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A REMARK ON WEIGHTED INEQUALITIES FOR GENERAL MAXIMAL OPERATORS

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ABSTRACT. Let $1 < p < \infty$, and let w, v be two nonnegative functions. We give a sufficient condition on w, v for which the general maximal operator $M_{\mathscr{B}}$ is bounded from $L^p(v)$ into $L^p(w)$. Our condition is stronger but closely related to the $A_{p,\mathscr{B}}$ condition for two weights.

1. INTRODUCTION

Let $\mathscr{Q}$ be the family of all open cubes in $\mathbf{R}^n$ with sides parallel to the axes, and let $M_{\mathscr{Q}} = M$ denote the Hardy-Littlewood maximal operator. According to a fundamental theorem of Sawyer [11], M is a bounded operator from $L^p(v)$ into $L^p(w)$, $1 < p < \infty$, if and only if $(w, v) \in S_p$, that is, there is a positive constant c such that

$$(1) \qquad \int_Q M(v^{1-p'}\chi_Q)(y)^p w(y)\,dy \le c\int_Q v(y)^{1-p'}\,dy, \qquad Q \in \mathscr{Q}.$$

On the other hand, it is well known that Muckenhoupt's A_p condition for two weights,

$$(2) \qquad \left(\frac{1}{|Q|}\int_Q w(y)\,dy\right)^{1/p}\left(\frac{1}{|Q|}\int_Q v(y)^{1-p'}\,dy\right)^{1/p'} \le c, \qquad Q \in \mathscr{Q},$$

is not equivalent to S_p unless $v = w$ (cf. [3; 2, p. 433]). One problem with Sawyer's condition is that it is very difficult to test in practice since it involves the operator M on it. It would be interesting to obtain sufficient conditions close in form to (2). In [10] we initiated this program and showed that it is enough to consider conditions such as (2) but replacing the local $L^{p'}$ average norm involved on the weight v by appropriate stronger norms. An antecedent of these results can be found in Neugebauer's paper [8].

This note is devoted to studying the corresponding two weight problem

$$(3) \qquad \int_{\mathbf{R}^n} M_{\mathscr{B}}f(y)^p w(y)\,dy \le c\int_{\mathbf{R}^n} |f(y)|^p v(y)\,dy,$$

where $\mathscr{B}$ is a general basis. By a basis $\mathscr{B}$ in $\mathbf{R}^n$, we mean a collection of open sets in $\mathbf{R}^n$. The study of general maximal operators $M_{\mathscr{B}}$ arises in many

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situations in Fourier Analysis where the geometry involved is other than the one given by the cubes or balls. Of course, in one-parameter Fourier Analysis the basis $\mathscr{Q}$ plays a central role together with the basis $\mathscr{D}$ of all dyadic cubes. A corresponding role in multiparameter Fourier Analysis is played by the basis $\mathscr{R}$ of all rectangles with sides parallel to the axes. An example of an "exotic" but interesting basis is given by the Córdoba-Zygmund basis $\mathfrak{R}$ in $\mathbf{R}^3$ of all rectangles with sidelengths of the form $\{s, t, st\}$.

For a general basis $\mathscr{B}$, $M_{\mathscr{B}}$ denotes the maximal operator associated to $\mathscr{B}$ defined by

$$M_{\mathscr{B}}f(x) = \sup_{x\in B} \frac{1}{|B|}\int_B |f(y)|\,dy$$

if $x \in \bigcup_{B\in\mathscr{B}}$ and $M_{\mathscr{B}}f(x) = 0$, otherwise. We say that w is a weight associated to the basis $\mathscr{B}$ if w is a nonnegative measurable function in $\mathbf{R}^n$ such that $w(B) = \int_B w(y)\,dy < \infty$ for each B in $\mathscr{B}$. The weight w belongs to the class $A_{p,\mathscr{B}}$, $1 < p < \infty$, if there is a constant c such that

$$\left(\frac{1}{|B|}\int_B w(y)\,dy\right)\left(\frac{1}{|B|}\int_B w(y)^{1-p'}\,dy\right)^{p-1} \le c \tag{4}$$

for all $B \in \mathscr{B}$. For $p = \infty$, we set $A_{\infty,\mathscr{B}} = \bigcup_{p>1} A_{p,\mathscr{B}}$.

In [9] we introduced the following class of bases that we shall use later.

