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UNITARY $\mathbb{Z}^d$-ACTIONS WITH CONTINUOUS SPECTRUM

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ABSTRACT. For any unitary $\mathbb{Z}^d$-action on a Hilbert space with continuous spectrum weakly wandering vectors are dense. This wandering can be forced to occur along IP-sets. This is a generalization and strengthening of a result due to Krengel. Our method is based on the contractive mapping fixed-point theorem.

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Among other results Krengel obtained in [K] an interesting characterization of unitary operators with continuous spectrum. Let U be a unitary operator acting on a Hilbert space H. A vector $f \in H$ is called *weakly wandering with respect to* U if there exists a doubly infinite strictly increasing sequence $\{n_i\}_{i\in\mathbb{Z}}$ such that $\langle U^{n_i} f, U^{n_j} f\rangle = 0$ for all distinct i, j. Krengel proved that U has continuous spectrum (i.e., U has no eigenvectors) if and only if the weakly wandering vectors are dense in H. As a matter of fact, the proof in [K] is given for isometries; in that case the wandering occurs along one-sided increasing sequences. However, essentially the same proof when applied to unitary operators yields wandering along a double infinite sequence. Krengel's original proof is quite intricate and does not seem to be easily susceptible to generalization and refinement.

The purpose of this note is to suggest a new and quite simple approach which nonetheless allows us to strengthen Krengel's result in the following two directions.

First, we show that the result holds for unitary $\mathbb{Z}^d$-actions. The method actually works for a wide class of groups (see the concluding remarks in §4). We confine ourselves to $\mathbb{Z}^d$-actions to keep this note short and to make clear the core of our approach. A Krengel-type theorem for d-parametric groups was also proved by Graham [G]. However, the proof in [G] is even more complicated than that given in [K] and does not seem to be generalizable to, say, noncompactly generated abelian groups.

Second, we show that sequences along which wandering occurs may be guaranteed to have an additional arithmetic structure, namely, that of an IP-set.

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Definition 1. Given an infinite set $A \subset \mathbb{Z}^d$, the *IP-set* Γ *generated by* A is the set

$$\Gamma = \{x_1 + \cdots + x_k | x_i \in A,\ x_i \neq x_j \text{ for } i \neq j,\ 1 \leq i, j \leq k,\ k \in \mathbb{N}\}$$

of all finite sums of distinct elements from A. We shall often use the notation $\Gamma = FS(A)$. An IP-set Γ is called *symmetric* if $\Gamma = FS(B)$, where B is a symmetric subset in $\mathbb{Z}^d$ (i.e., $x \in B \Rightarrow -x \in B$).

IP-sets may be viewed as generalized subsemigroups of $\mathbb{Z}^d$. The notion of IP-set has proved useful in many situations in ergodic theory and topological dynamics, where it turns out to be helpful to study iterations of operators along IP-sequences (cf. [F]).

Definition 2. A family of operators $\{U_n\}_{n\in\mathbb{Z}^d}$ on a Hilbert space H is called a $\mathbb{Z}^d$-*action* if $U_{n+m} = U_n U_m$ $\forall n, m \in \mathbb{Z}^d$. Such an action is called *unitary* if all U_n, $n \in \mathbb{Z}^d$, are unitary operators.

A nonzero vector $f \in H$ is called an *eigenvector for the unitary* $\mathbb{Z}^d$-*action* $\{U_n\}_{n\in\mathbb{Z}^d}$ on H if there exist complex numbers λ_n, $n \in \mathbb{Z}^d$, such that $U_n f = \lambda_n f$ $\forall n \in \mathbb{Z}^d$.

A unitary $\mathbb{Z}^d$-action is said to *have continuous spectrum* if it has no eigenvectors.

Main Theorem. *A unitary* $\mathbb{Z}^d$*-action* $\{U_n\}_{n\in\mathbb{Z}^d}$ *on a Hilbert space* H *has continuous spectrum if and only if for any* $f \in H$ *and any* $\varepsilon > 0$ *there exits* $\tilde{f}$, $\|f - \tilde{f}\| < \varepsilon$, *and a symmetric IP-set* $\Gamma \subset \mathbb{Z}^d$ *such that*

$$\langle U_\alpha \tilde{f}, U_\beta \hat{f} \rangle = 0 \quad \textit{for all distinct } \alpha, \beta \in \Gamma.$$

In the next section we collect a few auxiliary facts. A proof of the main theorem is given in §3. Section 4 is devoted to further discussion and some concluding remarks.

