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WEAKLY COMPACT HOMOMORPHISMS FROM GROUP ALGEBRAS

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ABSTRACT. A locally compact group G has the weakly compact homomorphism property if every weakly compact homomorphism from the group algebra $L^1(G)$ into another Banach algebra $\mathfrak{B}$ has finite-dimensional range. It has been shown that compact and abelian groups have this property. We extend this to a large class of groups including all solvable groups.

1. INTRODUCTION

In a number of papers [2–4] the authors have considered the question of whether there are any weakly compact homomorphisms from $\mathfrak{A}$ into $\mathfrak{B}$ with infinite-dimensional range where $\mathfrak{A}$ and $\mathfrak{B}$ are two given Banach algebras. We are concerned here solely with the case $\mathfrak{A} = L^1(G)$ where G is a locally compact group. We say that G is WCHP (or has the weakly compact homomorphism property) if every weakly compact homomorphism from $L^1(G)$ into any Banach algebra $\mathfrak{B}$ has a finite-dimensional range. It has been proved that compact groups [2, Corollary 2.5] and abelian groups [2, Corollary 2.4] are WCHP. There is a process [4, Theorem 4.1] for proving that groups are WCHP which applies in a variety of cases. It seems probable that all locally compact groups are WCHP, but we have not been able to determine whether this is so. We give here (in §3) a number of results which show that if WCHP groups are combined in certain ways, the resulting group is WCHP. We would like to be able to prove that if G is a locally compact group containing a closed normal subgroup such that G/H and H are WCHP then G is WCHP. The main result of this paper, Theorem 4.3, is that if G/H satisfies a stronger condition, which we call WCHP$^+$ and which involves certain groups related to G/H also being WCHP, then G is WCHP. Having introduced this new property we consider its heredity; this is also done in §4. It turns out that most groups, for example, abelian and compact groups, which were known to be WCHP are also WCHP$^+$ and a variant of Theorem 4.1 of [4] can be proved for WCHP$^+$. These results are in §5. We are thus in the position that if G is a group with a finite series of closed subgroups $\{e\} = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$ for which G_{i-1} is normal in G_i and G_i/G_{i-1} is WCHP$^+$ at each step then G is WCHP$^+$. In particular, every solvable group

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is WCHP$^+$. Finally in §6 we review the WCHP problem in the light of the results in this paper.

In the remainder of this section we give some background results and introduce some notation, and in §2 we describe two constructions which we use repeatedly in later sections.

In the rest of the paper, unless we specifically say otherwise, G will be an arbitrary locally compact topological group.

Definition 1.1. An *eligible homomorphism* from G is a bounded continuous group homomorphism from G into a conditionally weakly compact subset of the regular group of some unital Banach algebra $\mathfrak{B}$. It is *dense* if the linear span of its range is dense in $\mathfrak{B}$ and *finite dimensional* if the linear span of its range is finite dimensional.

Given an eligible homomorphism ϕ we can put

$$\theta(\mu) = \int_G \phi(g)\, d\mu(g) \qquad (\mu \in M(G)). \tag{1.1}$$

θ is then a continuous homomorphism from $M(G)$ and hence $l^1(G)$ and $L^1(G)$ into $\mathfrak{B}$ which is continuous on bounded sets in $M(G)$ with respect to a number of topologies on its domain weaker than the norm topology. It follows that if any of ϕ, $\theta(M(G))$, $\theta(L^1(G))$, $\theta(l^1(G))$ is finite dimensional then so are the others and that if any of them is dense then so are the others. Galé [3, Proposition 2.2] has shown that if we start with a weakly compact homomorphism θ of $L^1(G)$ onto a dense subalgebra of a Banach algebra $\mathfrak{B}$ then $\mathfrak{B}$ is unital and there is an eligible homomorphism ϕ of G into $\mathfrak{B}$ related to θ as in (1.1). Thus to show that G is WCHP it is necessary and sufficient to show that every eligible homomorphism from G is finite dimensional. As a notational convenience given an eligible homomorphism ϕ we shall also denote the homomorphism θ constructed from ϕ by ϕ.

