ON THE SEMIGROUPS OF FULLY INVARIANT IDEALS
OF THE FREE GROUP ALGEBRA
AND THE FREE ASSOCIATIVE ALGEBRA

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ABSTRACT. Let R be an integral domain, K its field of fractions, F a free group. Let I and J be fully invariant (=verbal) ideals of the group algebra KF. We prove that over certain domains the equality $IJ \cap RF = (I \cap RF) \times (J \cap RF)$ need not be true. A similar result is valid for fully invariant ideals of the free associative algebra. This implies that the product of pure varieties of group representations over an integral domain need not be pure, that there exist pure nonprojective varieties of group representations and of associative algebras, and also answers some other questions raised in the literature.

1. INTRODUCTION

The present work was inspired by the following question. Let R be an integral domain, K its field of fractions, F a free group, and let I and J be fully invariant (i.e., invariant under all endomorphisms of F) ideals of the group algebra KF. Is it true that

$$(1) \qquad IJ \cap RF = (I \cap RF)(J \cap RF)?$$

This question is actually a reformulation, in group ring terms, of a problem on varieties of group representations raised by Plotkin in the early 1970s. Without going into details (which can be found in [PV, V2, V5]), recall that all varieties of group representations over a given commutative ring K form a semigroup under a naturally defined multiplication. This semigroup is canonically anti-isomorphic to the semigroup $\mathbb{V}(K)$ of verbal ideals of KF. If K is a field, the abstract structure of this semigroup was described in [PG] (see also [BL]): $\mathbb{V}(K)$ is a free semigroup with 0 and 1. However, for K not a field the structure of $\mathbb{V}(K)$ is by no means well understood.

From now on R is an integral domain and K is its field of fractions. It is natural that the first step in the study of the semigroup $\mathbb{V}(R)$ is to understand the relationship between $\mathbb{V}(R)$ and $\mathbb{V}(K)$. Let $\theta : \mathbb{V}(K) \to \mathbb{V}(R)$ be the map defined by $I^\theta = I \cap RF$. A straightforward verification shows that θ is injective

and that $\mathrm{Im}(\theta)$ consists of all verbal ideals J of RF such that RF/J is torsion-free as an R-module (these verbal ideals and the corresponding varieties of group representations are called *pure*). Our initial problem can now be stated as follows: *Is θ a homomorphism of semigroups?* There are other equivalent formulations of the same problem, for instance: *Is the product of pure varieties also pure?* All these formulations plus some related questions were repeatedly discussed in [P, V2, PV] and several other papers.

A positive solution of the problem for R a Dedekind domain was obtained by the author [V2, Corollary 9.10]. This result was later generalized by Bergman [B] who proved that if R is a Dedekind domain, then (1) is valid for every *right* ideal I and every *left* ideal J of KF; for an exposition of his result see [V5, §4.3]. However, the problem over an *arbitrary* domain has remained open for a rather long time. In §2 of this note we construct an example solving it in the negative.

Another problem of a similar character was posed in 1985 by L'vov. Let $R\langle X\rangle$ and $K\langle X\rangle$ be the free associative algebras on an alphabet X over R and K respectively, and let I and J be fully invariant ideals ($= T$-ideals) of $K\langle X\rangle$. *Is it true that*

$$(1') \qquad\qquad IJ \cap R\langle X\rangle = (I \cap R\langle X\rangle)(J \cap R\langle X\rangle)?$$

Again over a Dedekind domain the answer turned out to be positive (see, e.g., [V3]), but over an arbitrary domain this has remained unknown until the present. We show that the counterexample to (1) can literally be carried over to the free associative algebra, thus giving a counterexample to $(1')$ as well. Using these examples, in §3 we answer several questions on varieties of group representations and associative algebras which were previously raised in the literature.

