

## ON GENERATORS OF $L/R^2$ LIE ALGEBRAS

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**ABSTRACT.** Let  $L$  be a free Lie algebra of finite rank  $n$  and  $R$  its arbitrary ideal. A necessary and sufficient condition for  $n$  elements of the Lie algebra  $L/R^2$  to be a generating set is given. In particular, we have a criterion for  $n$  elements of a free Lie algebra of rank  $n$  to be a generating set which is similar to the corresponding group-theoretic result due to Birman (*An inverse function theorem for free groups*, Proc. Amer. Math. Soc. **41** (1973), 634–638).

### 1. INTRODUCTION

Let  $L = L_n$  be a free Lie algebra of finite rank  $n \geq 2$  over the field  $K$  of characteristic 0. All background and undefined notions here can be found in [1]. Given an ideal  $R$  of  $L$  we consider the Lie algebra  $L/R^2$ , i.e., a free commutative extension of  $L/R$ , and give a necessary and sufficient condition for some  $n$  elements of  $L/R^2$  to be its generating set. The similar question for groups has been treated in [4]. Here our main result is the following theorem. Before formulating this theorem we introduce some more notation.

Let  $U(L)$  be the universal enveloping algebra of  $L$ , i.e., a free associative algebra over the field  $K$ . For an ideal  $R$  of  $L$ , we denote by  $\varepsilon_R$  the natural homomorphism of  $L$  onto  $L/R$ . The induced homomorphism of  $U(L)$  onto  $U(L/R)$  we also denote by  $\varepsilon_R$ . By  $d_i(u)$  we denote the  $i$ th Fox derivative of an element  $u \in U(L)$  (see [3]).

**Theorem.** *Let  $R$  be an ideal of  $L$ , and let  $y_1, \dots, y_n$  be elements of  $L$ . Then Lie algebra  $L/R^2$  is generated by the images of  $y_1, \dots, y_n$  if and only if the matrix  $\|d_j(y_i)^{\varepsilon_R}\|_{1 \leq i, j \leq n}$  has a left inverse over  $U(L/R)$ .*

The following corollary is an analog of Birman's result [2] for groups.

**Corollary.** *A mapping  $\phi: x_i \rightarrow y_i$ ,  $1 \leq i \leq n$ , induces an automorphism of  $L$  if and only if the matrix  $\|d_j(y_i)\|_{1 \leq i, j \leq n}$  has a left inverse over  $U(L)$ .*

It should be mentioned that a criterion similar to the assertion of this corollary has been obtained recently in [5].

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The arrangement of the paper is as follows. In §2 we present our technical tool—the free differential calculus—and use it in §3 to prove the theorem.

## 2. FREE DIFFERENTIAL CALCULUS

Let  $L = L_n$  be a free Lie algebra over the field  $K$  of characteristic 0 with the set  $\{x_i\}_{1 \leq i \leq n}$  of free generators. We write  $(u, v)$  for the Lie product of two elements  $u$  and  $v$ . By  $\Delta$  we denote the augmentation ideal of its universal enveloping algebra  $U(L)$ , i.e., the kernel of the augmentation map  $\varepsilon: U(L) \rightarrow K$  defined by  $\varepsilon(x_i) = 0$ ,  $1 \leq i \leq n$ . If  $R \neq L$  is an ideal of  $L$  then by  $\Delta_R$  we denote the kernel of the natural homomorphism  $\varepsilon_R: U(L) \rightarrow U(L/R)$ .

For arbitrary elements  $y_1, \dots, y_n, v$  of  $U(L)$  we denote by  $v[y_1, \dots, y_n]$  the result of the substitution of  $y_1, \dots, y_n$  instead of  $x_1, \dots, x_n$  in  $v$ .

In [3] Fox gave a detailed account of the differential calculus in a free group ring. Since any free associative algebra is naturally embedded in a free group algebra (over the same field), most of the technical results remain valid for free associative algebras.

To be more specific, we introduce here free derivations as the mappings  $d_i: U(L) \rightarrow U(L)$ ,  $1 \leq i \leq n$ , satisfying the following conditions whenever  $\alpha, \beta \in K$ ,  $u, v \in U(L)$ :

- (1)  $d_i(x_j) = \delta_{ij}$  (Kronecker delta);
- (2)  $d_i(\alpha u + \beta v) = \alpha d_i(u) + \beta d_i(v)$ ;
- (3)  $d_i(uv) = d_i(u)\varepsilon(v) + u d_i(v)$ .

Instead of  $d_i(u)$  we sometimes write  $\partial u / \partial x_i$ .

It is an obvious consequence of the definitions that  $d_i(1) = 0$ .