Suppose now that ϕ is a finite-dimensional homomorphism from G and $\mathfrak{C}$ is the linear span of $\phi(G)$. Thus $\Gamma = \phi(G)^-$ is a compact group and the identity map gives a homomorphism ψ from Γ into $\mathfrak{C}$ which extends to $M(\Gamma)$ and $L^1(\Gamma)$. Because of the structure of $L^1(\Gamma)$, $\mathfrak{C}$ is the direct sum of a finite number of full matrix algebras and $\mathfrak{C} = \phi(l^1(G)) = \phi(L^1(G)) = \phi(M(G)) = \psi(l^1(\Gamma)) = \psi(L^1(\Gamma)) = \psi(M(\Gamma))$. Moreover, we can make $\mathfrak{C}$ a C^*-algebra in such a way that the elements of Γ are unitary.

2. Two constructions

A. **The pullback construction.** Let G_1, G_2, and H be three locally compact groups, and let $\alpha_i\colon G_i \to H$ be continuous homomorphisms. The pullback group Γ is the closed subgroup $\{(g_1, g_2) : g_i \in G_i,\ \alpha_1(g_1) = \alpha_2(g_2)\}$ of $G_1 \times G_2$ together with the maps $\beta_i\colon \Gamma \to G_i$ which are the restrictions of the coordinate projections. We have $\alpha_1\beta_1 = \alpha_2\beta_2$.

Lemma 2.1. *In the above situation the following hold.*

(a) β_1 *is surjective if and only if* $\alpha_2(G_2) \supseteq \alpha_1(G_1)$. *In particular, if* α_2 *is surjective so is* β_1.
(b) β_1 *is injective if and only if* α_2 *is.*
(c) *If* α_2 *is open so is* β_1.
(d) *If* α_2 *is open and* α_1 *has dense range then* β_2 *has dense range.*

The proof of this result is a routine verification. A continuous homomorphism α of G into another locally compact group G' is an open surjection if and only if α lifts to an isomorphism of $G/(\ker \alpha)$ onto G'. We call such maps *quotient maps.* From the lemma we see that if α_2 is a quotient map then so is β_1.

B. **The WCHP subgroup construction.** Let ϕ be an eligible dense homomorphism of the locally compact group G into $\mathfrak{B}$, and suppose H is a closed normal WCHP subgroup. Then $\phi(l^1(H))$ is the direct sum of a finite number of full matrix algebras and $\phi|_{l^1(H)}$ decomposes as a direct sum $\phi_1 \oplus \cdots \oplus \phi_r$ of maps onto these summands. For each $g \in G$, $\alpha(g)\cdot b \mapsto \phi(g)b\phi(g)^{-1}$ is an automorphism of $\mathfrak{B}$ which maps $\phi(H)$ and hence $\phi(l^1(H))$ onto itself. Thus this map permutes the minimal central idempotents $e_j = \phi_j(e)$ of $\phi(l^1(H))$; that is, there is a permutation $\sigma(g)$ of $\{1, \dots, r\}$ with $\phi(g)e_j\phi(g)^{-1} = e_k$ where $k = \sigma(g)j$. We have $\sigma(g)\sigma(g') = \sigma(gg')$ and hence G acts on $\{1, \dots, r\}$. Partition $\{1, \dots, r\}$ into distinct orbits $O_1, \dots, O_s$ under this action, and put $\tilde{e}_j = \sum(k \in O_j)e_k$. Then $\phi(g)\tilde{e}_j\phi(g)^{-1} = \tilde{e}_j$, so each $\tilde{e}_j$ is a central idempotent in $\mathfrak{B}$ and ϕ can be decomposed as $\psi_1 \oplus \cdots \oplus \psi_s$ where $\psi_i(g) = \tilde{e}_i\phi(g)$. If i and j belong to the same orbit O_l then there is an automorphism $b \mapsto \phi(g)b\phi(g^{-1})$ which maps e_i to e_j and hence $\phi_i(l^1(H)) = e_i\phi(l^1(H))$ onto $\phi_j(l^1(H)) = e_j\phi(l^1(H))$ so these are isomorphic matrix algebras. Take a fixed element j_0 in the orbit and for each element j in the orbit choose $g_j \in G$ with $\phi(g_j)e_j\phi(g_j)^{-1} = e_{j_0}$. Then $b \mapsto [\phi(g_j)e_jbe_k\phi(g_k)^{-1}]_{j,k}$ is an isomorphism of $\tilde{e}_l\mathfrak{B}$ into the algebra of $p \times p$ matrices with entries from $e_{j_0}\mathfrak{B}e_{j_0}$, i.e., $M_p \otimes e_{j_0}\mathfrak{B}e_{j_0}$, where p is the number of elements in O_l. As $e_{j_0}\mathfrak{B}e_{j_0} \cap \phi(l^1(H)) = M_n$, the algebra of $n \times n$ matrices, for some n, we see, using the matrix units e_{1j} in place of the $\phi(g_j)$, that $e_{j_0}\mathfrak{B}e_{j_0} \simeq M_n \otimes \mathfrak{B}_0$ where $\mathfrak{B}_0 = e_{11}\mathfrak{B}e_{11}$. Thus $\mathfrak{B}$ is the direct sum of a finite number of algebras of the form $M_p \otimes M_n \otimes \mathfrak{B}_0$. We shall think of these as algebras of $p \times p$ matrices with entries from $M_n \otimes \mathfrak{B}_0$. The component of $\phi(a)$ where $a \in l^1(H)$ is a diagonal matrix with entries from $M_n \otimes e_0$ where e_0 is the identity in $\mathfrak{B}_0$. The components of $\phi(g)$ are $p \times p$ matrices with only one nonzero entry in each row and column. If $j = \sigma(g)i$ then $\phi_j(h) = e_j\phi(h) = \phi(g)e_i\phi(g^{-1})\phi(h) = \phi(g)\phi_i(g^{-1}hg)\phi(g)^{-1}$, so ϕ_j is equivalent to $h \mapsto \phi_i(g^{-1}hg)$ in the sense that there is an isomorphism between their ranges which transforms one map into the other.