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The following Proposition is well known for $\mathbb{Z}$-actions; it goes back to Koopman and von Neumann [KN]. The extension of $\mathbb{Z}^d$-actions is routine.

We recall that a set $S \subset \mathbb{Z}^m$ is said to *have density* 1 if

$$d(S) = \lim_{N\to\infty} \frac{\#(S \cap [-N, N]^m)}{(2N+1)^m} = 1.$$

Proposition. *For a unitary* $\mathbb{Z}^m$*-action* $\{U_n\}_{n\in\mathbb{Z}^m}$ *on a Hilbert space* H, *the following statements are equivalent*:

(i) *The action* $\{U_n\}_{n\in\mathbb{Z}^m}$ *has continuous spectrum.*

(ii) *For any* $f, g \in H$ *there exists a set* $S \subset \mathbb{Z}^m$ *with* $d(S) = 1$ *such that*

$$\lim_{\substack{n\to\infty \\ n\in S}} \langle U_n f, g \rangle = 0.$$

(iii) *For all* $f, g \in H$

$$\lim_{N\to\infty} \frac{1}{(2N+1)^m} \sum_{n\in[-N,N]^m} |\langle U_n f, g \rangle| = 0.$$

(iv) *If for* $f \in H$ *the orbit* $\{U_n f\}_{n\in\mathbb{Z}^d}$ *is precompact in the strong topology, then* $f = 0$.

We use the following notation. For an IP-set $\Gamma \subset \mathbb{Z}^d$, put $\Gamma' := \Gamma \setminus \{0\}$. For $S \subset \mathbb{Z}^d$ and $c \in \mathbb{Z}$, $c \neq 0$, put $S/c := \{n \in \mathbb{Z}^d | cn \in S\}$. For $S \subset \mathbb{Z}^d$ and $n \in \mathbb{Z}^d$, put $S - n := \{l \in \mathbb{Z}^d | l + n \in S\}$. For $S \subset \mathbb{Z}^d$ and $c \in \mathbb{N}$, put

$$cS := \{s_1 + s_2 + \cdots + s_k | s_i \in S,\ 1 \leq i \leq k,\ 1 \leq k \leq c\}.$$

In other words cS is the set of all k-element sums formed by elements from S for $k = 1, 2, \ldots, c$.

Lemma 1. *Let the set* $S \subset \mathbb{Z}^d$ *have density* 1. *Then* $d(S/c) = 1$ *for any* $c \in \mathbb{Z}$, $c \neq 0$, *and* $d(S - n) = 1$ *for any* $n \in \mathbb{Z}^d$. *If the sets* $S_1, \ldots, S_k$ *have density* 1, *then* $D(\bigcap_{1\leq i\leq k} S_i) = 1$.

Proof. The proof is straightforward.

Lemma 2. *Let* $S \subset \mathbb{Z}^d$ *have density* 1. *For any* $c \in \mathbb{N}$ *there exists a symmetric* IP*-set* $\Gamma = FS(\{\pm n_i\}_{i=1}^{\infty})$ *such that* $(c\Gamma)' \subset S$.

Proof. By Lemma 1 the set $S_1 = \bigcap_{0<|i|\leq c} S/i$ has density 1 and, in particular, is nonempty. Pick $n_1 \in S_1$. Then $in_1 \in S$ for any i, $0 < |i| \leq c$. Define $S_2 = \bigcap_{0\leq|i|\leq c,\ 0<|j|\leq c}(S - in_1)/j$. By Lemma 1 there exists $n_2 \in S_2$. Then

$$in_1 + jn_2 \in S \cup \{0\} \quad \forall i, j,\ -c \leq i, j \leq c.$$

Assume that $n_1, n_2, \ldots, n_k$ have been already picked in such a way that $\sum_{j=1}^{k} i_j n_j \subset S \cup \{0\}$ for all $\{i_j\}_1^k$, $|i_j| \leq c$. Let

$$S_{k+1} = \bigcap \left(S - \sum_{j=1}^{k} i_j n_j \right) / i_{k+1},$$

where the intersection is taken over all i_j, $1 \leq j \leq k + 1$, such that $-c \leq i_1, \ldots, i_k \leq c$ and $0 < |i_{k+1}| \leq c$.