Definition 1.1. A basis $\mathscr{B}$ is a *Muckenhoupt* basis if for each $1 < p < \infty$, and every $w \in A_{p,\mathscr{B}}$,

$$M_{\mathscr{B}}: L^p(w) \to L^p(w).$$

It is shown in [9] that this definition is equivalent to saying that for each $1 < p < \infty$

$$M_{\mathscr{B},w}: L^p(w) \to L^p(w), \qquad w \in A_{\infty,\mathscr{B}}. \tag{5}$$

Here $M_{\mathscr{B},w}$ denotes the weighted maximal operator defined by

$$M_{\mathscr{B},w}f(x) = \sup_{x\in B}\frac{1}{w(B)}\int_B |f(y)|w(y)\,dy.$$

Most of the important bases are Muckenhoupt bases, and, in particular, those mentioned above: $\mathscr{Q}$, $\mathscr{D}$, $\mathscr{R}$, and $\mathfrak{R}$.

Now let (w, v) be a couple of weights associated to the basis $\mathscr{B}$. Extending a previous result of Sawyer in [12] for the strong maximal operator, Jawerth [4] gave a necessary and sufficient condition for $M_{\mathscr{B}}$ to be bounded from $L^p(v)$ into $L^p(w)$ under no restriction on $\mathscr{B}$. Let $\mathscr{F}$ be the family of all finite unions $G = \bigcup_{j=1}^N B_j$ of sets in $\mathscr{B}$; Jawerth's condition is that for some positive constant c

$$\int_G M_{\mathscr{B}}(v^{1-p'}\chi_G)(y)^p w(y)\,dy \le c\int_G v(y)^{1-p'}\,dy, \qquad G \in \mathscr{F}. \tag{6}$$

This condition is even harder to verify than Sawyer's condition S_p. In Theorem 1.3 we shall provide a simpler sufficient condition that is not necessary.

We shall use the following class of weights.

Definition 1.2. We say that a weight w associated to the basis $\mathscr{B}$ satisfies condition (A) if there are constants $0 < \lambda < 1$, $0 < c = c(\lambda) < \infty$ such that for all measurable sets E

$$\text{(A)} \qquad w(\{x \in \mathbf{R}^n : M_{\mathscr{B}}(\chi_E)(x) > \lambda\}) \leq c w(E).$$

Before stating our main theorem we shall make some remarks concerning condition (A).

This class of weights was considered by Jawerth in [4], although the unweighted version goes back to the work of Córdoba in [1]. One reason that makes condition (A) interesting is the fact that it is weaker than the $A_{\infty,\mathscr{B}}$ condition whenever the basis $\mathscr{B}$ is a Muckenhoupt basis. To see this let $w \in A_{\infty,\mathscr{B}}$. Then $w \in A_{r,\mathscr{B}}$ for some $1 < r < \infty$ and, by standard properties of the $A_{p,\mathscr{B}}$ weights,

$$\frac{|E|}{|B|} \leq c\left(\frac{w(E)}{w(B)}\right)^{1/r},$$

for each measurable subset $E \subset B \in \mathscr{B}$. It follows then that $M_{\mathscr{B}}(\chi_E)(x) \leq c(M_{\mathscr{B},w}(\chi_E)(x))^{1/r}$, and, therefore, if $\mathscr{B}$ is a Muckenhoupt basis then (5) yields, for all $\lambda > 0$,

$$\begin{aligned} w(\{x \in \mathbf{R}^n : M_{\mathscr{B}}(\chi_E)(x) > \lambda\}) &\leq w\left(\left\{x \in \mathbf{R}^n : M_{\mathscr{B},w}(\chi_E)(x) > \frac{\lambda^r}{c^r}\right\}\right) \\ &\leq \frac{c^r}{\lambda^r}\int_{\mathbf{R}^n} \chi_E(x)^r w(x)\,dx = c(\lambda)w(E), \end{aligned}$$

which is condition (A).

In fact Jawerth and Torchinsky have shown in [5] that the $A_{\infty,\mathscr{R}}$ condition is strictly stronger than condition (A). As an example they show (cf. [5, p. 270]) that if the weight w in $\mathbf{R}^n$, $n > 1$, is A_∞ in each variable except one where it is merely doubling, then w satisfies condition (A) while w does not belong to $A_{\infty,\mathscr{R}}$, as is well known.

A first result on the two weight problem was remarked by the author in [9]. We pointed out that the following generalized Fefferman-Stein's type inequality

$$\text{(7)} \qquad \int_{\mathbf{R}^n} M_{\mathscr{B}}f(y)^p w(y)\,dy \leq c\int_{\mathbf{R}^n} |f(y)|^p M_{\mathscr{B}}w(y)\,dy$$

holds assuming that $\mathscr{B}$ is a Muckenhoupt basis, and that the weight w satisfies condition (A). We recall that the classical Fefferman-Stein inequality for the Hardy-Littlewood maximal operator M has no restriction on w.

At this point we mention that a particular instance of this result was previously obtained by Lin in [6]. His result is for the strong maximal operator $M_{\mathscr{R}}$ in dimension $n = 2$ and with the weight w satisfying the $A_{\infty,\mathscr{R}}$ condition.