2. The Counterexample

Let $X = \{x_1, x_2, \ldots\}$ be a set of free generators of the group F and let $Z = \{z_1, z_2, \ldots\}$ be any set of the same cardinality. Then for any commutative ring R there is a natural embedding, going back to Magnus [M], of the free group algebra RF into the algebra $R\langle\langle Z\rangle\rangle$ of formal power series in the variables $z_1, z_2, \cdots$. Under this embedding the free generators x_i of F are mapped into $1 + z_i$; the rest is uniquely determined. We will systematically identify RF with its image in $R\langle\langle Z\rangle\rangle$; thus every $u \in RF$ is uniquely presented as a series

$$u = u_{(0)} + u_{(1)} + \cdots + u_{(n)} + \cdots$$

where $u_{(n)} = \sum \lambda_{i_1\cdots i_n} z_{i_1} \cdots z_{i_n}$, the *homogeneous component of degree n of u*, is a finite R-linear combination of monomials of degree n in the z_i. In particular,

$$x_i = 1 + z_i \quad (\text{and so} \quad z_i = x_i - 1),$$

$$x_i^{-1} = 1 - z_i + z_i^2 - z_i^3 + \cdots.$$

An ideal I of RF is called *homogeneous* if for any $u \in I$ its homogeneous components $u_{(n)}$ are contained in I. Furthermore, each $u_{(n)}$ is uniquely presented as a sum of *multihomogeneous* components of u (i.e., homogeneous in each variable), and I is called multihomogeneous if for any element from I all its multihomogeneous components belong to I as well.

Denote by Δ the augmentation ideal of RF. It is well known that an element $u \in RF$ belongs to Δ^n if and only if, in the above notation, $u_{(0)} = u_{(1)} = \cdots = u_{(n-1)} = 0$. This implies, in particular, that if I is an ideal of KF such that $\Delta^n \supseteq I \supseteq \Delta^{n+1}$, then I is homogeneous.

Recall finally that an ideal I of RF is called *fully invariant* (or *verbal*) if it is invariant under all endomorphisms of RF which are induced by endomorphisms of the group F. In this paper we will usually use the shorter term "verbal".

Let $K = k(t)$ be the field of rational functions in a variable t over a field k. Noting that K is the field of fractions of its subring $R = k[t^2, t^3]$, we can now construct the desired counterexample.

Theorem 1. *Let k be a field of characteristic $\neq 2, 3$, $K = k(t)$, and $R = k[t^2, t^3]$. Take in the group algebra KF two elements*

$$(2) \qquad u = z_1^2 z_2 - t z_2 z_1^2, \qquad v = z_1 z_2 z_3 z_4$$

(where $z_i = x_i - 1$), and denote by I the verbal ideal they generate in KF. Then

$$I^2 \cap RF \neq (I \cap RF)^2.$$

Proof. 1. Consider in KF the element

$$(3) \qquad w = (tu)^2 = t^2 z_1^2 z_2 z_1^2 z_2 - t^3 z_1^2 z_2^2 z_1^2 - t^3 z_2 z_1^4 z_2 + t^4 z_2 z_1^2 z_2 z_1^2.$$

It belongs to I^2, and since all its coefficients are in R, we have $w \in I^2 \cap RF$. Thus it remains to show that

$$(4) \qquad w \notin (I \cap RF)^2.$$

Assume the contrary: $w \in (I \cap RF)^2$. Then w can be presented in the form

$$(5) \qquad w = u^1 v^1 + u^2 v^2 + \cdots + u^n v^n$$

where $u^i, v^i \in I \cap RF$ (do not confuse the upper indices with exponents!). Now note that

$$(6) \qquad \Delta^3 \supset I \supset \Delta^4$$

where Δ is the augmentation ideal of KF. It follows that each of the elements u^i, v^i has no homogeneous component of degree less than 3. Noting that the element w is homogeneous of degree 6 and using (5), we conclude that

$$w = w_{(6)} = u^1_{(3)} v^1_{(3)} + \cdots + u^n_{(3)} v^n_{(3)}.$$

By (6), I is a homogeneous ideal. Therefore, $u^i_{(3)}, v^i_{(3)} \in I$, and since these elements are also contained in RF, we have $u^i_{(3)}, v^i_{(3)} \in I \cap RF$. Therefore, we may assume that all the u^i, v^i in (5) are homogeneous of degree 3. Furthermore, applying to (5) the endomorphism φ of F defined by

$$x_i^\varphi = \begin{cases} x_i & \text{if} \quad i = 1, 2, \\ 1 & \text{if} \quad i > 2, \end{cases}$$

we see that all the u^i, v^i may be assumed to involve only the variables z_1 and z_2.