3. Inheritance results for WCHP

Theorem 3.1. *If G is a locally compact group and H is a closed subgroup of finite index then G is WCHP if and only if H is.*

Proof. Suppose that H is WCHP and ϕ is a dense eligible homomorphism of G. Select elements $t_1, t_2, \dots, t_n$, one from each right coset of H. Then $\phi | H$ is finite dimensional and $\phi(G) = \bigcup_{i=1}^n \phi(H)\phi(t_i)$, so ϕ is finite dimensional.

Conversely suppose G is WCHP and ϕ is a dense eligible homomorphism of H into $\mathfrak{B}$. Select $t_1, \dots, t_n$ as above with $t_1 = e$. For each $g \in G$ there is a permutation $\sigma(g)$ of $\{1, \dots, n\}$ such that for each $i \in \{1, \dots, n\}$ there is $h_{i,g} \in H$ with $t_i g = h_{i,g}t_j$ where $j = \sigma(g)i$. This defines $\sigma(g)$ and the $h_{i,g}$ uniquely. Consider the algebra $\mathfrak{B}' = M_n \otimes \mathfrak{B}$ of $n \times n$ matrices with elements from $\mathfrak{B}$. It can be given a norm equivalent to $\mathfrak{B}^{n^2}$ with the sup norm under which it is a unital Banach algebra. Define $b_g = \operatorname{diag}\phi(h_{i,g})$, and let s_g be the permutation matrix with $(s_g)_{ij} = \delta_{ik}1$ where $k = \sigma_g^{-1}(j)$ and 1 is the identity in $\mathfrak{B}$. Put $\Phi(g) = b_g s_g$. We have $t_i gg' = (t_i g)g'$, so $h_{i,gg'} = h_{i,g}h_{j,g'}$ where $j = \sigma(i)$ and $\sigma_{gg'} = \sigma_{g'}\sigma_g$. Thus $s_{gg'} = s_g s_{g'}$ and $\Phi(g)\Phi(g') = b_g s_g b_{g'} s_g^{-1} s_{gg'} = b_{gg'} s_{gg'} = \Phi(gg')$. The range of Φ consists of matrices with entries from $\Phi(H)$ and so is conditionally weakly compact. Also $\Phi(e) = I_n \otimes 1$ and Φ is continuous, so Φ is an eligible homomorphism from G. Consequently it is finite dimensional. For $h \in H$, $t_1 h = h$, so $h_{1,h} = h$ and $\sigma(h)1 = 1$. Thus $\Phi(h)_{11} = \phi(h)$. The map $c \to c_{11}$ of $\mathfrak{B}'$ onto $\mathfrak{B}$ is linear, so $\phi(H) \subseteq \Phi(G)_{11}$ is finite dimensional.