By Lemma 1, S_{k+1} has density 1. Pick $n_{k+1} \in S_{k+1}$. Then

$$\sum_{j=1}^{k+1} i_j n_j \subset S \cup \{0\} \quad \forall i_j,\ 0 \leq j \leq k+1 \text{ with } 0 \leq |i_j| \leq c.$$

This process gives us a sequence $\{n_i\}_{i=1}^{\infty}$ such that $c\Gamma \subset S \cup \{0\}$ for $\Gamma = FS(\{\pm n_i\}_{i=1}^{\infty})$. □

Lemma 3. *Let* $\{U_n\}_{n\in\mathbb{Z}^d}$ *be a unitary* $\mathbb{Z}^d$*-action on a Hilbert space* H *with continuous spectrum. Let* $f \in H$, $c \in \mathbb{N}$. *Then for any* $\eta > 0$ *there exists a symmetric* IP*-set* $\Gamma = FS(\{\pm n_i\}_{i=1}^{\infty})$ *such that* $\sum_{n\in(c\Gamma)'} |\langle U_n f, f\rangle| < \eta$.

Proof. By the Proposition the set $\{n \in \mathbb{Z}^d | \, |\langle U_n f, f\rangle| < \theta\}$ has density 1 for any $\theta > 0$. Then by Lemma 1, for any $\theta > 0$ and any finite set $F \subset H$, the set

$$A_{F,\theta} = \left\{ n \in \mathbb{Z}^d \,\middle|\, \sum_{0<|i|\leq c} \sum_{g\in F} |\langle U_{in} g, f\rangle| < \theta \right\}$$

also has density 1. Let $n_1 \in A_{F_1, \eta/2}$, where F_1 is the singleton $\{f\}$. If for $j \geq 1$ the points $n_1, \dots, n_j$ have already been picked, put $\Gamma_j = FS\{\pm n_i\}_{i=1}^{j}$, then take $F_j = \{U_n f | n \in \Gamma_j\}$, and pick $n_{j+1} \in A_{F_j}, \eta/2^j$. Continuing in this way we get an infinite sequence $\{n_i\}_{i=1}^{\infty}$ such that $\Gamma = FS(\{\pm n_i\}_{i=1}^{\infty})$ gives us the desired symmetric IP-set. □

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Proof of Main Theorem. We give the "only if" part only. The "if" part follows easily from (iv) of the proposition in §2.

Without loss of generality we may assume that $\langle f, f\rangle = 1$.

Let ε, $0 < \varepsilon < 1$, be given. Put $c = 6$ and $\eta = \varepsilon/4$, and by Lemma 3 choose a symmetric IP-set

$$\Gamma = FS(\{\pm n_i\}_{i=1}^{\infty}) \quad \text{such that} \sum_{n \in (6\Gamma')} |\langle U_n f, f\rangle| < \varepsilon/4.$$

We shall show that there exist complex numbers h_α,

$$h_{-\alpha} = \overline{h_\alpha}, \qquad \alpha \in E = \Gamma - \Gamma, \tag{1}$$

such that

$$\sum_{\alpha \in E} |h_\alpha| < \varepsilon \tag{1'}$$

and

$$\langle U_n \tilde{f}, U_m \tilde{f}\rangle = 0 \quad \text{for any distinct } n, m \in \Gamma, \tag{2}$$

where

$$\tilde{f} = f + \sum_{\alpha \in E} h_\alpha U_\alpha f. \tag{3}$$

Inequality (1′) implies that $\|f - \tilde{f}\| < \varepsilon$ and $\|\tilde{f}\| > 1 - \varepsilon > 0$, but we require more, namely,

$$\langle \tilde{f}, \tilde{f}\rangle = 1. \tag{4}$$

If $\tilde{f}$ satisfies (3), then conditions (1), (2), (4) are equivalent to the infinite system of nonlinear (quadratic) equations in the indeterminates h_α, $\alpha \in E$. Indeed, for n, $m \in \Gamma$,

$$\begin{aligned}\langle U_n \tilde{f}, U_m \tilde{f}\rangle &= \left\langle U_n \left(f + \sum_{\alpha \in E} h_\alpha U_\alpha f\right), \; U_m \left(f + \sum_{\beta \in E} h_\beta U_\beta f\right)\right\rangle \\ &= \langle U_{n-m} f, f\rangle + \sum_{\alpha \in E} h_\alpha \langle U_{n-m+\alpha} f, f\rangle + \sum_{\beta \in E} \overline{h}_\beta \langle U_{n-m-\beta} f, f\rangle \\ &\quad + \sum_{\alpha, \beta \in E} h_\alpha \overline{h}_\beta \langle U_{n-m+\alpha-\beta} f, f\rangle.\end{aligned}$$