Thus every u^i and v^i is an R-linear combination of monomials

$$(7) \qquad z_1^3, \quad z_1^2 z_2, \quad z_1 z_2 z_1, \quad z_2 z_1^2, \quad z_2^2 z_1, \quad z_2 z_1 z_2, \quad z_1 z_2^2, \quad z_2^3.$$

Analyzing (3), (5), and (7), we see that the product of two monomials from (7) may yield a monomial of w only if these two monomials are both of degree 2 in z_1 and of degree 1 in z_2. It follows that $w = u_*^1 v_*^1 + \cdots + u_*^n v_*^n$ where u_*^i and v_*^i are respectively the multihomogeneous components of u^i and v^i of degree 2 in z_1 and of degree 1 in z_2. However, it is known that *if I is a homogeneous verbal ideal of KF and u is a homogeneous element from I such that* $\deg u < \operatorname{char} K$, *then all multihomogeneous components of u belong to I* (this fact was proved in [V4] over a field of characteristic zero, but the proof remains valid under our assumptions). Since $\deg u^i = \deg v^i = 3$ and $\operatorname{char} K > 3$, we obtain that $u_*^i, v_*^i \in I$. Thus we have eventually proved that w can be written in the form (5) where all $u^i, v^i \in I \cap RF$ involve only z_1 and z_2 and are homogeneous of degree 2 in z_1 and homogeneous of degree 1 in z_2.

Denote, for a moment, by F_0 the subgroup of F generated by x_1, x_2, by Δ_0 the augmentation ideal of KF_0, and by I_0 the verbal ideal of KF_0 generated by u and Δ_0^4. A standard for variety theory argument shows that $I_0 = I \cap KF_0$. Combining this observation with the above, we see that the element w (which belongs to KF_0) can be written in the form (5) where all the u^i, v^i belong to $I_0 \cap RF_0$. In other words, *from now on we may (and will!) assume that F is the free group of rank 2 with free generators x_1, x_2, Δ is the augmentation ideal of its group algebra KF, and I is the verbal ideal of KF generated by u and Δ^4.*

2. Our next objective is to find a K-basis of I modulo Δ^4. First of all, we note that $\dim_K(KF/\Delta^4) = 15$ because the 15 monomials

$$(8) \qquad 1; \quad z_1, \, z_2; \quad z_1^2, \, z_1 z_2, \, z_2 z_1, \, z_2^2; \quad z_1^2 z_2, \, z_2^2 z_1, \, z_1^3, \ldots$$

form a natural basis in KF modulo Δ^4. Second, let us show that $\dim_K(KF/I) = 9$. A direct proof of this fact would require long and rather boring calculations; instead we will deduce it from one result proved in [PG]. Let V be a 9-dimensional vector space over K with a basis $b_1, \ldots, b_9$, and let G be a subgroup of $\mathrm{GL}_9(K)$ generated by two unitriangular matrices

$$g_1 = e + e_{12} + e_{24} + e_{37} + te_{59} - (1+t)e_{68} + e_{78},$$
$$g_2 = e + e_{13} + e_{26} + e_{35} + te_{48} + e_{69} - (1+t)e_{79}$$

(here e is the identity matrix and e_{ij} are the standard matrix units). Consider the natural representation $\rho = (V, G)$ of G on V. It is easy to see that V is generated by b_1 as a KG-module and so ρ is a cyclic representation. Therefore if $\operatorname{Reg}_K F = (KF, F)$ is the regular representation of F, then there exists an obvious epimorphism of representations

$$\phi: (KF, F) \to (V, G).$$

The representation $\rho = (V, G)$ was constructed and studied in [PG]. In particular, it was proved there that ρ satisfies the identities (2). Hence ϕ can be factored through KF/I giving an epimorphism of K-spaces $KF/I \to V$. It follows that $\dim_K(KF/I) \geq 9$.