The main result of this paper is

Theorem 1.3. *Let $1 < p < \infty$, and let $\mathscr{B}$ be a general basis satisfying $M_{\mathscr{B}}$: $L^s(\mathbf{R}^n) \to L^s(\mathbf{R}^n)$ for all $1 < s < \infty$. Suppose that (w, v) is a couple of weights such that w satisfies* (A), *and that for some $1 < r < \infty$ there is a contant c such that*

$$\text{(8)} \qquad \frac{1}{|B|}\int_B w(y)\,dy \left(\frac{1}{|B|}\int_B v(y)^{(1-p')r}\,dy\right)^{(p-1)/r} \leq c,$$

for all $B \in \mathscr{B}$. *Then*

$$M_{\mathscr{B}}: L^p(v) \to L^p(w). \tag{9}$$

It is easy to check that $(w, M_{\mathscr{B}}w)$ satisfies (8), and then we have

Corollary 1.4. *Let* $1 < p < \infty$, *and let* $\mathscr{B}$ *be a general basis satisfying* $M_{\mathscr{B}}$: $L^s(\mathbf{R}^n) \to L^s(\mathbf{R}^n)$ *for all* $1 < s < \infty$. *Suppose that* w *is a weight that satisfies condition* (A). *Then*

$$\int_{\mathbf{R}^n} M_{\mathscr{B}}f(y)^p w(y)\,dy \le c \int_{\mathbf{R}^n} f(y)^p M_{\mathscr{B}}w(y)\,dy.$$

In particular, if $\mathscr{B}$ is a Muckenhoupt basis then we always have that $M_{\mathscr{B}}$: $L^s(\mathbf{R}^n) \to L^s(\mathbf{R}^n)$, $1 < s < \infty$, yielding (7) as a corollary.

Corollary 1.5. *Let* $1 < p < \infty$, *and let* $\mathscr{B}$ *be a Muckenhoupt basis. Suppose that* w *is a weight that satisfies condition* (A). *Then*

$$\int_{\mathbf{R}^n} M_{\mathscr{B}}f(y)^p w(y)\,dy \le c \int_{\mathbf{R}^n} f(y)^p M_{\mathscr{B}}w(y)\,dy.$$

2. Proof of the theorem

First we claim that (8) implies

$$M_{\mathscr{B}}: L^p(v) \to L^{p,\infty}(w), \tag{10}$$

where $L^{p,\infty}(w)$ is the weighted Lorentz space defined by all the functions f such that $\sup_{\lambda>0}(\lambda^p w(\{x \in \mathbf{R}^n : M_{\mathscr{B}}f(x) > \lambda\})) < \infty$.

To prove (10) we just need to prove that there is a constant c such that

$$w(K) \le \frac{c}{t^p} \int_{\mathbf{R}^n} f(y)^p v(y)\,dy,$$

for each $t > 0$, $f \ge 0$, and for any compact subset K of $\{x \in \mathbf{R}^n : M_{\mathscr{B}}(f)(x) > t\}$. By the compactness of K and the definition of $M_{\mathscr{B}}f$, we can find a finite collection $B_1, \dots, B_N$ such that

$$K \subset \bigcup_{j=1}^{N} B_j \quad \text{and} \quad t < \frac{1}{|B_j|} \int_{B_j} f(y)\,dy, \tag{11}$$

for each $j = 1, \dots, N$. We now follow a well-known selecting procedure argument (cf. [2, p. 463] for instance). Let $\widetilde{B}_1 = B_1$ and, once $\widetilde{B}_1, \dots, \widetilde{B}_{k-1}$ have been selected, we choose $\widetilde{B}_k$ to be the first set in the given sequence (if any) such that

$$\left| \widetilde{B}_k \cap \left(\bigcup_{j=1}^{k-1} \widetilde{B}_j \right) \right| < \lambda |\widetilde{B}_k|.$$

Now we claim that

$$\bigcup_{j=1}^{N} B_j \subset \{x \in \mathbf{R}^n : M_{\mathscr{B}}(\chi_{\bigcup_{j=1}^{M} \widetilde{B}_j})(x) \ge \lambda\}. \tag{12}$$

Let $x \in \bigcup_{j=1}^{N} B_j$; if x belongs to some $\widetilde{B}_k$ it is of course obvious that it is contained on the set to the right since $\lambda < 1$. If, on the other hand, $x \in B_j$

for some B_j that has been discarded in the selection process, we must have $|B_j \cap (\bigcup_{j=1}^M \widetilde{B}_j)| \geq \lambda |B_j|$, and therefore $M_{\mathscr{B}}(\chi_{\bigcup_{j=1}^M \widetilde{B}_j})(x) \geq \lambda$. Now, since w satisfies condition (A), (12) yields

$$w\left(\bigcup_{j=1}^N B_j\right) \leq cw\left(\bigcup_{j=1}^M \widetilde{B}_j\right).$$

