

EXTENDIBILITY CRITERION FOR A PROJECTIVE MODULE OF RANK ONE OVER $R[T]$ AND $R[T, T^{-1}]$

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ABSTRACT. In this note we give a criterion for a finitely generated projective module \mathcal{P} of constant rank one over $R[T]$ or $R[T, T^{-1}]$ to be extended from R in terms of invertible ideals, when R is an integral domain. We show that if I is an invertible ideal of $R[T]$ or $R[T, T^{-1}]$ such that $I \cap R \neq 0$, then I is extended from R if and only if $I \cap R$ is an invertible ideal of R .

1. INTRODUCTION

Let R be a commutative ring, and let A be the polynomial algebra $R[T]$ or the Laurent polynomial algebra $R[T, T^{-1}]$. Let \mathcal{P} be a finitely generated projective A -module. We say that “ \mathcal{P} is extended from R ” if there exists an R -module \mathcal{Q} such that $\mathcal{P} \simeq \mathcal{Q} \otimes_R A$ as A -modules. In this note we investigate the question: when is a finitely generated projective module \mathcal{P} of (constant) rank one over A extended from R ? It is easy to see that for this question we can assume without loss of generality that R is a reduced ring. Hence, throughout the paper we will assume that R is a *reduced commutative ring*.

If R has only finitely many minimal prime ideals (e.g., R is an integral domain or R is a noetherian ring) then $Q(R)$, the *total quotient ring* of R , is a finite direct product of fields. In this case, since all finitely generated projective modules of (constant) rank one over $Q(R)[T]$ and $Q(R)[T, T^{-1}]$ are free, it is easy to see that there exists an invertible ideal I of A such that

- (1) $I \cap R$ contains a non-zero-divisor of R ,
- (2) $I \simeq \mathcal{P}$ as A -modules.

See [1, Chapter II, §5] for details. Therefore, in this situation, one is reduced to consider the following question:

Question. Let R be a reduced ring with only finitely many minimal prime ideals. Let A denote the polynomial algebra $R[T]$ or the Laurent polynomial algebra $R[T, T^{-1}]$. Let I be an invertible ideal of A such that $I \cap R$ contains a non-zero-divisor of R . Then, when is I extended from R ?

In this paper we settle this question as follows:

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Theorem (A). *Let R be a reduced ring with only finitely many minimal prime ideals. Let I be an invertible ideal of $R[T, T^{-1}]$ such that $J = I \cap R$ contains a non-zero-divisor of R . Then I is extended from R if J is an invertible ideal of R . Moreover, if R is an integral domain and I is extended from R , then $J = I \cap R$ is an invertible ideal of R .*

Theorem (B). *Let R be a reduced ring with only finitely many minimal prime ideals. Let I be an invertible ideal of $R[T]$ such that $J = I \cap R$ contains a non-zero-divisor of R . Then I is extended from R if and only if J is an invertible ideal of R .*

We also give an example of a reduced noetherian ring and an invertible ideal I of $A = R[T, T^{-1}]$ such that $J = I \cap R$ contains a non-zero-divisor of R , I is extended from R as an A -module, but J is not an invertible ideal of R .

In case of an ideal I of $A = R[T]$ or $R[T, T^{-1}]$, there are naturally two notions of extendibility, namely,

- (1) *ideal-extendibility*, i.e., $I = \mathcal{I}A$ for some ideal \mathcal{I} of R ,
- (2) *module-extendibility*, i.e., there exists an R -module M such that $I \approx M \otimes_R A$ as A -modules.

Obviously ideal-extendibility implies module-extendibility, but the converse need not be true.

Theorem (A) and Theorem (B) are proved by first showing that if $A = R[T, T^{-1}]$ (R a domain) or $A = R[T]$ (R a reduced ring), then for an ideal I of A the two notions of extendibility are equivalent if $I \cap R$ contains a non-zero-divisor (Lemmas 2.4 and 2.8). Example 2.7 shows that for $A = R[T, T^{-1}]$, the two notions need not be the same even for an invertible ideal I of A containing a non-zero-divisor of R if R is not a domain.