On the other hand, among the 15 monomials (8) only the first nine can be linearly independent modulo I. Indeed, denoting by $\equiv$ equality modulo I, we have

(9) $$z_1^2 z_2 \equiv t z_2 z_1^2 \quad \text{and} \quad z_2^2 z_1 \equiv t z_1 z_2^2.$$

Applying to any of these identities the endomorphism $x_1 \mapsto x_2$ of F, we obtain $z_2^3 \equiv 0$ and so $z_1^3 \equiv 0$. Further, applying to the latter the endomorphism $x_1 \mapsto x_1 x_2 x_1$ of F, we obtain

$$
\begin{aligned}
0 &\equiv (x_1 x_2 x_1 - 1)^3 \\
&= (2z_1 + z_2 + z_1 z_2 + z_1^2 + z_2 z_1 + z_1 z_2 z_1)^3 \\
&= 8z_1^3 + 6z_1^2 z_2 + 6z_1 z_2 z_1 + 6z_2 z_1^2 + z_2^3 + \text{ higher degree terms} \\
&= 6(z_1^2 z_2 + z_1 z_2 z_1 + z_2 z_1^2) + \text{ higher degree terms.}
\end{aligned}
$$

Since I is a homogeneous ideal, it follows that $6(z_1^2 z_2 + z_1 z_2 z_1 + z_2 z_1^2) \equiv 0$. Since char $K \neq 2, 3$, we have

(10) $$z_1^2 z_2 + z_1 z_2 z_1 + z_2^2 z_1 \equiv 0.$$

By (9) and (10),

$$z_1 z_2 z_1 \equiv -z_1^2 z_2 - z_2 z_1^2 \equiv -(1 + t)z_2 z_1^2.$$

All these relations show that the first nine of the monomials (8) span KF modulo I. Thus $\dim_K(KF/I) = 9$.

Since $\dim_K(KF/\Delta^4) = 15$, we conclude that $\dim_K(I/\Delta^4) = 6$. Now it is easy to find a basis for I modulo Δ^4. In view of the above, the following six elements belong to I:

(11) $$
\begin{aligned}
&z_1^3, \quad z_2^3, \quad z_1^2 z_2 - t z_2 z_1^2, \quad z_2^2 z_1 - t z_1 z_2^2, \\
&z_1 z_2 z_1 + (1 + t) z_2 z_1^2, \quad z_2 z_1 z_2 + (1 + t) z_1 z_2^2.
\end{aligned}
$$

These elements are linearly independent modulo Δ^4 and so they form a desired basis.

3. We return now to the element w. It was proved in step 1 that if $w \in (I \cap RF)^2$, then w can be presented in the form (5) where each of u^i, v^i belongs to $I \cap RF$, is homogeneous of degree 2 in z_1, and is homogeneous of degree 1 in z_2. Since (11) is a basis of I modulo Δ^4, it follows that each of u^i, v^i is a K-linear combination of *two* elements:

$$z_1^2 z_2 - t z_2 z_1^2 \quad \text{and} \quad z_1 z_2 z_1 + (1 + t) z_2 z_1^2.$$

Hence for each $i = 1, \ldots, n$ there exist $a_i, b_i, c_i, d_i \in K$ such that

(12) $$
\begin{aligned}
u^i &= a_i(z_1^2 z_2 - t z_2 z_1^2) + b_i(z_1 z_2 z_1 + (1 + t) z_2 z_1^2) \\
&= a_i z_1^2 z_2 + b_i z_1 z_2 z_1 + ((b_i - a_i)t + b_i)z_2 z_1^2,
\end{aligned}
$$

(13) $$
\begin{aligned}
v^i &= c_i(z_1^2 z_2 - t z_2 z_1^2) + d_i(z_1 z_2 z_1 + (1 + t) z_2 z_1^2) \\
&= c_i z_1^2 z_2 + d_i z_1 z_2 z_1 + ((d_i - c_i)t + d_i)z_2 z_1^2.
\end{aligned}
$$

Since $u^i, v^i \in RF$, the coefficients in (12) and (13) must belong to R. Therefore,