This together with (11), (8), and Hölder's inequality yields the estimate

$$\begin{aligned}
w(K) &\leq cw\left(\bigcup_{j=1}^M \widetilde{B}_j\right) \leq c\sum_j \left(\frac{1}{t|\widetilde{B}_j|}\int_{\widetilde{B}_j} f(y)\,dy\right)^p w(\widetilde{B}_j) \\
&= c\frac{1}{t^p}\sum_j \left(\frac{1}{|\widetilde{B}_j|}\int_{\widetilde{B}_j} f(y)v(y)^{1/p}v(y)^{-1/p}\,dy\right)^p w(\widetilde{B}_j) \\
&\leq c\frac{1}{t^p}\sum_j \left(\frac{1}{|\widetilde{B}_j|}\int_{\widetilde{B}_j} f(y)^{(p'r)'}v(y)^{(p'r)'/p}\,dy\right)^{p/(p'r)'} \\
&\quad\times \left(\frac{1}{|\widetilde{B}_j|}\int_{\widetilde{B}_j} v(y)^{-p'r/p}\,dy\right)^{p/p'r} \frac{w(\widetilde{B}_j)}{|\widetilde{B}_j|}|\widetilde{B}_j| \\
&\leq cK\frac{1}{t^p}\sum_j \left(\frac{1}{|\widetilde{B}_j|}\int_{\widetilde{B}_j} f(y)^{(p'r)'}v(y)^{(p'r)'/p}\,dy\right)^{p/(p'r)'} |\widetilde{B}_j|.
\end{aligned}$$

Denote $E_j = \widetilde{B}_j \setminus \bigcup_{i=1}^{j-1} \widetilde{B}_i$, so that $\{E_j\}$ is a disjoint family with $E_j \subset \widetilde{B}_j$ and $|\widetilde{B}_j| < \frac{1}{1-\lambda}|E_j|$. Then

$$\begin{aligned}
w(K) &\leq c\frac{1}{t^p}\sum_j \left(\frac{1}{|\widetilde{B}_j|}\int_{\widetilde{B}_j} f(y)^{(p'r)'}v(y)^{(p'r)'/p}\,dy\right)^{p/(p'r)'} |E_j| \\
&\leq \frac{c}{t^p}\sum_j \int_{E_j} M_{\mathscr{B}}(f^{(p'r)'}v^{(p'r)'/p})(y)^{p/(p'r)'}\,dy \\
&\leq \frac{c}{t^p}\int_{\mathbf{R}^n} M_{\mathscr{B}}(f^{(p'r)'}v^{(p'r)'/p})(y)^{p/(p'r)'}\,dy \\
&\leq \frac{c}{t^p}\int_{\mathbf{R}^n} f(y)^p v(y)\,dy,
\end{aligned}$$

since, by hypothesis, $M_{\mathscr{B}}: L^s(\mathbf{R}^n) \to L^s(\mathbf{R}^n)$, $1 < s < \infty$.

To conclude the proof of the theorem we observe first that we always have

(13) $$M_{\mathscr{B}}: L^\infty(v) \to L^\infty(w).$$

Now, denoting condition (8) by $A_{p,r}$, we see that (w, v) satisfies $A_{\overline{p},\overline{r}}$ for some $1 < \overline{p} < p$, $1 < \overline{r} < \infty$; in fact $(p-1)/r + 1 < \overline{p} < p$, $1 < \overline{r} <$

$(\overline{p}-1)r/(p-1)$ will do it:

$$\frac{1}{|B|}\int_B w(y)\,dy\left(\frac{1}{|B|}\int_B (v(y)^{-1})^{\overline{r}/(\overline{p}-1)}\,dy\right)^{(\overline{r}/(\overline{p}-1))^{-1}}$$

$$\leq \frac{1}{|B|}\int_B w(y)\,dy\left(\frac{1}{|B|}\int_B (v(y)^{-1})^{r/(p-1)}\,dy\right)^{(r/(p-1))^{-1}} \leq K.$$

By the above argument $A_{\overline{p},\overline{r}}$ and $M_{\mathscr{B}}\colon L^s(\mathbf{R}^n)\to L^s(\mathbf{R}^n)$, $1<s<\infty$, yield

$$M_{\mathscr{B}}\colon L^{\overline{p}}(v)\to L^{\overline{p},\infty}(w).$$

This together with (13) implies $M_{\mathscr{B}}\colon L^p(v)\to L^p(w)$ by the Marcinkiewicz interpolation theorem. This concludes the proof of the theorem. □

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