Theorem 3.2. *Let G and G' be two locally compact groups, and let α be a continuous homomorphism of G onto a dense subgroup of G'. If G is WCHP then so is G'.*

Proof. If ϕ is an eligible homomorphism from G' then $[\phi\alpha(G)]^- = [\phi(G')]^-$ (norm closure), $\phi\alpha$ is an eligible homomorphism from G, and $[\phi\alpha(G)]^- \supseteq \phi(G')$. So if $\phi\alpha$ is finite dimensional then ϕ is.

Note that this applies when $G' = G$ algebraically and α is the identity map. In particular, if G is WCHP as a discrete group then it is WCHP with its given topology.

Theorem 3.3. *Let G be a locally compact group and let $H_1, \ldots, H_n$ be closed subgroups such that $G = [H_1 H_2 \cdots H_n]^-$. If the H_i are all WCHP then so is G.*

Proof. By $G = [H_1 H_2 \cdots H_n]^-$ we mean that every element of a dense subset E_0 of G can be expressed as a product $h_1 h_2 \cdots h_n$ with $h_i \in H_i$, $i = 1, \ldots, n$. Let ϕ be an eligible homomorphism from G. Then $\phi(H_i)$ is finite dimensional for each i and hence $\phi(E_0) \subseteq \phi(H_1)\phi(H_2)\cdots\phi(H_n)$ and $\phi(G) \subseteq \phi(E_0)^-$ are.

It follows from this result that a semidirect product of WCHP groups is WCHP.

4. WCHP$^+$ GROUPS

Definition 4.1. The locally compact group G is said to be WCHP$^+$ if every locally compact group Γ containing a closed abelian normal subgroup H with $\Gamma/H \cong G$ is WCHP.

This definition is introduced to enable us to prove Theorem 4.3. Additional discussion of the definition follows the proof of that theorem.

Theorem 4.2. (i) *Let H be a closed subgroup of finite index in G. If H is WCHP$^+$ then so is G.*

(ii) *Theorems* 3.2 *and* 3.3 *apply with WCHP$^+$ in place of WCHP.*

Proof. (i) is a simple variation of the easy part of Theorem 3.1. For the WCHP$^+$ version of Theorem 3.2 we take a locally compact group Γ' containing a closed abelian normal subgroup H' with $\Gamma'/H' \simeq G'$ and an eligible homomorphism ϕ' from Γ'. We form the pullback Γ of the maps $\alpha\colon G \to G'$ and the quotient map $q'\colon \Gamma' \to G'$ and denote the maps $\Gamma \to G$, $\Gamma \to \Gamma'$ by q and α' respectively. Then q is a quotient map and its kernel is $\{(e, h') : h' \in H'\} = H$. Thus H is a closed normal abelian subgroup of Γ.

Also $\phi'\alpha'$ is an eligible homomorphism from Γ and hence is finite dimensional. Since α' has dense range, this implies that ϕ' is finite dimensional.

For the WCHP$^+$ version of Theorem 3.3 we use the same notation as in that theorem and Definition 4.1 and denote the quotient map $\Gamma \to G$ by q. Put $H_i' = q^{-1}(H)$ and let $\gamma \in q^{-1}(E_0)$. Then $q(\gamma) = h_1 h_2 \cdots h_n$ with $h_i \in H_i$. Thus $\gamma = h_1' h_2' \cdots h_n'$ where $h_1', \ldots, h_{n-1}'$ are arbitrary selections from $H_1', \ldots, H_{n-1}'$ with $q(h_i') = h_i$ and $h_n' = \gamma(h_1' \cdots h_{n-1}')^{-1} \in q^{-1}(h_n)$. The H_i' are WCHP because the H_i are WCHP$^+$, and the result follows from Theorem 3.3.

Theorem 4.3. *Let G be a locally compact group and let H be a closed normal subgroup. If H is WCHP and G/H is WCHP$^+$ then G is WCHP.*

