 0$; for example, $[1/(1-r^2)](z-r^2/\bar{z})$ has this property. However, Nitsche [11] has shown that there is an $R_0(r) < 1$ such that there is
no univalent harmonic mapping from $A(r, 1)$ onto $A(R, 1)$ whenever $R_0 < R < 1$.

There is no univalent logharmonic mappings from $A(r, 1)$, $0 < r < 1$, onto $A(0, 1)$. This is a direct consequence of (2.1) in our Theorem 2.1. But, on the other hand, we have for $R$ neither a positive lower bound nor a uniform upper bound strictly less than one. Indeed, $f(z) = z |z|^{2\beta}$, $\Re\beta > -\frac{1}{2}$, is univalent on $A(r, 1)$ and its image is $A(R^{1+2\Re\beta}, 1)$.

Unlike the case of univalent harmonic mappings, univalent logharmonic ring mappings need not have a continuous extension onto the closure of $A(r, 1)$. Indeed,

$$f(z) = \frac{z(z-1)}{(1-z)}$$

is a univalent logharmonic ring mapping from $A(\frac{1}{2}, 1)$ onto itself whose cluster sets on the outer boundary are $C(f, e^{it}) = \{-1\}$, if $z = e^{it}$, $0 < t < 2\pi$, and $C(f, 1) = \{w; |w| = 1\}$.

Theorems 2.1 and 2.2 give a complete characterization of univalent logharmonic mappings in $s_r$. It turns out that these are exactly the mappings of the form

$$f(z) = \frac{\varphi(z)}{|\varphi(z)|^2 z^{2\gamma}}$$

where $\Re\gamma > 0$, $\varphi \in H(A(r, 1))$, and where $p(z) = z \varphi'(z)/\varphi(z) = 1 + \sum_{k \in \mathbb{Z} \setminus \{0\}} p_k z^k$ satisfies $\Re p(z) > 0$ on $A(r, 1)$. It follows that functions in $s_r$, map concentric circles onto concentric circles. Such a question was raised by Nitsche [12] for univalent harmonic ring mappings and was negatively answered in [6].

Theorem 2.3 is an existence and uniqueness result for a fixed given dilatation function $a$. This is possible, since the cluster set of a boundary point of $A(r, 1)$ is either a singleton or an interval on the circle $|w| = R$ (resp. $|w| = 1$).

We shall use the following notation. Denote by $\mathcal{P}_r$ the class

$$\mathcal{P}_r = \left\{ p \in H(A(r, 1)); \Re p > 0 \text{ on } A(r, 1) \text{ and } \frac{1}{2\pi i} \int_{|z|=\rho} p(z) \frac{dz}{z} = 1 \text{ for some } \rho \in (r, 1) \right\}.$$

This class has been studied by several authors, for example, in [3, 7–10, 13]. In particular, mappings in $\mathcal{P}_r$ can be represented by the generalized Herglotz formula

$$p(z) = \int_{-\pi}^{\pi} p^*(z e^{-it}) d\mu(t) + \int_{-\pi}^{\pi} p^* \left( \frac{r}{z e^{-it}} \right) d\nu(t) - 1$$

where

$$p^*(z) = \frac{1 + z}{1 - z} + 2 \sum_{k=1}^{\infty} \frac{r^{2k}}{1 - r^{2k}} (z^k - z^{-k})$$

and where $\mu$ and $\nu$ are probability measures on the Borel $\sigma$-algebra over $[-\pi, \pi)$. Note that (1.3) reduces to the classical Herglotz formula if $r = 0$. The mapping $p^*$ is univalent on $A(r, 1)$ and maps $A(r, 1)$ onto the right half-plane cut along a vertical slit, which lies on $\{w; \Re w = 1\}$ and is bisected by the real axis.
Let \( S^*(r, 1) \) be the set of all univalent analytic functions \( \varphi \) on \( A(r, 1) \) having the properties:

(i) \( p(z) = z\varphi'(z)/\varphi(z) \in H(A(r, 1)) \),

(ii) \( \Re p(z) > 0 \) on \( A(r, 1) \).