Put

$$b'_\lambda = \begin{cases} -\frac{1}{2} b_\lambda, & \lambda \neq 0, \\ 0, & \lambda = 0, \end{cases}$$

where $b_\lambda = \langle U_\lambda f, f\rangle$, $\lambda \in \mathbb{Z}^d$. (Notice that $|b_\lambda| \le 1$ and $b_{-\lambda} = \overline{b_\lambda}$, $\lambda \in \mathbb{Z}^d$.) With this notation conditions (2), (4) become the system of equations

$$\sum_{\alpha\in E} h_\alpha b_{\lambda+\alpha} + \frac{1}{2}\sum_{\alpha,\beta\in E} h_\alpha \overline{h}_\beta b_{\lambda+\alpha-\beta} = b'_\lambda, \qquad \lambda \in E. \tag{5}$$

Notice that condition (4) corresponds to the case $\lambda = 0$ in (5).

Now consider the space

$$l^1_{\text{symm}}(E) = \left\{ x = (x_\alpha)_{\alpha\in E},\ x_{-\alpha} = \overline{x}_\alpha \,\middle|\, \|x\| = \sum_{\alpha\in E}|x_\alpha| < \infty \right\}$$

of symmetric summable functions (sequences) on E.

The function b'_λ, $\lambda \in E$, is bounded (by $1/2$) on E and symmetric; it follows that the operator Φ: $x = (x_\alpha) \mapsto y = (y_\lambda)$, where

$$y_{-\lambda} = b'_\lambda - \sum_{\substack{\alpha\in E\\ \alpha\ne-\lambda}} b_{\lambda+\alpha}x_\alpha - \frac{1}{2}\sum_{\alpha,\beta\in E} b_{\lambda+\alpha-\beta}x_\alpha\overline{x}_\beta \tag{6}$$

is well defined as an operator from $l^1(E)$ into $l^\infty(E)$ and the image $y = \Phi x$ is a symmetric function if $x \in l^1_{\text{symm}}(E)$.

We shall show that:

(i) Φ maps $l^1_{\text{symm}}(E)$ into itself.

(ii) The ball $B_\tau = \{x \in l^1_{\text{symm}}(E) : \|x\| \le \tau\}$ is invariant under Φ if $\frac{\varepsilon}{5} \le \tau \le \varepsilon$.

(iii) Φ is contractive in B_τ if $\tau < (1-\frac{\varepsilon}{4})/(1+\frac{\varepsilon}{4})$; more precisely,

$$\|\Phi(x) - \Phi(x')\| \le \theta_1\|x - x'\|, \qquad x, x' \in B_\tau,$$

where $\theta_1 = \theta + \frac{\varepsilon}{4}(1-\theta)$, $\theta := \tau(1+\frac{\varepsilon}{4})/(1-\frac{\varepsilon}{4}) < 1$.

To prove (i) let us estimate $\|\Phi(x)\|$. By (6) we have

$$\|\Phi(x)\| = \sum_{\lambda\in E}|y_\lambda| \le \|\Sigma_0\| + \|\Sigma_1\| + \|\Sigma_2\|, \tag{7}$$

where Σ_0, Σ_1, Σ_2 are constant, linear, and quadratic parts of $\{y_\lambda\}$ respectively. Recall that $E = \Gamma - \Gamma$, $0 \in \Gamma$, and Γ is symmetric; therefore, $E = \Gamma + \Gamma = 2\Gamma \subset 2E \subset 3E \subset 6\Gamma$. Then

$$\begin{aligned}\|\Sigma_1\| &= \sum_{\lambda\in E}\left|\sum_{\substack{\alpha\in E\\ \alpha\ne-\lambda}} b_{\lambda+\alpha}x_\alpha\right| \le \sum_{\alpha\in E}|x_\alpha| \sum_{\substack{\lambda\in E\\ \lambda\ne-\alpha}}|b_{\lambda+\alpha}| \\ &\le \sum_{\alpha\in E}|x_\alpha|\left(\sum_{\lambda\in(2E)'}|b_\lambda|\right) \le \frac{\varepsilon}{4}\|x\|\end{aligned}$$