Proof. First of all we introduce some notation. If E is a set and p a positive integer then $S(p, E)$ is the set of $p \times p$ matrices which contain one element of E and $p-1$ zeroes in each row and column. If E has a multiplication (we are interested in the case in which E is a group or an algebra) then matrix multiplication defines a product on $S(p, E)$, and if E has an identity element 1 then $S_p = S(p, 1) \subseteq S(p, E)$. We denote the subset of $S(p, E)$ of diagonal matrices by $D(p, E) \simeq E^p$. We have $\sigma d \sigma^{-1} \in D(p, E)$ whenever $\sigma \in S_p$ and $d \in D(p, E)$, and, in fact, $S(p, E)$ is the semidirect product of S_p and E^p with this action of S_p on E^p. Suppose ϕ is an eligible representation of G on $\mathfrak{B}$, and make the WCHP subgroup construction for the subgroup H. If we can show that $g \mapsto \tilde{e}_l \phi(g)$ has finite-dimensional range for each orbit O_l then it will follow that ϕ has finite-dimensional range. Thus we can reduce to the case in which there is a single orbit. We have $\phi(l^1(H)) = D(p, M_n \otimes e_0)$ and $\phi(G) \subseteq S(p, M_n \otimes \mathfrak{B}_0)$. Every automorphism of $D(p, M_n \otimes e_0)$ is of the form $d \mapsto u d u^{-1}$ where $u \in S(p, n) = S(p, Sl(n, \mathbb{C}) \otimes e_0)$. Thus for each $g \in G$ there is $m(g) \in S(p, n)$ with $\phi(g) d \phi(g)^{-1} = m(g) d m(g)^{-1}$ for all $d \in \phi(l^1(H))$. Thus $b(g) = m(g)^{-1}\phi(g) \in D(p, I_n \otimes \mathfrak{B}_0)$ and $\phi(g) = m(g) b(g)$. We can factor $m(g) = \sigma(g)\mu(g)$ where $\sigma(g) \in S_p$ and $\mu(g) \in D(p, Sl(n, \mathbb{C}) \otimes e_0)$. Denote $Sl(n, \mathbb{C})/Z$, where Z is the centre of $Sl(n, \mathbb{C})$, by $Sl'(n, \mathbb{C})$, $S(p, Sl'(n, \mathbb{C}))$ by $S'(p, n)$, and the quotient map $S(p, n) \to S'(p, n)$ by q. The definition of $m(g)$ involved an arbitrary selection but $qm(g)$ is uniquely determined by ϕ and g. $S(p, n)$ and $S'(p, n)$ are Lie groups and, given $g_0 \in G$, $m(g)$ could have been chosen continuously in a neighborhood of g_0, so we see that qm is continuous. Also $\phi(gg')d\phi(gg')^{-1} = m(g)m(g')dm(g')^{-1}m(g)^{-1}$ for all $d \in \phi(l^1(H))$, so $m(g)m(g')$ is a possible choice for $m(gg')$ and hence $qm(g)qm(g') = qm(gg')$ $(g, g' \in G)$.

Let G_2 be the pull-back of the maps q and qm. We denote the maps $G_2 \to G$ and $G_2 \to S(p, n)$ by q' and m' respectively. q' is a quotient map because q is. We have $qmq' = qm'$, so $mq'(g) = m'(g)\omega(g)$ for all $g \in G_2$ where ω is a function from G_2 into $D(p, \mathbb{T}_n I_n \otimes e_0)$ and $\mathbb{T}_n$ is the group of nth roots of 1. Now put $\beta(g) = \omega(g) b(q'g) \in D(p, I_n \otimes \mathfrak{B}_0)$ and $\theta(g) = \sigma(q'g)\beta(g) \in S(p, I_n \otimes \mathfrak{B}_0)$. θ is clearly continuous, and we will show that it is a group homomorphism. Put $\phi' = \phi \circ q'$, and factor $m'(g) = \sigma'(g)\mu'(g)$ where $g \in G_2$, $\sigma'(g) \in S_p$, and $\mu'(g) \in D(p, Sl(n, \mathbb{C}) \otimes e_0)$. We have $\phi'(g) = m(q'g)b(q'g) = m'(g)\beta(g) = \sigma'(g)\mu'(g)\beta(g)$. The identity

$\phi'(gg') = \phi'(g)\phi'(g')$ yields the identity

$$\begin{aligned}&\sigma'(gg')\mu'(gg')\beta(gg')\\ &\quad = \sigma'(gg')[\sigma'(g')^{-1}\mu(g)\sigma'(g')\mu'(g')][\sigma'(g')^{-1}\beta(g)\sigma'(g')\beta(g')],\end{aligned}$$

using the fact that elements of $D(p, I_n \otimes \mathfrak{B}_0)$ commute with elements of $D(p, Sl(n, \mathbb{C})\otimes e_0)$. A similar calculation starting with $m'(gg') = m'(g)m'(g')$ yields