The condition of \( \varphi \) to be univalent can be replaced by the property \( p(z) = 1 + \sum_{k \in \mathbb{Z} \setminus \{0\}} p_k z^k \). For completeness we give a short proof.

**Lemma 1.1.** \( \varphi \in S^*(r, 1) \) if and only if \( p(z) = z\varphi'(z)/\varphi(z) \in \mathcal{P}_r \).

**Proof.** If \( \varphi \in S^*(r, 1) \), it is sufficient to show that \( p_0 = 1 \). Indeed, we have for \( \rho \in (r, 1) \)

\[
p_0 = \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{z\varphi'(z)\,dz}{\varphi(z)} = \frac{1}{2\pi} \oint_{|z|=\rho} d\arg(\rho e^{it}) = 1.
\]

Next, suppose that \( p(z) = z\varphi'(z)/\varphi(z) = 1 + \sum_{k \in \mathbb{Z} \setminus \{0\}} p_k z^k \in H(A(r, 1)) \) and that \( \Re p(z) > 0 \) on \( A(r, 1) \). Then \( \varphi \in H(A(r, 1)) \). We show that \( \varphi \) is univalent on \( A(r, 1) \). Fix \( \rho \in (r, 1) \). Since \( \Re p(z) > 0 \) on \( A(r, 1) \) and \( p_0 = 1 \), we have

\[
\frac{1}{2\pi} \oint_{|z|=\rho} d\arg(\varphi(z)) = \frac{1}{2\pi} \oint_{|z|=\rho} d\arg(\varphi(z)) = p_0 = 1.
\]

Moreover, we conclude from the relation

\[
\frac{\partial}{\partial t} \frac{\arg(\rho e^{it})}{\arg(\varphi(z))} = \Re \left\{ \frac{\rho e^{it}\varphi'(\rho e^{it})}{\varphi(\rho e^{it})} \right\} > 0
\]

that \( \varphi \) is univalent on \( \{z : |z| = \rho\} \) and that \( \Gamma_\rho = \{w : w = \varphi(\rho e^{it}) ; 0 \leq t \leq 2\pi\} \) is a simple closed strictly starlike curve winding once around the origin. Finally, for \( r < \rho_1 < \rho_2 < 1 \), we have \( \Gamma_{\rho_1} \cap \Gamma_{\rho_2} = \emptyset \). Indeed, \( \varphi(\rho_1 < |z| < \rho_2) \) has to be a bounded domain. Therefore, \( \varphi \) is univalent on \( \{z : p_1 < |z| < p_2\} \) for all \( r < p_1 < p_2 < 1 \) and, hence, also on \( A(r, 1) \).

Let \( f \) be in \( \mathcal{A}_c \). Then \( F(\zeta) = \log f(e^{i\zeta}) \) is a univalent harmonic mapping from the vertical strip \( \mathcal{S}_c = \{\zeta : -c < \Re \zeta < 0\} \), \( c = -\ln r \), onto the vertical strip \( \mathcal{S}_{-\log R} \). Furthermore, \( F \) is a solution of the elliptic partial differential equation

\[
(1.5) \quad F_{\zeta} = aF_\zeta,
\]

where the second dilatation function \( a \) satisfies

\[
(1.6) \quad a \in H(\mathcal{S}_c), \quad |a| < 1, \quad \text{and} \quad a(\zeta + 2\pi i) = a(\zeta) \text{ on } \mathcal{S}_c.
\]

Such mappings \( F \) appear in a natural way in studying the nonparametric minimal surfaces over a vertical strip whose normal directions are periodic. (For more details see [4, 12].)

Suppose now that \( a \) satisfies (1.6) and that \( F = U + iV \) is a univalent solution of (1.5) which maps \( \mathcal{S}_c \) onto \( \mathcal{S}_d \) for some \( d > 0 \) such that \( \lim_{\zeta \to \infty} U(\zeta + in) = 0 \) and \( \lim_{\zeta \to -\infty} U(\zeta + in) = -d \). We show in Theorem 3.1 that there is a \( \delta > 0 \) such that the function \( f(z) = \exp[\frac{2\pi i}{d} F(\log z)] \) belongs to \( \mathcal{A}_c \).
In what follows we shall frequently use the expression $|z|^{2\beta}$. We always take the branch corresponding to $1^{2\beta} = 1$.