and

$$\begin{aligned}\|\Sigma_2\| &= \frac{1}{2}\sum_{\lambda\in E}\left|\sum_{\alpha,\beta\in E} b_{\lambda+\alpha-\beta}x_\alpha\overline{x_\beta}\right| \le \frac{1}{2}\sum_{\alpha,\beta\in E}|x_\alpha|\,|\overline{x}_\beta|\left(1 + \sum_{\substack{\lambda\in E\\ \lambda\ne\beta-\alpha}}|b_{\lambda+\alpha+\beta}|\right) \\ &\le \frac{1}{2}\sum_{\alpha,\beta\in E}|x_\alpha|\,|x_\beta|\left(1 + \sum_{\lambda\in(3E)'}|b_\lambda|\right) \le \frac{1}{2}\left(1+\frac{\varepsilon}{4}\right)\|x\|^2.\end{aligned}$$

Of course,

$$\|\Sigma_0\| \le \frac{1}{2} \sum_{\lambda\in E'} |b_\lambda| \le \frac{\varepsilon}{8}.$$

Therefore,

$$\|\Phi(x)\| \le \frac{\varepsilon}{8} + \frac{\varepsilon}{4}\|x\|_1 + \frac{1}{2}\left(1+\frac{\varepsilon}{4}\right)\|x\|^2 \quad \forall x \in l^1_{\mathrm{symm}}(E),$$

and $\Phi(x)$ is symmetric; this proves (i).

If $x \in B_\tau$, i.e., $\|x\| \le \tau$, then

$$\begin{aligned}\|\Phi(x)\| &\le \frac{\varepsilon}{8} + \frac{\varepsilon}{4}\tau + \frac{1}{2}\left(1+\frac{\varepsilon}{4}\right)\tau^2 \\ &= \frac{\varepsilon}{8}(1+\tau)^2 + \frac{1}{2}\tau^2 \le \tau \quad \text{for } \frac{\varepsilon}{5} \le \tau \le \varepsilon.\end{aligned}$$

This proves (ii).

To prove (iii) take $x = \{x_\alpha\}$, $x' = \{x'_\alpha\}$ in B_τ, and let $d = \|x - x'\|$.

We have, similarly to (7),

$$\|\Phi(x) - \Phi(x')\| \le \|\sigma_1\| + \|\sigma_2\|,$$

where σ_1 and σ_2 correspond respectively to linear and quadratic terms. Notice that the constant terms are the same in $\Phi(x)$ and $\Phi(x')$,

$$\|\sigma_1\| \le \sum_{\lambda\in\Gamma} \sum_{\substack{\alpha\in E\\ \alpha\ne-\lambda}} |b_{\lambda+\alpha}|\,|x_\alpha - x'_\alpha| = \sum_{\alpha\in E} |x_\alpha - x'_\alpha| \left(\sum_{\substack{\lambda\in E\\ \lambda\ne-\alpha}} |b_{\lambda+\alpha}|\right) \le \frac{\varepsilon}{4} d;$$

$$\begin{aligned}\|\sigma_2\| &\le \frac{1}{2} \sum_{\lambda\in E} \sum_{\alpha,\beta\in E} |b_{\lambda+\alpha-\beta}|\,|x_\alpha \overline{x}_\beta - x'_\alpha \overline{x}'_\beta| \\ &\le \frac{1}{2} \sum_{\lambda,\alpha,\beta\in E} |b_{\lambda+\alpha-\beta}|\{|x_\alpha\overline{x}_\beta - x_\alpha\overline{x}'_\beta| + |x_\alpha\overline{x}'_\beta - x'_\alpha\overline{x}'_\beta|\} \\ &\le \frac{1}{2} \sum_{\alpha,\beta\in E} |x_\alpha|\,|x_\beta - x'_\beta| \cdot \sum_{\lambda\in E} |b_{\lambda+\alpha-\beta}| \\ &\quad + \frac{1}{2} \sum_{\alpha,\beta\in E} |x'_\beta|\,|x'_\alpha - x'_\alpha| \cdot \sum_{\lambda\in E} |b_{\lambda+\alpha-\beta}| \le \left(1+\frac{\varepsilon}{4}\right)\tau d.\end{aligned}$$

Finally,

$$\begin{aligned}\|\Phi(x) - \Phi(x')\| &\le \|\sigma_1\| + \|\sigma_2\| \le \frac{\varepsilon}{4} d + \tau\left(1+\frac{\varepsilon}{4}\right) d \\ &= d\left(\tau\left(1+\frac{\varepsilon}{4}\right) + \frac{\varepsilon}{4}\right) = d\left(\theta\left(1-\frac{\varepsilon}{4}\right) + \frac{\varepsilon}{4}\right) = \theta_1 \cdot d.\end{aligned}$$