$$\sigma'(gg')\mu(gg') = \sigma'(gg')\sigma'(g')^{-1}\mu'(g)\sigma'(g')\mu'(g')$$

so that $\beta(gg') = \sigma'(g')^{-1}\beta(g)\sigma'(g')\beta(g')$, from which it follows that $\theta(gg') = \theta(g)\theta(g')$. If e is the identity in G_2 then $\phi'(e) = m'(e)\beta(e) = \sigma(e)\mu'(e)\beta(e)$ and $m'(e)$ are both the identity in $\mathfrak{B}$ and so $\sigma'(e)$, $\beta(e)$, and $\theta(e)$ are all the identity element of $\mathfrak{B}$. We also have $\phi'(g) = \theta(g)\mu'(g)$.

It remains to show that $\theta(G_2)$ is conditionally weakly compact in $\mathfrak{B}_1 = M_p\otimes I_n\otimes\mathfrak{B}_0 \subseteq M_p\otimes M_n\otimes\mathfrak{B}_0$. Because $\ker\sigma'$ has finite index in G_2, it is enough to show that $\theta(\ker\sigma')$ is conditionally weakly compact. For $g \in \ker\sigma'$ we have $\theta(g) \in D(p, I_n\otimes\mathfrak{B}_0)$, $u'(g) \in D(p, M_n\otimes e_0)$, so it suffices to show that for each j the diagonal entries $\{e_j\theta(g)e_j : g \in \ker\sigma'\}$ form a conditionally weakly compact set in $e_j\mathfrak{B}e_j \simeq M_n \otimes \mathfrak{B}_0$. The norm on $e_j\mathfrak{B}e_j$ is equivalent to the tensor norm $\| \|^\frown$ on $M_n\hat{\otimes}\mathfrak{B}_0$. We have $\|e_j\phi'(g)e_j\|^\frown = \|e_j\theta(g)e_j\| \, \|e_j\mu'(g)e_j\|$. Also $e_j\mu'(g)e_j = x\otimes e_0$ where x is a matrix with determinant 1 and so $\|e_j\mu'(g)e_j\| \geq 1$. This shows that $\|e_j\theta(g)e_j\| \leq K$ where K is a bound for $\{\|e_j\phi'(g)e_j\|^\frown : g \in \ker\sigma'\}$ and hence $\|e_j\mu'(g)e_j\| \leq \|e_j\theta(g^{-1})e_j\| \, \|e_j\phi'(g)e_j\|^\frown \leq K^2$. Thus

$$e_j\theta(\ker\sigma')e_j \subseteq e_j\phi'(\ker\sigma')e_j \cdot e_j\mu'(\ker\sigma')e_j$$

where the first set on the right is conditionally weakly compact and the second is a bounded subset of a finite-dimensional space, so $e_j\theta(\ker\sigma')e_j$ is conditionally weakly compact. We have shown that θ is an eligible homomorphism from G_2 into $\mathfrak{B}_1$.

For $h \in H$ we have $m(h)$ and $\phi(h) \in D(p, M_n\otimes e_0)$, so $\sigma(h)$ is the identity and $b(h) \in D(p, \mathbb{C}I_n \otimes e_0) = \mathfrak{C}$. Consider now $K = \ker\theta$ and $H_2 = q'^{-1}(H)$ and form $G_3 = G_2/K\cap H_3$. θ generates a bounded continuous homomorphism θ' of G_3 into $\mathfrak{B}_*$. Let H_3 be the closed normal subgroup of G_3 corresponding to H_2 in G_2. Then $G_3/H_3 \simeq G_2/H_2 \simeq G/H$ and $H_3 \simeq H_2/K\cap H_2$. If $g \in H_2$ then $mq'(g) \in \mathfrak{C}$ and $\omega(g) \in \mathfrak{C}$, so $m'(g) \in \mathfrak{C}$. Also $\phi'(g) \in \mathfrak{C}$ and $\phi'(g) \in \mathfrak{C}$, so $\theta(g) = \phi(g)\mu'(g) \in \mathfrak{C}$. This shows that θ' maps H_3 into $\mathfrak{C}$. Clearly it is injective on H_3 and maps into $D(p, \mathbb{T}I_n \otimes e_0)$ because it is bounded. This shows that H_3 is abelian. Thus G_3 is WCHP because G is WCHP$^+$ and so $\theta'(G_3)$ is finite dimensional; that is, $\theta(G_2)$ is finite dimensional. This shows that $\phi(G) = \phi'(G_2) \subseteq \theta(G_2)\mu'(G_2)$ is finite dimensional.