(14) $$a_i, \ b_i, \ c_i, \ d_i, \ (b_i - a_i)t, \ (d_i - c_i)t \ \in R.$$

Further, expanding the products $u^i v^i$, we have

$$w = \sum_{i=1}^{n} u^i v^i = \sum_{i=1}^{n} [a_i z_1^2 z_2 + b_i z_1 z_2 z_1 + ((b_i - a_i)t + b_i)z_2 z_1^2]$$

$$\times [c_i z_1^2 z_2 + d_i z_1 z_2 z_1 + ((d_i - c_i)t + d_i)z_2 z_1^2]$$

(15)

$$= \sum_{i=1}^{n} (a_i c_i z_1^2 z_2 z_1^2 z_2 + b_i c_i z_1 z_2 z_1^3 z_2$$

$$+ a_i d_i z_1^2 z_2 z_1 z_2 z_1 + b_i d_i z_1 z_2 z_1^2 z_2 z_1 + w_i),$$

where each w_i is a linear combination of monomials *different* from $z_1^2 z_2 z_1^2 z_2$, $z_1 z_2 z_1^3 z_2$, $z_1^2 z_2 z_1 z_2 z_1$, $z_1 z_2 z_1^2 z_2 z_1$. Comparing the coefficients at the last four monomials in (15) with those in (3), we conclude that

(16) $$a_1 c_1 + \cdots + a_n c_n = t^2,$$

(17) $$b_1 c_1 + \cdots + b_n c_n = 0,$$

(18) $$a_1 d_1 + \cdots + a_n d_n = 0,$$

(19) $$b_1 d_1 + \cdots + b_n d_n = 0.$$

Subtracting (18) from (16) and (19) from (17), we obtain

(20) $$a_1(c_1 - d_1) + \cdots + a_n(c_n - d_n) = t^2,$$

(21) $$b_1(c_1 - d_1) + \cdots + b_n(c_n - d_n) = 0.$$

Now subtract (21) from (20):

(22) $$(a_1 - b_1)(c_1 - d_1) + \cdots + (a_n - b_n)(c_n - d_n) = t^2.$$

By (14), $(a_i - b_i)t \in k[t^2, t^3]$ and $(c_i - d_i)t \in k[t^2, t^3]$. This means that for each i the constant term of the polynomials a_i and b_i coincide; the same is true for c_i and d_i. Since $a_i, b_i, c_i, d_i \in k[t^2, t^3]$, it follows that the degree of each polynomial $a_i - b_i$ and $c_i - d_i$ in t is at least 2, and so (22) is impossible. This contradiction completes the proof. □

A similar result is valid for free associative algebras. Recall that if $K\langle X \rangle$ is the free associative K-algebra on a set $X = \{x_1, x_2, \ldots\}$ of free variables, then its ideal T is called *fully invariant* (or *T-ideal*) if it is invariant under all K-endomorphisms of $K\langle X \rangle$. Such ideals are in a natural one-to-one correspondence with varieties of associative K-algebras.

Theorem 2. *Let k be a field of characteristic $\neq 2, 3$, $K = k(t)$, and $R = k[t^2, t^3]$. If T is the T-ideal of $K\langle X \rangle$ generated by $x_1^2 x_2 - t x_2 x_1^2$ and $x_1 x_2 x_3 x_4$, then*

$$T^2 \cap R\langle X \rangle \neq (T \cap R\langle X \rangle)^2.$$

The proof is virtually the same as that of Theorem 1. The only slight difference is in step 2. To prove the inequality $\dim_K(K\langle X \rangle / T) \geq 9$, one cannot apply the result of [PG] because the latter deals specifically with identities of group representations. Instead one should use the formula $\dim_K(KF/I) = 9$

which has been already established in the proof of Theorem 1, and verify that KF/I satisfies the identities $x_1^2 x_2 - t x_2 x_1^2$ and $x_1 x_2 x_3 x_4$ as a K-algebra. Since $K\langle X\rangle/T$ is the free algebra of rank $|X|$ in the variety defined by these identities, there exists an epimorphism $K\langle X\rangle/T \to KF/I$, whence the desired inequality follows.

3. Corollaries

1. Let $\mathscr{X}$ be a variety of group representations over an integral domain R and let $I = \mathrm{Id}\,\mathscr{X}$ be the corresponding verbal ideal of RF. The variety $\mathscr{X}$ is called *pure* (*projective*) if RF/I is torsion-free (projective) as an R-module. *Is every pure variety projective?* This question has been open since the early 1970s; to answer it, we first note that the result of §2 can be reformulated as follows.