2. EXTENDIBILITY CRITERION

In this section we will prove Theorems (A) and (B) stated above (Theorems 2.11 and 2.13, respectively). We begin with the following definition:

Definition 2.1. Let A be a reduced ring, and let $Q(A)$ denote the total quotient ring of A . An A -submodule M of $Q(A)$ is said to be *invertible* if there exists an A -submodule N of $Q(A)$ such that $MN = A$.

We note that such an N is unique and we denote it by M^{-1} . If an ideal I of A is invertible, we say that I is an *invertible ideal* of A .

Let B be an A -subalgebra of $Q(A)$, and let I be an invertible ideal of A . Then it follows immediately from the definition that IB is an invertible ideal of B .

Now we state a lemma, a proof of which can be found in [1, Chapter II, §5].

Lemma 2.2. *Let A be a reduced ring and S be a multiplicative set of non-zero-divisors of A . Let $B = S^{-1}A$. If all finitely generated projective B -modules of constant rank one are free, then given a finitely generated projective A -module \mathcal{P} of constant rank one there exists an invertible ideal I of A such that $I \cap S \neq \emptyset$ and $I \simeq \mathcal{P}$ as A -modules.*

As a consequence of the above lemma we have the following:

Lemma 2.3. *Let R be a reduced ring with only finitely many minimal prime ideals. Let S denote the set of all non-zero-divisors of R . Let $A = R[T]$ or $R[T, T^{-1}]$, and let \mathcal{P} be a finitely generated projective A -module of constant rank one. Then there exists an invertible ideal I of A such that $I \cap S \neq \emptyset$ and $I \simeq \mathcal{P}$.*

Now we prove Theorems (A) and (B) stated in the introduction. For the proof of these theorems we need some lemmas.

Lemma 2.4. *Let R be a domain, and let I be a finitely generated ideal of $R[T, T^{-1}]$. Assume that $J = I \cap R$ is nonzero. Then the following statements are equivalent:*

- (1) $I = JR[T, T^{-1}]$.
- (2) $I \simeq J \otimes_R R[T, T^{-1}]$ as $R[T, T^{-1}]$ -modules.
- (3) There exists an R -module M such that $I \simeq M \otimes_R R[T, T^{-1}]$ as $R[T, T^{-1}]$ -modules.

Proof. The implications $1 \Rightarrow 2$ and $2 \Rightarrow 3$ hold for any ring R (not necessarily a domain). So it remains to prove the implication $3 \Rightarrow 1$.

Let M be an R -module such that $I \simeq M \otimes_R R[T, T^{-1}]$ as $R[T, T^{-1}]$ -modules. Since I is a finitely generated nonzero ideal of $R[T, T^{-1}]$ and $R[T, T^{-1}]$ is a free R -module, it follows that M is a finitely generated torsion free R -module of rank one. Hence there exists a finitely generated nonzero ideal \mathcal{S} of R such that $M \simeq \mathcal{S}$ as R -modules. Thus

$$\mathcal{S}R[T, T^{-1}] \simeq \mathcal{S} \otimes_R R[T, T^{-1}] \simeq M \otimes_R R[T, T^{-1}] \simeq I$$

as $R[T, T^{-1}]$ -modules. Let $\theta: \mathcal{S}R[T, T^{-1}] \rightarrow I$ be an isomorphism. Let $b \in \mathcal{S}$ be a nonzero element of R . Then we claim that $bI = \theta(b)\mathcal{S}R[T, T^{-1}]$.

Let $g \in I$ and $h \in \mathcal{S}R[T, T^{-1}]$ be such that $\theta(h) = g$. Then $bg = b\theta(h) = \theta(bh) = h\theta(b)$ and this proves the claim.

Let $c \in I$ be a nonzero element of R . Then $cb = \theta(b)f$ for some $f \in \mathcal{S}R[T, T^{-1}]$. But since R is a domain, this shows that $\theta(b)T^n = a \in R$ for some integer n . Now the equality $bI = \theta(b)\mathcal{S}R[T, T^{-1}] = a\mathcal{S}R[T, T^{-1}]$ gives that $bJ = b(I \cap R) = a\mathcal{S}$. Therefore $bI = bJR[T, T^{-1}]$ and hence $I = JR[T, T^{-1}]$. \square

Remark 2.5. Lemma 2.4 is true if R is a finite direct product of domains and $I \cap R$ contains a non-zero-divisor of R .