2. Logharmonic ring mappings

We start with a characterization of univalent logharmonic ring mappings.

**Theorem 2.1.** A function $f$ belongs to $\mathcal{A}$ if and only if

$$f(z) = z|z|^{2\beta} h(z)/\overline{h(z)},$$

where

(a) $h \in H(A(r, 1))$ and $0 \notin h(A(r, 1))$;
(b) $\Re(z h'(z)/h(z)) > -1/2$ on $A(r, 1)$;
(c) $\frac{1}{2\pi} \int_{|z|=\rho} d\arg h(z) = 0$, $r < \rho < 1$;
(d) $\Re \beta > -1/2$.

In particular, functions belonging to $\mathcal{A}$ map concentric circles onto concentric circles.

**Proof.** Let $f \in \mathcal{A}$, and suppose that $f(A(r, 1)) = A(R, 1)$. Then we have $f(z) = H(z)G(z)$ where $H$ and $G$ are locally analytic (possibly multivalued) functions on $A(r, 1)$. Furthermore, because $f(z) \neq 0$ on $A(r, 1)$, the functions

$$\frac{f_z}{f} = \frac{H'}{H} = \sum_{k \in \mathbb{Z}} a_k z^k, \quad \frac{\overline{f_z}}{f} = \frac{G'}{G} = \sum_{k \in \mathbb{Z}} b_k z^k$$

are analytic on $A(r, 1)$. By integration and exponentiation we get formally

$$f(z) = z^{a-1} |z|^{2\beta} h(z)g(z)$$

where $h$ and $g$ belong to $H(A(r, 1))$ and satisfy

$$\frac{1}{2\pi} \int_{|z|=\rho} d\arg h(z) = \frac{1}{2\pi} \int_{|z|=\rho} d\arg g(z) = 0$$

for all $\rho \in (r, 1)$. Therefore, the conditions (a) and (c) hold.

The identity $f(ze^{2\pi i}) = f(z)$ implies that $e^{2\pi i(a_{-1} - b_{-1})} = 1$ and that $a_{-1} - b_{-1} = n \in \mathbb{Z}$. So far, we have the representation

$$f(z) = z^n |z|^{2\beta} h(z)g(z), \quad n \in \mathbb{Z}, \ \beta = \overline{b_{-1}}.$$ 

Since

$$|h\overline{g}|_{|z|=r} \equiv R/r^{n+2} \Re \beta \quad \text{and} \quad |h\overline{g}|_{|z|=1} \equiv 1,$$

we conclude that $hg = e^{i\alpha} z^m$, $\alpha \in \mathbb{R}$, $m \in \mathbb{Z}$. But (2.2) shows that $m = 0$, and we are lead to the representation

$$f(z) = e^{i\alpha} z^n |z|^{2\beta} \frac{h(z)}{\overline{h(z)}}, \quad h \in H(A(r, 1)).$$

Finally, we have

$$1 = \int_{|z|=\rho} d\arg f(z) = n + 2 \int_{|z|=\rho} d\arg h(z) = n$$
for all \( \rho \in (r, 1) \), from which it follows that \( n = 1 \). Replacing \( h(z) \) by \( e^{-i\alpha/2}h(z) \), we get the representation formula

\[
f(z) = z|z|^{2\beta}h(z)/h(z).
\]

The relation (2.1) implies that \( |f(re^{i\theta})| = R = r^{1+2\Re \beta} \), and the relation \( \Re \beta = (1/2)[(\log R)/\log r - 1] \) > \(-1/2 \) shows that (d) holds. In order to prove (b), first observe that \( |f(z)||z| = p^{1+2\Re \beta} \), \( p \in (r, 1) \). Since \( f \) is univalent on \( A(r, 1) \) and \( \int_{|z| = \rho} d\arg f(z) = 1 \), we get for \( z = \rho e^{it} \in A(r, 1) \)

\[
(2.5) \quad \frac{\partial \arg f(\rho e^{it})}{\partial t} = \Im \left( \frac{zf_z}{f} - \frac{\overline{z}f_{\overline{z}}}{f} \right) = \Re \left( \frac{zf_z}{f} - \frac{\overline{z}f_{\overline{z}}}{f} \right)
= \Re \left( z \left( \frac{1 + \beta}{z} + \frac{h'}{h} - \frac{\beta}{z} + \frac{h'}{h} \right) \right) = \Re \left( 1 + 2 \frac{zh'(z)}{h(z)} \right) > 0,
\]

and (b) is proved. The necessity part is established.