It is easy to prove (see [3]) that these derivations have another nature as well. The ideal  $\Delta$  is a free left  $U(L)$ -module with free basis  $\{x_i\}_{1 \leq i \leq n}$ , and the mappings  $d_i$  are projections to the corresponding free cyclic direct summands. Thus any element  $u \in \Delta$  can be uniquely written in the form  $u = \sum d_i(u)x_i$ . Moreover, for any elements  $y_1, \dots, y_n$  of  $U(L)$  one can always find an element  $u \in U(L)$  with  $d_i(u) = y_i$ ,  $1 \leq i \leq n$ .

We need some technical lemmas to be used throughout §3. The first lemma is an immediate consequence of the definitions.

**Lemma 1.** *Let  $J$  be an arbitrary ideal of  $U(L)$  and let  $u \in \Delta$ . Then  $u \in J\Delta$  if and only if  $d_i(u) \in J$  for each  $i$ ,  $1 \leq i \leq n$ .*

The next lemma can be found in [6].

**Lemma 2.** *Let  $R$  be an ideal of  $L$  and let  $u \in L$ . Then  $u \in \Delta_R\Delta$  if and only if  $u \in R^2$ .*

We have the following “chain rule” for Fox derivations (see [3]).

**Lemma 3.** *Let  $y_1, \dots, y_n, w$  be some elements of  $U(L)$  and let  $v = w[y_1, \dots, y_n]$ . Then*

$$d_j(v) = \sum_{1 \leq k \leq n} \frac{\partial v}{\partial y_k} \cdot \frac{\partial y_k}{\partial x_j}.$$

By  $\partial v / \partial y_k$  we mean here  $d_k(w)[y_1, \dots, y_n]$ .

Finally, we prove the following lemma which seems to be well known although I was not able to find an appropriate reference.

**Lemma 4.** *Let  $M$  be an arbitrary Lie algebra over a field and  $U(M)$  its universal enveloping algebra. Let  $v_1, \dots, v_m$  and  $u$  be elements of  $M$ . Suppose  $u$  belongs to the left ideal of  $U(M)$  generated by  $v_1, \dots, v_m$ . Then  $u$  belongs to the Lie algebra  $N$  generated by  $v_1, \dots, v_m$ .*

*Proof.* We choose a totally ordered (additive) basis  $B$  of  $M$  containing a basis of  $N$ , such that  $u, v \in B, u \in N, v \notin N \Rightarrow u < v$ . Let  $I$  be the left ideal of  $U(M)$  generated by  $N$ . By the Poincarè-Birkhoff-Witt theorem,  $I$  is linearly generated by the monomials of the form  $w_1 \cdots w_k, k \geq 1, w_i \in B, w_k \in N$ . It is easy to check now, using the identity  $uv = vu + (u, v)$ , that  $I$  is linearly generated by the monomials of the form  $w_1 \cdots w_k, k \geq 1, w_i \in B, w_k \in N, w_1 \geq \cdots \geq w_k$ , as well. Applying again the Poincarè-Birkhoff-Witt theorem we see that these decreasing products are linearly independent, so if some  $s$  belongs to  $M \cap I$  then  $s$  is a linear combination of monomials from  $B \cap N$ , i.e.,  $s$  is in  $N$ .

### 3. PROOF OF THE THEOREM

1. We prove first the “only if” part. Let the images of  $y_1, \dots, y_n$  generate the Lie algebra  $L/R^2$ . Then we can find elements  $v_1, \dots, v_n$  of  $U(L)$  such that  $v_i[y_1, \dots, y_n] = x_i \pmod{R^2}$ . Applying Lemmas 3 and 2 we see that the matrix  $\|\partial v_i / \partial y_k\|_{1 \leq i, k \leq n}^{eR}$  is left inverse to  $\|d_j(y_i)^{eR}\|_{1 \leq i, j \leq n}$  in the ring of matrices with entries from  $U(L/R)$ .

2. Now we prove the “if” part. Let  $J = \|d_j(y_i)^{eR}\|_{1 \leq i, j \leq n}$ , and let  $\tilde{X}$  be the column  $(\tilde{x}_1, \dots, \tilde{x}_n)$  and  $\tilde{Y}$  the column  $(\tilde{y}_1, \dots, \tilde{y}_n)$ , where we write  $\tilde{u}$  instead of  $u^{eR}$ . Then we have  $J\tilde{X} = \tilde{Y}$ , and, multiplying by  $J^{-1}$  on the left,  $\tilde{X} = J^{-1}\tilde{Y}$ .

It follows that in the ring  $U(L/R)$  the elements  $\tilde{x}_1, \dots, \tilde{x}_n$  belong to the left ideal generated by  $\tilde{y}_1, \dots, \tilde{y}_n$ . Hence  $\tilde{x}_1, \dots, \tilde{x}_n$  belong to the Lie algebra generated by  $\tilde{y}_1, \dots, \tilde{y}_n$  in view of Lemma 4, so  $\tilde{y}_1, \dots, \tilde{y}_n$  generate  $L/R$ .