Remark 2.6. The following example shows that Lemma 2.4 need not be true if R is not a direct product of domains.

Example 2.7. Let $R = k[[X, Y]]/(XY) = k[[x, y]]$. Let $f = x + yT$ be an element of $R[T, T^{-1}]$ and let $I = fR[T, T^{-1}]$. Then it is easy to see that $I \cap R = (x^2, y^2)$, which contains a non-zero-divisor $x^2 - y^2$. Moreover, since f is a non-zero-divisor of $R[T, T^{-1}]$, the ideal I is a free module of rank one over $R[T, T^{-1}]$ and hence it is extended from R as an $R[T, T^{-1}]$ -module. But obviously $fR[T, T^{-1}] \neq (x^2, y^2)R[T, T^{-1}]$ as (x^2, y^2) is not an invertible ideal of R . \square

In the case of a polynomial algebra $R[T]$ we get the following generalisation of Lemma 2.4.

Lemma 2.8. *Let R be a reduced ring, and let I be a finitely generated ideal of $R[T]$. Assume that $J = I \cap R$ contains a non-zero-divisor. Then the following statements are equivalent:*

- (1) $I = JR[T]$.
- (2) $I \simeq J \otimes_R R[T]$ as $R[T]$ -modules.
- (3) *There exists an R -module M such that $I \simeq M \otimes_R R[T]$ as $R[T]$ -modules.*

Proof. As above the implications $1 \Rightarrow 2$ and $2 \Rightarrow 3$ are obvious. Thus it remains to prove $3 \Rightarrow 1$.

Let $s \in J$ be a non-zero-divisor of R . Then it is easy to see that M is a finitely generated torsion free R -module and M_s is a free R_s -module of rank one. Therefore there exists a finitely generated ideal \mathcal{S} of R such that $\mathcal{S} \simeq M$ as R -modules. Moreover, since \mathcal{S}_s is a free R_s -module of rank one, without loss of generality we can assume that $t = s^n \in \mathcal{S}$ for some positive integer n . Let $\theta: \mathcal{S}R[T] \rightarrow I$ be an isomorphism of $R[T]$ -modules. Then we claim that $tI = \theta(t)\mathcal{S}R[T]$.

Let $g \in I$ and $f \in \mathcal{S}R[T]$ be such that $\theta(f) = g$. Then $tg = t\theta(f) = \theta(tf) = f\theta(t)$. This proves the claim.

Since $s \in I$, the equality $tI = \theta(t)\mathcal{S}R[T]$ shows that $ts = \theta(t)g$ for some $g \in \mathcal{S}R[T]$. Now by Lemma 2.9 (stated below) we have $\theta(t) \in R$. Therefore

$$tJ = tI \cap R = \theta(t)\mathcal{S}R[T] \cap R = \theta(t)\mathcal{S}.$$

Hence $tI = tJR[T]$. But t is a non-zero-divisor of R . Therefore $I = JR[T]$. \square

Lemma 2.9. *Let R be a reduced ring and s be a non-zero-divisor of R . Let $f \in R[T]$ be such that $s \in fR[T]$. Then $f \in R$.*

Proof. Let $s = f(T)g(T)$ for some $g(T) \in R[T]$. Write $f(T) = a_0 + a_1T + \cdots + a_nT^n$ for some $a_i \in R$, $0 \leq i \leq n$, with $a_n \neq 0$. We want to show that $n = 0$.

Since $s = f(0)g(0) = a_0g(0)$ and s is a non-zero-divisor, a_0 is either a unit or a non-zero-divisor in R . If $n > 0$ then since R is reduced there exists a minimal prime ideal p of R such that $a_n \notin p$. Since a_0 is a unit or a non-zero-divisor, obviously $a_0 \notin p$. Let ‘bar’ denote ‘modulo p ’. Then we have $\bar{s} = \bar{f}(T)\bar{g}(T)$ in $\bar{R}[T]$. But since $\bar{f}(T)$ is a polynomial of positive degree this is absurd and hence $n = 0$. \square

When R is reduced (but not necessarily a domain) one has the following weaker version of Lemma 2.4.