Conversely, assume that \( f(z) = z|z|^{2\beta}h(z)/h(z) \) and that the conditions (a)-(d) are satisfied. We have to prove that \( f \) is a univalent logharmonic mapping from \( A(r, 1) \) onto \( A(R, 1) \) for some \( R \in (0, 1) \) and that \( \int_{|z| = \rho} d\arg f(z) = 1 \) for all \( \rho \in (r, 1) \). First of all, we see that \( f \) maps any circle \( \{|z| = \rho\} \), \( \rho \in (r, 1) \), into the circle \( \{|w| = p^{1+2\Re \beta}\} \). Let us show that \( f \) is a solution of (1.1). By (2.1), we have

\[
a = \frac{\overline{f_z}/f}{\overline{f_{\overline{z}}}/f} = \frac{\overline{\beta} - zh'/h}{1 + \beta + zh'/h} \in H(A(r, 1)),
\]

and, since \( \Re \beta > -1/2 \) and \( \Re(zh'(z)/h(z)) > -1/2 \) on \( A(r, 1) \), elementary calculations show that \( |a| < 1 \) on \( A(r, 1) \). Next, we show that \( f \) is univalent on \( A(r, 1) \). Using the relation (2.5) and condition (b), we conclude that each circle \( \{z : |z| = \rho\} \), \( \rho \in (r, 1) \), is mapped monotonically onto the circle \( \{w : |w| = p^{1+2\Re \beta}\} \), and, from condition (c), we deduce that \( f \) is univalent in \( A(r, 1) \) and that \( \int_{|z| = \rho} d\arg f = 1 \). Therefore, \( f \in \mathcal{A}_r \). This completes the proof.

Another version of Theorem 2.1 is the following representation.

**Theorem 2.2.** A function \( f \) is in \( \mathcal{A}_r \) if and only if it is of the form

\[
f(z) = \left( \frac{\varphi(z)}{|\varphi(z)|} |z|^{2\gamma} \right),
\]

where \( \Re \gamma > 0 \) and \( \varphi \in S^*(r, 1) \).

**Proof.** Let \( f \) be in \( \mathcal{A}_r \). Then, by Theorem 2.1, we have

\[
f(z) = z|z|^{2\beta}h(z)/h(z),
\]

where \( \Re \beta > -1/2 \), \( \Re(zh'(z)/h(z)) > -1/2 \) on \( A(r, 1) \), and \( \int_{|z| = \rho} d\arg h(z) = 0 \) for all \( \rho \in (r, 1) \). Put \( \varphi(z) = zh^2(z) \). It follows that

\[
f(z) = z|z|^{2\beta}h^2(z)/|h(z)|^2 = \left( \frac{\varphi(z)}{|\varphi(z)|} |z|^{2\beta+1} \right) = \left( \frac{\varphi(z)}{|\varphi(z)|} |z|^{2\gamma} \right),
\]

Another version of Theorem 2.1 is the following representation.
where \( \Re \gamma > 0 \). On the other hand, we have

\[
(2.8) \quad \Re(z\varphi'(z)/\varphi(z)) = 1 + 2 \Re(zh'(z)/h(z)) > 0
\]

for all \( z \in A(r, 1) \). Moreover, putting \( p(z) = z\varphi'(z)/\varphi(z) \) we get

\[
(2.9) \quad p_0 = \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{p(z)}{z} \, dz = \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{\varphi'(z)}{\varphi(z)} \, dz = \frac{1}{2\pi} \oint_{|z|=\rho} d \arg \varphi(z)
\]

which implies (Lemma 1.1) that \( \varphi \in S^*(r, 1) \).