Therefore we can find elements  $u_1, \dots, u_n$  of  $U(L)$  such that

$$\|d_j(u_i)[\tilde{y}_1, \dots, \tilde{y}_n]\|_{1 \leq i, j \leq n} = J^{-1}.$$

Let  $w_i = u_i[y_1, \dots, y_n]$ . Clearly we may assume that  $w_i^e = 0$ . Then we have by Lemma 3

$$(1) \quad \|d_j(w_i)^{eR}\|_{1 \leq i, j \leq n} = J^{-1}J = I,$$

the identity matrix.

Thus (1) gives  $\tilde{w}_i = \tilde{x}_i$ . Since  $w_i - x_i \in \Delta_R$ , we should have for some  $v_j \in R, h_j \in U(L)$

$$(2) \quad w_i - x_i = \sum_j v_j h_j.$$

Denote the right-hand side of (2) by  $S$ . We prove now that we can replace  $S$  with some element  $t$  of  $L$  with  $d_j(S) = d_j(t) \pmod{\Delta_R}$  for any  $j, 1 \leq j \leq n$ .

For, we write

$$(3) \quad S = \sum_j v_j(h_j - h_j^e + 1) + \sum_j (h_j^e - 1)v_j.$$

The second sum in the right-hand side of (3) is an element of  $L$  since each  $v_j$  is. Let now  $h_j - h_j^e + 1 = g_j$ . Then

$$d_i(v_j g_j) = d_i(v_j)g_j^e + v_j d_i(g_j).$$

But  $g_j^e = 1$ , and  $v_j \in R$ , so

$$d_i(v_j g_j) = d_i(v_j) \pmod{\Delta_R}.$$

Hence  $d_i(S) = d_i(t) \pmod{\Delta_R}$ , where

$$t = \sum_j v_j + \sum_j (h_j^e - 1)v_j = \sum_j h_j^e v_j.$$

Now we take  $\hat{u}_i = x_i + \sum_j h_j^e v_j$  instead of  $u_i$  and  $\hat{w}_i = \hat{u}_i[y_1, \dots, y_n]$ . Then

$$\|d_j(\hat{u}_i)[\bar{y}_1, \dots, \bar{y}_n]\|_{1 \leq i, j \leq n} = J^{-1}$$

since  $d_j(\hat{u}_i) = d_j(u_i) \pmod{\Delta_R}$ . It follows that

$$\|d_j(\hat{w}_i)^{eR}\|_{1 \leq i, j \leq n} = J^{-1} J = I$$

in view of Lemma 3. This means that  $d_j(\hat{w}_i)^{eR} = \delta_{ij}$ , i.e.,

$$d_i(\hat{w}_i) - 1 \in \Delta_R, \quad d_j(\hat{w}_i) \in \Delta_R \quad \text{for } j \neq i.$$

Hence

$$(4) \quad d_i(\hat{w}_i)x_i - x_i \in \Delta_R \Delta \quad \text{and} \quad d_j(\hat{w}_i)x_j \in \Delta_R \Delta \quad \text{for } j \neq i.$$

Now take a sum of all the inclusions (4) for a fixed  $i$ . This yields  $\hat{w}_i - x_i \in \Delta_R \Delta$  in view of  $u = \sum d_i(u)x_i$  for any  $u \in U(L)$ . Hence, as  $\hat{w}_i - x_i \in L$ , Lemma 2 gives  $\hat{w}_i - x_i \in R^2$ , or  $\hat{w}_i = x_i \pmod{R^2}$ , and it follows that the natural images of  $y_1, \dots, y_n$  generate  $L/R^2$ . The theorem is proved.

Now taking  $R = \{0\}$  we obtain the following.

**Corollary.** *A mapping  $\phi: x_i \rightarrow y_i$ ,  $1 \leq i \leq n$ , induces an automorphism of  $L$  if and only if the matrix  $\|d_j(y_i)\|_{1 \leq i, j \leq n}$  has a left inverse over  $U(L)$ .*

*Remark.* The analog of this corollary does not hold in a free associative algebra. Indeed, consider the free associative algebra  $F$  over a field of characteristic 0 with free generators  $x_1$  and  $x_2$ . Take the mapping  $\phi: x_1 \rightarrow x_1 + x_1 x_2$ ,  $x_2 \rightarrow x_2$ . Then  $J_\phi$  is invertible in the ring of  $2 \times 2$  matrices with the entries from  $F$ , while  $\phi$  is not an automorphism of  $F$ .

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