Lemma 2.10. *Let R be a reduced ring, and let I be a finitely generated ideal of $R[T, T^{-1}]$ such that $J = I \cap R$ contains a non-zero-divisor of R . If I is extended from R as a module, then there exists an element f of $R[T, T^{-1}]$ such that f is not a zero-divisor of $R[T, T^{-1}]$ and fI is extended from R as an ideal.*

This easily follows from the proof of Lemma 2.4.

Now we prove the main theorems.

Theorem 2.11. *Let R be a reduced ring with only finitely many minimal prime ideals. Let I be an invertible ideal of $R[T, T^{-1}]$ such that $J = I \cap R$ contains a non-zero-divisor of R . Then I is extended from R if J is an invertible ideal of R . Moreover, if R is an integral domain and I is extended from R , then $J = I \cap R$ is an invertible ideal of R .*

Proof. Let I be an invertible ideal of $R[T, T^{-1}]$ such that $J = I \cap R$ contains a non-zero-divisor s of R and J is an invertible ideal of R . If $I = R[T, T^{-1}]$ then clearly $I = JR[T, T^{-1}]$, where $J = I \cap R = R$. So we can assume that I is a proper ideal of $R[T, T^{-1}]$. Now we will show that $I = JR[T, T^{-1}]$. Clearly $JR[T, T^{-1}] \subseteq I$ and to show the equality it is enough to show that for every maximal ideal \mathfrak{J} of R , $J_{\mathfrak{J}}R_{\mathfrak{J}}[T, T^{-1}] = I_{\mathfrak{J}}$. But then $I_{\mathfrak{J}} \cap R_{\mathfrak{J}} = J_{\mathfrak{J}}$, and as $J_{\mathfrak{J}}$ is an invertible ideal in the local ring $R_{\mathfrak{J}}$, $J_{\mathfrak{J}}$ is principal and hence $J_{\mathfrak{J}} = tR_{\mathfrak{J}}$ for some $t \in R$ which is not a zero-divisor of R . Thus by replacing R with $R_{\mathfrak{J}}$ we are reduced to proving the following:

Let R be a reduced local ring with only finitely many minimal prime ideals. Let I be an invertible ideal of $R[T, T^{-1}]$ which is a proper ideal such that $I \cap R = tR$ for some non-zero-divisor t of R . Then $I = tR[T, T^{-1}]$.

Since I is an invertible ideal of $R[T, T^{-1}]$ containing a non-zero-divisor t of R , it is easy to see that the canonical epimorphism $I/(T-1)I \rightarrow I + (T-1)/(T-1)$ is an isomorphism and hence

$$(*) \quad I/(T-1)I \simeq I + (T-1)/(T-1) \simeq I/I \cap (T-1).$$

This shows that $I + (T-1)/(T-1)$ is an invertible ideal of R which contains the element t of R .

If $I \neq tR[T, T^{-1}]$ then there exists an element $g_1 \in R[T, T^{-1}]$ such that $g_1 \in I \setminus tR[T, T^{-1}]$. Without loss of generality we may assume that g_1 is a polynomial in $R[T]$ and is of least degree (among such elements of I). Let us write g_1 as $g_1 = a_0 + a_1(T-1) + \dots + a_r(T-1)^r$ with $a_i \in R$ and $a_r \neq 0$. Obviously $t \nmid a_0$. Otherwise $a_0 = ta$ for some $a \in R$. Hence $g_1 - a_0 = g_1 - ta = (T-1)f$ for some $f \in R[T]$. As $T-1$ is a non-zero-divisor modulo I (by $(*)$) we have $f \in I \setminus tR[T, T^{-1}]$. But $\deg f < \deg g_1$, contradicting the minimality of degree of g_1 . Hence $t \nmid a_0$.

Let $\{t, g_1, g_2, \dots, g_n\} \subseteq R[T, T^{-1}]$ be a set of generators of I , where $g_i \in R[T]$ for $1 \leq i \leq n$. Then $I + (T-1)/(T-1) = (t, g_1(1) = a_0, g_2(1), \dots, g_n(1))$. Since $I + (T-1)/(T-1)$ is an invertible ideal of R and R is local, $I + (T-1)/(T-1)$ is a principal ideal of R generated, say, by b and $b \in \{t, a_0, g_2(1), \dots, g_n(1)\}$. But since $t \nmid a_0$ we have $t \nmid b$. So $b = g_i(1)$ for some i , $1 \leq i \leq n$. Moreover $t/b = d$ belongs to the maximal ideal of R .