Conversely, let \( f = |z|^{2\gamma} \varphi/|\varphi|, \Re \gamma > 0 \), and \( \varphi \in S^*(r, 1) \). We have to show that \( h(z) = \sqrt[2]{\varphi(z)/z} \) can be defined as a single-valued function on \( A(r, 1) \). Observe first that

\[
h(z) = \frac{[z\varphi'/\varphi - 1]}{2} \in H(A(r, 1)).
\]

By (2.9), we have

\[
\frac{1}{2\pi} \oint_{|z|=\rho} d \arg h = \frac{1}{2} \left\{ \left[ \frac{1}{2\pi} \oint_{|z|=\rho} d \arg \varphi \right] - 1 \right\} = \frac{p_0 - 1}{2} = 0.
\]

Therefore, we can define \( \log h \) and, hence, also \( h \) as an analytic function on \( A(r, 1) \). Evidently, \( h \) does not vanish on \( A(r, 1) \); so we have shown properties (a) and (c) of \( h \) in Theorem 2.1. The properties (b) and (d) of \( h \) and the form (2.1) of \( f \) follow from the relations (2.7) and (2.8). Finally, we apply Theorem 2.1 and the proof is complete.

For the next result, we fix the second dilatation function \( a \in H(A(r, 1)) \), \( |a(z)| < 1 \) for all \( z \in A(r, 1) \).

**Theorem 2.3.** For a given \( a \in H(A(r, 1)) \), \( |a(z)| < 1 \) for all \( z \in A(r, 1) \) and for a given \( z_0 \in A(r, 1) \), there exists one and only one univalent solution \( f \) of (1.1) in \( \mathcal{A}_r \) such that \( f(z_0) > 0 \).

**Remark.** Theorem 2.3 is not true for univalent harmonic ring mappings (see [6, Theorem 7.3]).

In the proof of Theorem 2.3 we shall need the following lemma.

**Lemma 2.4.** There exists a \( \beta \) depending on \( a \) such that \( \Re \beta > -1/2 \) and

\[
I = \int_0^{2\pi} \frac{\beta - (1 + \beta)a(\rho e^{it})}{1 + a(\rho e^{it})} \, dt = 0
\]

for all \( \rho \in (r, 1) \).

**Proof.** Observe that \( I \) does not depend on \( \rho \) since the integrand is analytic on \( A(r, 1) \). Furthermore (writing \( a(\rho e^{it}) = a \)), the condition

\[
(2.10) \quad I = \int_0^{2\pi} \frac{\beta - |a|^2 - \beta a - a + \bar{a} \beta - \beta |a|^2}{|1 + a|^2} \, dt = 0
\]

is satisfied for

\[
\Re \beta = \left( \int_0^{2\pi} \frac{\Re a + |a|^2}{|1 + a|^2} \, dt \right) \left( \int_0^{2\pi} \frac{1 - |a|^2}{|1 + a|^2} \, dt \right)^{-1}
\]

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and
\[ \text{Im } \beta = -\frac{(1 + 2 \text{Re } \beta)}{2\pi} \int_0^{2\pi} \frac{\text{Im } a}{|1 + a|^2} \, dt. \]

Furthermore, we have
\[ 1 + 2 \text{Re } \beta = 2\pi \left( \int_0^{2\pi} \frac{1 - |a|^2}{|1 + a|^2} \, dt \right)^{-1} > 0, \]
and the lemma is shown.

**Proof of Theorem 2.3.** Consider the function
\[ h(z) = \exp \left( \int \frac{\beta - (1 + \beta)a(z)}{z(1 + a(z))} \, dz \right), \]
where \( \beta \) is defined in Lemma 2.4. Since
\[ \frac{1}{2\pi} \oint_{|z|=\rho} d\arg h(z) = \frac{1}{2\pi} \oint_{|z|=\rho} \frac{\beta - (1 + \beta)a(z)}{z(1 + a(z))} \, dz = \frac{I}{2\pi} = 0, \]
it follows that \( h \in H(A(r, 1)) \) and that
\[ \text{Re} \left( \frac{zh'(z)}{h(z)} + \frac{1}{2} \right) = \frac{1 - |a|^2}{|1 + a|^2} \left( \frac{1}{2} + \text{Re } \beta \right) > 0 \quad \text{on } A(r, 1). \]

This implies that \( \text{Re}(zh'(z)/h(z)) > -1/2 \) on \( A(r, 1) \). Applying Theorem 2.1, we conclude that the function \( \hat{f} \) defined by
\[ \hat{f}(z) = z|z|^{2\beta} h(z)/h(z) \]
belongs to \( \mathcal{A}_\beta \) and is a solution of (1.1). Finally, take
\[ f(z) = (\hat{f}(z_0)/|\hat{f}(z_0)|) \hat{f}(z), \]
and the proof of the existence is established.