Corollary 1 (solution of Problem 8.5.2 from [P]). *There exists an integral domain R (for example, $R = k[t^2, t^3]$) such that the product of pure varieties of group representations over R need not be pure.*

Proof. The assertion that the product of pure varieties over R is pure means that if I_0 and J_0 are verbal ideals of RF such that RF/I_0 and RF/J_0 are R-torsion-free, then $RF/I_0 J_0$ is also R-torsion-free. The latter is equivalent to saying that (1) is true for any verbal ideals I and J of KF. But this is impossible in view of Theorem 1. $\square$

Let R be an integral domain whose existence is guaranteed by Corollary 1, and assume that every pure variety over R is projective. Let $\mathscr{X}$ and $\mathscr{Y}$ be arbitrary pure varieties, and let $\rho = \mathrm{Fr}\,\mathscr{X}$ and $\sigma = \mathrm{Fr}\,\mathscr{Y}$ be their free cyclic representations of countable rank. By assumption, the modules of these representations are projective over R. Hence, by the main result of [V1] (see also [V2, Theorem 6.2]),

$$\mathrm{var}(\rho \bigtriangledown \sigma) = \mathscr{X}\,\mathscr{Y}$$

where $\rho \bigtriangledown \sigma$ is the triangular product of ρ and σ. Since the module of the representation $\rho \bigtriangledown \sigma$ is torsion-free over R, the variety generated by this representation is pure. Therefore, the product of arbitrary pure varieties over R must be pure, which is in contradiction with Corollary 1. Thus we have proved

Corollary 2 (solution of Problem 18.2.8 from [PV]). *A pure variety of group representations over an integral domain need not be projective.* $\square$

It is still unknown whether this is true over $\mathbb{Z}$ [P, Problem 15]. We repeat this problem in terms of group rings. Let I be a verbal ideal of the free group ring $\mathbb{Z}F$. *Is it true that if $\mathbb{Z}F/I$ is a torsion-free abelian group, then it is a free abelian group?*

2. As was mentioned in §1, over a Dedekind domain our initial problem has a positive solution. This fact was proved in [V1] using the technique of triangular products. Namely, let as usual R be a domain and let K be its field of fractions. To each representation $\rho = (V, G)$ over R there naturally corresponds a representation $\rho_K = (K \otimes V, G)$, also regarded as a representation over R. Theorem 9.8 in [V2] states that if R is a Dedekind domain and ρ and σ are pure representations over R, then

$$(23) \qquad\qquad \mathrm{var}(\rho_K \bigtriangledown \sigma_K) = \mathrm{var}\,\rho \cdot \mathrm{var}\,\sigma.$$

An immediate corollary is that for R a Dedekind domain the product of pure varieties is pure; in other words, (1) is true.

Problem 9.14 in [V2] asks if (23) is valid over *any* domain. Now we see that it is not the case: otherwise the product of pure varieties would *always* be pure which is impossible in view of Corollary 1.

3. Our proofs of Corollaries 1 and 2 were entirely based on Theorem 1 and on [V2, Theorems 6.2 and 9.8]. Since these three theorems have natural analogs for associative algebras (Theorem 2 and [V3, Theorems 1 and 2] respectively), the proofs of the corollaries remain valid for varieties of algebras as well. In terms of T-ideals the corresponding results can be stated as follows.

Corollary 3. *There exists an integral domain R such that the free algebra $R\langle X\rangle$ contains a T-ideal T with the following properties:*

(i) *$R\langle X\rangle/T$ is R-torsion-free while $R\langle X\rangle/T^2$ is not;*
(ii) *$R\langle X\rangle/T$ is not R-projective.*

The results of this note may give the impression that the situation of the free group algebra is completely analogous to that of the free associative algebra, at least when the above problems are concerned. But it is not the case. For example, it was noted that it is still unknown whether Corollary 2 is valid over $\mathbb{Z}$. On the other hand, the corresponding question for T-ideals can be solved very easily. Indeed, let $\mathbb{Z}\langle X\rangle/T$ be a torsion-free abelian group for some T-ideal T. A standard argument with the use of the Vandermonde determinant shows that T is a multihomogeneous ideal. Therefore, the factor-algebra $\mathbb{Z}\langle X\rangle/T$ is a direct sum of its multihomogeneous components. Each of these components is a finitely generated torsion-free abelian group and, hence, a free abelian group. Consequently $\mathbb{Z}\langle X\rangle/T$ is a free abelian group itself.

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