Let \bar{R} be the normalisation of R in its total quotient ring. Note that \bar{R} is a finite direct product of domains. Since \bar{R} is normal and $I\bar{R}[T, T^{-1}]$ is invertible, it is extended from \bar{R} as a module. Therefore, as $I\bar{R}[T, T^{-1}]$ contains a non-zero-divisor of \bar{R} , namely t , by Lemma 2.4 and Remark 2.5, $I\bar{R}[T, T^{-1}] = L\bar{R}[T, T^{-1}]$, where $L = I\bar{R}[T, T^{-1}] \cap \bar{R}$.

Let $\{a_1, a_2, \dots, a_m\} \subseteq \bar{R}$ be a set of generators for L . Recall that $\{t = g_0, g_1, \dots, g_n\}$ is a set of generators for $I\bar{R}[T, T^{-1}]$. Then we get

the following relations:

$$g_i = \sum_{j=1}^m h_{ij} a_j \quad \text{for } i = 0, 1, \dots, n,$$

$$a_k = \sum_{l=0}^n f_{kl} g_l \quad \text{for } k = 1, 2, \dots, m,$$

for some h_{ij} and f_{kl} in $\bar{R}[T, T^{-1}]$. Let R' be the R -subalgebra of \bar{R} generated by $\{a_i\} \cup$ coefficients of $\{h_{ij}, f_{kl}\}$, and let L' be the ideal of R' generated by $\{a_1, a_2, \dots, a_m\}$. Clearly R' is a finitely generated R -subalgebra of \bar{R} . Therefore R' is a finite R -module and hence R' is semilocal. Then the equality $IR'[T, T^{-1}] = L'R'[T, T^{-1}]$ shows that L' is an invertible ideal of R' and hence (R' being semilocal) is a principal ideal, generated by, say, r . Thus we have $IR'[T, T^{-1}] = (t, g_1, \dots, g_n)R'[T, T^{-1}] = rR'[T, T^{-1}]$. Therefore $rR' = (t, g_1(1), \dots, g_n(1))R' = bR'$. Hence without loss of generality we can assume that $r = b$.

Now we claim that there exists a finitely generated R -subalgebra \tilde{R} of R' such that

- (1) $I\tilde{R}[T, T^{-1}] = b\tilde{R}[T, T^{-1}]$,
- (2) d^l ($d = t/b$) $\in \mathcal{E}_{\tilde{R}/R}$ for some positive integer l , where $\mathcal{E}_{\tilde{R}/R}$ denotes the conductor ideal of \tilde{R} in R .

We will now complete the proof of the theorem by assuming this claim.

Since $(t = g_0, g_1, \dots, g_n)\tilde{R}[T, T^{-1}] = b\tilde{R}[T, T^{-1}]$, we can write $b = \sum_{i=0}^n h_i g_i$, where $h_i \in \tilde{R}[T, T^{-1}]$. Then for $c \in \mathcal{E} = \mathcal{E}_{\tilde{R}/R}$, we have $cb = c \sum_{i=0}^n h_i g_i = \sum_{i=0}^n (ch_i) g_i$. Since $c \in \mathcal{E}$, we have $ch_i \in R[T, T^{-1}]$ and hence $cb \in I \cap R = tR$. This shows that $(b/t)\mathcal{E} = \mathcal{S}$ is an ideal of R . Clearly \mathcal{S} is an ideal of \tilde{R} and hence $\mathcal{S} \subseteq \mathcal{E}$. This shows that $\mathcal{E} \subseteq d\mathcal{E}$ and therefore $\mathcal{E} = d\mathcal{E}$. Hence $\mathcal{E} = d\mathcal{E} = d^2\mathcal{E} = \dots = d^l\mathcal{E} \subseteq d^l R \subseteq \mathcal{E}$. Therefore $\mathcal{E} = d^l R = d^{l+1} R$, which is absurd since d is an element of the maximal ideal of R which is a non-zero-divisor.

Therefore $I = tR[T, T^{-1}]$ as required.