We now show the uniqueness of the mapping. Suppose that \( f_1 \) and \( f_2 \) are two solutions of (1.1) satisfying the conditions of Theorem 2.3.

First, we show that \( f_1(z_0) = f_2(z_0) \), which implies that \( \text{Re } \beta \) is unique, since
\[ f_1(z_0) = f_2(z_0) = r^{1 + 2 \text{Re } \beta}. \]

Suppose that the contrary holds. Put
\[ \frac{f_1(z_0)}{f_2(z_0)} = |z_0|^{2\text{Re}(\beta_1 - \beta_2)} = s > 0 \quad \text{and} \quad \Phi(z) = s \cdot \frac{f_2(z)}{f_1(z)}. \]

The function \( \Phi \) is again a nonconstant solution of (1.1) with respect to the same dilatation function \( a \) and is hence an orientation-preserving open mapping. We have \( |\Phi(\rho e^{it})| = (\rho/|z_0|)^{2 \text{Re}(\beta_2 - \beta_1)} \) for all \( \rho \in (r, 1) \). With no loss of generality, we may assume that \( \text{Re}(\beta_2 - \beta_1) > 0 \); if not, consider \( 1/\Phi \). Then we have \( |\Phi(e^{it})| > 1 \) and \( |\Phi(\rho e^{it})| < 1 \). Choose \( r < \rho_1 < \rho_2 < 1 \) such that \( |\Phi(\rho_1 e^{it})| < 1 \) and \( |\Phi(\rho_2 e^{it})| > 1 \). Then, by the argument principle, we have
\[ J = \frac{1}{2\pi} \oint_{|z|=\rho_2} d\arg \Phi(z) - 1 - \frac{1}{2\pi} \oint_{|z|=\rho_1} d\arg \Phi(z) - 1 \geq 1, \]
since $\rho_1 < |z_0| < \rho_2$ and $\Phi(z_0) = 1$. On the other hand,
\[
\frac{1}{2\pi} \oint_{|z| = \rho_2} d\arg(f_1) - \frac{1}{2\pi} \oint_{|z| = \rho_1} d\arg(f_1) = 0
\]
and, therefore,
\[
J = \frac{1}{2\pi} \oint_{|z| = \rho_2} d\arg(sf_2 - f_1) - \frac{1}{2\pi} \oint_{|z| = \rho_1} d\arg(sf_2 - f_1).
\]
Since, for $|z_0| \neq \rho$,
\[
s|f_2(\rho e^{it})| - |f_1(\rho e^{it})| = \rho^{1+2\Re(\beta_1)} \left( \left( \frac{|z_0|}{\rho} \right)^{2\Re(\beta_1 - \beta_2)} - 1 \right) \neq 0,
\]
we conclude that $J = 0$, which leads to a contradiction. Hence, $f_1(z_0) = f_2(z_0)$, which implies that $|\phi| \equiv 1$ on $A(r, 1)$ and $\Phi(z_0) = 1$. Since $\Phi$ is a solution of (1.1) but is not an open mapping, it follows that $\Phi \equiv 1$, and, therefore, $f_1 \equiv f_2$.

The next result gives an algebraic structure of the class $\mathcal{M}$.

**Theorem 2.5.** The family $\mathcal{M}$ is “logarithmically convex”, i.e., if $f_1, f_2 \in \mathcal{M}$, then $F = f_1 f_2^{1-\lambda} \in \mathcal{M}$ for all $\lambda \in (0, 1)$.