We can choose τ to satisfy both inequalities in (ii) and (iii). Indeed put $\tau = \lambda\varepsilon$. Then we need $\frac{1}{5} \le \lambda \le 1$ and $\lambda < \lambda(\varepsilon) = (1 - \frac{\varepsilon}{4})/(\varepsilon(1+\frac{\varepsilon}{4}))$. But $\min_{0<\varepsilon\le 1} \lambda(\varepsilon) = \lambda(1) = \frac{3}{5}$; therefore, any τ, $\frac{1}{5}\varepsilon \le \tau \le \frac{3}{5}\varepsilon$, $0 < \varepsilon < 1$, is good for both (ii) and (iii).

By the Contractive Mapping Theorem, Φ has a (unique) fixed point. It proves the main theorem. □

4

The method used in the proof of the main theorem is applicable to a wide class of (countable) groups. Analogous results hold, for instance, for divisible abelian groups and for direct sums of cyclic groups of prime orders. The complete description of the class of countable (not necessarily abelian) groups admitting the analog of the main theorem is an attractive problem.

Let U be a unitary operator on a Hilbert space H. A vector $f \in H$ is called *rigid* with respect to U if $\lim_{k\to\infty} \|U^{n_k} f - f\| = 0$ for some sequence $\{n_k\}$, $n_k \nearrow \infty$. A unitary operator is *mildly mixing* if it has no nonzero rigid vectors. A unitary operator U is *strongly mixing* if $\{U^n f\}$ weakly tends to 0 as $n \to \infty$ for all $f \in H$. (We use here, for general unitary operators, the terminology which is traditional in the case of operators induced by measure-preserving transformations.) It is known that

strong mixing $\Rightarrow$ mild mixing $\Rightarrow$ weak mixing ($\equiv$ continuous spectrum).

The classes of mildly mixing and strongly mixing operators can be characterized in terms of properties of sequences along which weak wandering occurs for a dense set of vectors in H. Namely, *a unitary operator U is mildly mixing if and only if for any IP-set $E \subset \mathbb{N}$, $f \in H$, and $\varepsilon > 0$ there exist $\tilde{f} \in H$, $\|f - \tilde{f}\| < \varepsilon$, and an IP-set $\tilde{E} \subset E$ such that $\langle U^n \tilde{f}, U^m \tilde{f}\rangle = 0$ for all distinct n, $m \in \tilde{E}$.*

Moreover, *U is strong mixing if and only if for any infinite subset $S \subset \mathbb{N}$, $f \in H$, and $\varepsilon > 0$ there exist $\tilde{f} \in H$, $\|f - \tilde{f}\| < \varepsilon$, and an infinite subset $A \subset S$ such that $\langle U^n \tilde{f}, U^m \tilde{f}\rangle = 0$ for any distinct n, $m \in FS(A)$.*

Question. What is the description of the family P of unitary operators with continuous spectrum acting on the Hilbert space H which have the property that for any $f \in H$ and $\varepsilon > 0$ there exist $\tilde{f}$, $\|\tilde{f} - f\| \le \varepsilon$, and a set $S \subset \mathbb{N}$ *having positive upper density* (i.e., $\limsup_{K\to\infty}[\#(S \cap [1, K])/K] > 0$) such that $\langle U^n \tilde{f}, U^m \tilde{f}\rangle = 0$ for all distinct n, $m \in S$?

The following remark shows that there exist unitary operators with continuous spectrum which do not belong to P. Let us call a unitary operator *rigid* if every $f \in H$ is rigid. There exist rigid operators with continuous spectrum, and it can be shown that for such operators no nonzero vector can wander along a sequence of positive upper density.

Let T be a measure-preserving transformation of a probability space (Ω, μ), and let U be the unitary operator on $L^2(\Omega, \mu)$ induced by T. In this special case instead of vectors in $L^2(\Omega, \mu)$, one can consider measurable partitions of Ω into a finite number of sets. It is shown in [K] that T is weakly mixing if and only if weakly independent partitions (i.e., partitions ξ for which there exists an infinite sequence $\{n_i\}_{i=1}^{\infty}$ such that $\{T^{n_i}\xi\}$ are mutually independent) are dense in the space of all partitions. We do not know if the method of this note is applicable to the case of partitions.

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