Proof of the claim. Since $(t = g_0, g_1, \dots, g_n)R'[T, T^{-1}] = bR'[T, T^{-1}]$, $g_i = b g'_i$ ($0 \leq i \leq n$), where $g'_i \in R'[T, T^{-1}]$. In fact, since $g_i \in R[T]$ and b is not a zero-divisor of R' , we have $g'_i \in R'[T]$. Moreover $g'_0 = d$.

Let $K = b^{-1}I$. Then K is an invertible $R[T, T^{-1}]$ -submodule of $R'[T, T^{-1}]$ generated by $\{g'_0, g'_1, \dots, g'_n\}$. Since $KR'[T, T^{-1}] = R'[T, T^{-1}]$, we have $K^{-1} \subseteq R'[T, T^{-1}]$ and $K^{-1}R'[T, T^{-1}] = R'[T, T^{-1}]$.

Let $K^{-1} = (u_0, u_1, \dots, u_n)R'[T, T^{-1}]$, where $u_i \in R'[T, T^{-1}]$ for $0 \leq i \leq n$. Let \tilde{R} denote the finitely generated R -subalgebra of R' generated by the coefficients of $\{u_i\}_{i=0}^n$. Then $u_i \in \tilde{R}[T, T^{-1}]$ for all i . Since R' is integral over \tilde{R} and $K^{-1}R'[T, T^{-1}] = R'[T, T^{-1}]$, we get that $K^{-1}\tilde{R}[T, T^{-1}] = \tilde{R}[T, T^{-1}]$. This shows that $K \subseteq \tilde{R}[T, T^{-1}]$ and $K\tilde{R}[T, T^{-1}] = \tilde{R}[T, T^{-1}]$. Since $K = b^{-1}I$, we get that $I\tilde{R}[T, T^{-1}] = b\tilde{R}[T, T^{-1}]$. This proves the first part of the claim.

Since \tilde{R} is generated as an R -algebra by coefficients of u_i , generators of \tilde{R} as an R -module can be chosen to be elements which are monomials in coefficients

of u_i . Since \tilde{R} is a finite R -module, finitely many such monomials will generate \tilde{R} as an R -module. Since $d = g'_0 \in K$, we have $du_i \in R[T, T^{-1}]$ for all i , $i = 0, 1, \dots, n$. This shows that $d^l \tilde{R} \subseteq R$ for some positive integer l . Thus the proof of the claim is complete.

Now assume that R is a domain. If I is extended from R as a module, then by Lemma 2.4, $I = JR[T, T^{-1}]$, since $J = I \cap R \neq 0$. Therefore J is an invertible ideal of R . \square

Remark 2.12. If R is not necessarily a domain and if I is extended from R as a module, then under the hypothesis of the theorem we get, by Lemma 2.10, that there exists an element $f \in R[T, T^{-1}]$ such that $fI \cap R$ is an invertible ideal of R .

Theorem 2.13. *Let R be a reduced ring with only finitely many minimal prime ideals. Let I be an invertible ideal of $R[T]$ such that $J = I \cap R$ contains a non-zero-divisor of R . Then I is extended from R as a module if and only if J is an invertible ideal of R .*

Proof. If I is extended from R then, by Lemma 2.8, $I = JR(T)$. Therefore it follows that J is an invertible ideal of R . To prove the converse, without loss of generality, we can assume that I is a proper ideal.

Let us assume that J is an invertible ideal of R . Then as I contains a non-zero-divisor, say s , of R , it is easy to see that the canonical epimorphism $I/IT \rightarrow I+(T)/(T) \simeq I/I \cap (T)$ is an isomorphism. This implies that T is not a zero-divisor of $R[T]/I$. Therefore $IR[T, T^{-1}] \cap R[T] = I$. Now the ideal $IR[T, T^{-1}]$ is an invertible ideal of $R[T, T^{-1}]$ such that $IR[T, T^{-1}] \cap R = I \cap R = J$. Therefore, by Theorem 2.11, we get that $IR[T, T^{-1}] = JR[T, T^{-1}]$. This implies that for any $f \in I$, $T^n f \in JR[T]$ for some positive integer n . This shows that $f \in JR[T]$ and hence $I = JR[T]$ as required. \square

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