**Proof.** Let $\varphi_j$, $j = 1, 2$, be the univalent starlike mappings defined in Theorem 2.2 corresponding to $f_j$, $j = 1, 2$. Then the functions $p_j(z) = z\varphi_j'(z)/\varphi_j(z)$ belong to $\mathcal{P}_r$. Since $p = \lambda p_1 + (1 - \lambda)p_2$ has the same property, we conclude from Lemma 1.1 that $\varphi = \varphi_1^{\lambda} \varphi_2^{1-\lambda} \in S^*(r, 1)$. Finally, by Theorem 2.2, we have
\[
f(z) = f_1^\lambda(z) \cdot f_2^{1-\lambda}(z) = \left( \frac{\varphi(z)}{|\varphi(z)|} |z|^{2(\lambda \gamma_1 + (1-\lambda)\gamma_2)} \right) = \left( \frac{\varphi(z)}{|\varphi(z)|} |z|^{2\gamma} \right)
\]
for all $z \in A(r, 1)$, and, therefore, $f$ belongs to $\mathcal{M}$.

### 3. Univalent Harmonic Strip Mappings with Periodic Dilatations

Denote by $\mathcal{S}_c$ the vertical strip $\{ \zeta : -c < \Re \zeta < 0 \}$, $\zeta = \xi + i\eta$, and let $\Sigma_c$ be the set of all univalent harmonic and orientation-preserving mappings $F = U + iV$ from $\mathcal{S}_c$ onto $\mathcal{S}_d$ for some $d > 0$ such that $\lim_{\zeta \to 0} U(\zeta + i\eta) = 0$ and $\lim_{\zeta \to 0} U(\zeta + i\eta) = -d$. It follows that $U(\zeta) = d\xi/c$ and that there is an $a \in H(\mathcal{S}_c)$, $|a| < 1$ on $\mathcal{S}_c$, such that $F$ is a solution of the elliptic partial differential equation
\[
\frac{\partial F}{\partial \zeta} = aF_{\xi}.
\]
Furthermore, $F$ admits the representation
\[
F(\zeta) = \frac{d}{c} \left[ \xi + i \Im \int_{\zeta}^{\zeta} \frac{1 - a(s)}{1 + a(s)} \, ds \right].
\]
Indeed, (3.1) is equivalent to the equation
\[
iV_{\xi} = \frac{1 - a}{1 + a} U_{\xi}.
\]
Let $\mathcal{F}_c$ be the set of all mappings $F$ which are of the form (3.2). First, we observe that each $F \in \mathcal{F}_c$ is univalent harmonic and orientation-preserving on $\mathcal{F}_c$. Evidently, $F$ is harmonic, and we conclude from

$$F_\zeta = \frac{d}{c} \cdot \frac{a}{1+a} \quad \text{and} \quad F_\zeta = \frac{d}{c} \cdot \frac{1}{1+a}$$

that $F$ is a solution of (3.1) and, therefore, is orientation-preserving. The univalence of $F$ follows from the fact that the vertical lines $\{\zeta : \Re \zeta = \zeta_0\}$, $\zeta_0 \in (-c, 0)$, are mapped into the vertical lines $\{w : \Re w = d\zeta_0/c\}$ and that \( \frac{\partial \nu}{\partial \eta} = \Re \left( \frac{1-a}{1+a} \right) > 0 \) on $\mathcal{F}_c$.

There are mappings in $\mathcal{F}_c$ which do not belong to $\Sigma_c$. For instance,

$$F(\zeta) = \frac{i}{2}e^{2\eta} \sin(2\zeta)$$

belongs to $\mathcal{F}_{\pi/2}$ but not to $\Sigma_{\pi/2}$. Indeed, $F$ is of the form (3.2) with

$$a(s) = \frac{1 + ie^{-2is}}{1 - ie^{-2is}}, \quad s \in \mathcal{F}_{\pi/2},$$

and

$$F(\mathcal{F}_{\pi/2}) = \{w ; -\pi/2 < \Re w < 0 \text{ and } \Im w > 0\}$$

is not a vertical strip. On the other hand, by similar arguments as in [5, Theorem 2.7], $\Sigma_c$ is dense in $\mathcal{F}_c$ with respect to the topology of the locally uniform convergence.

Suppose now that $a$ satisfies the periodicity condition

$$a(\zeta + 2\pi i) = a(\zeta), \quad \zeta \in \mathcal{F}_c.$$ 

Then (3.2) defines a univalent solution of (3.1), which maps $\mathcal{F}_c$ into $\mathcal{F}_d$. Let us show that $F = U + iV \in \Sigma_c$, i.e., that $F$ maps $\mathcal{F}_c$ onto $\mathcal{F}_d$. Indeed, $F(\zeta + 2\pi i)$ is also a univalent solution of (3.1), and, since

$$F(\zeta + 2\pi i) - F(\zeta) = \frac{d}{c} i \Im \int_{\zeta}^{\zeta+2\pi i} \frac{1-a(s)}{1+a(s)} \, ds$$

is not an open mapping, it follows that

(3.3) \hspace{1cm} F(\zeta + 2\pi i) \equiv F(\zeta) + i\delta \quad \text{on } \mathcal{F}_c\]

for some $\delta \in \Re$. The univalence of $F$ and the property $\partial V/\partial \eta > 0$ imply that $\delta > 0$, from which we conclude that the vertical lines $\{\zeta : \Re \zeta = \zeta_0\}$, $\zeta_0 \in (-c, 0)$, are mapped onto the vertical lines $\{w : \Re w = d\zeta_0/c\}$.

Summarizing we have shown that:

**Theorem 3.1.** Let $a \in H(\mathcal{F}_c)$, $|a| < 1$, and $a(\zeta + 2\pi i) = a(\zeta)$ on $\mathcal{F}_c$. Then (3.2) defines a univalent solution $F$ of (3.1) which belongs to $\Sigma_c$. Furthermore, any solution of (3.1) which belongs to $\Sigma_c$ is of the form (3.2), and there is a $\delta > 0$ such that (3.3) holds.

Define

$$\Sigma_{c, \mathcal{F}} = \{F \in \Sigma_c ; \ a(\zeta + 2\pi i) = a(\zeta) \text{ on } \mathcal{F}_c\}. $$
For \( F \in \Sigma_{c,\mathcal{F}} \), consider the mapping

\[
(3.4) \quad f(z) = \exp \left[ \frac{2\pi}{\delta} F(\log z) \right],
\]

where \( \delta \) is defined in (3.3). Then \( f \) is a univalent and logharmonic mapping on \( A(e^{-c}, 1) \) and belongs to the class \( \mathcal{A}_r \), \( r = e^{-c} \). Conversely, if \( f \in \mathcal{A}_r \) then \( F(\zeta) = \log f(e^{\delta}) \in \Sigma_{c,\mathcal{F}}, \ c = -\log r \). It is easy to transfer results from one class to the other. For instance, we may get a second proof of the existence in Theorem 2.3 as follows:

Let \( a_1 \in H(\mathcal{A}(r, 1)), |a_1| < 1 \) on \( \mathcal{A}(r, 1) \), and let \( z_0 \in \mathcal{A}(r, 1) \) be given. Put \( a(s) = a_1(e^s) \), and let \( F \) be defined by (3.2) and \( f \) by (3.4). Then

\[
(\frac{|f(z_0)|}{|f(z_0)|}) = f \text{ is the desired solution.}
\]

Observe that \( \Sigma_{c,\mathcal{F}} \) and \( \mathcal{A}_r \) are not compact families with respect to the topology of locally uniform convergence. However, we may still estimate some functionals over the classes \( \Sigma_{c,\mathcal{F}} \) and \( \mathcal{A}_r \), since \( \mathcal{P}_r \) is compact. For instance, the problem

\[
M(\rho, r) = \max_{f \in \mathcal{P}_r(\mathcal{A}(r, 1))} \max_{0 \leq t \leq 2\pi} \left\{ \frac{\arg e^{-it} f(p e^{it})| \bmod 2\pi} \right\}, \quad r < \rho < 1,
\]

has been solved in [3]. Using Theorem 2.2, we get immediately that

\[
|\arg e^{-it} f(p e^{it})| \bmod 2\pi \leq M(\rho, r) + 2|\Im \gamma| \ln \rho
\]

for all \( f \in \mathcal{A}_r \).

**References**


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