Note. In the above if $g \in G_2$ and $c \in \mathfrak{C}$ then $\theta(g)c\theta(g)^{-1} = \sigma'(g)c\sigma'(g)^{-1}$. We have seen that H_3 is algebraically isomorphic with a subgroup of $\mathbb{T}^p$, and it follows from what we have just said that the action of G_3 on H_3 consists of permuting the coordinates. Thus in the above theorem instead of assuming G/H is WCHP$^+$ it is enough to assume that any extension of G/H by a subgroup of $\mathbb{T}^p$ for which the action is of this type is WCHP. This hypothesis is equivalent to G/H being WCHP$^+$, the implication in one direction being obvious and in the other direction following from the amended version of the theorem and the fact that abelian groups are WCHP [2, Corollary 2.4].

There is another hypothesis on G/H which can be used instead of WCHP$^+$ in Theorem 4.3.

Definition 4.4. We say that the locally compact group G is WCHP^{++} if whenever G_1 is a closed normal subgroup of finite index in G and G_3 is a locally compact group containing a closed normal subgroup H_3 algebraically isomorphic with a subgroup of $\mathbb{T}$ in its centre with $G_3/H_3 \simeq G_1$ and ϕ is an eligible representation of G_3 with $\phi(H_3) \subseteq \mathbb{C}1$, then $\phi(G_3)$ is finite dimensional.

To prove Theorem 4.3 with WCHP^{++} in place of WCHP$^+$, after the WCHP subgroup construction we replace G by the subgroup G_1 of elements such that $\phi(g)$ commutes with the minimal central idempotents in $\phi(l^1(H))$. The effect of this is that we can take $p = 1$ in the rest of the argument and show that $\phi(G_1)$ is finite dimensional. It then follows as in Theorem 3.1 that $\phi(G)$ is finite dimensional.

Again it follows from the amended version and [2, Corollary 2.4] that every WCHP^{++} group is WCHP$^+$. For connected groups the reverse implication holds trivially because the only possible value for G_1 is G.

Corollary 4.5. *Let G be a locally compact group and H a closed normal subgroup. If G/H and H are WCHP$^+$ then so is G.*

Proof. Let Γ be a locally compact group containing a closed normal abelian subgroup K with $\Gamma/K \simeq G$ and let q be the quotient map $\Gamma \to G$. Put $H' = q^{-1}(H)$. Then $\Gamma/H' \simeq G/H$ is WCHP$^+$ and H' is WCHP because $K \subseteq H'$ and $H'/K \simeq H$ which is WCHP. The result now follows immediately from Theorem 4.4.

5. WCHP$^+$ FOR CERTAIN CLASSES OF GROUPS

The next few results are proved for discrete groups and hence, by Theorem 3.2, are independent of the topology on G.

Lemma 5.1. *Let G be a nilpotent group of height 2. Then G is WCHP.*

Proof. G contains a central subgroup H such that G/H is abelian. Suppose ϕ is an eligible representation of G on $\mathfrak{B}$. We apply the WCHP subgroup construction. In this situation $\phi(l^1(H))$ lies in the centre of $\mathfrak{B}$ and so the orbits O_j are singletons. Because H is abelian the matrix algebras M_n are one dimensional and so we reduce the argument to the special case $n = p = 1$. This simplifies the argument in Theorem 4.3 because $Sl(1, \mathbb{C}) = \{1\}$, so $G_2 = G$, $H_2 = H$, H_3 is a central subgroup in G_3, and $G_3/H_3 = G/H$ is abelian. Let H_4 be a maximal abelian subgroup of G_3; it contains H_3. Because G_3/H_3 is abelian, H_4 is normal. It is also abelian so we can apply the WCHP subgroup construction to θ'. Because H_4 is abelian, $n = 1$ and so there are characters $\chi_1, \dots, \chi_r$ on H_4 with $\theta'(h) = \chi_1(h)e_1 + \cdots + \chi_r(h)e_r$.

If $G_4 = \{g : g \in G_3,\ \sigma(g) = \text{identity}\}$ then $\chi_i(ghg^{-1}) = \chi_i(h)$ for all $g \in G_4$, $h \in H_4$, $i = 1, \dots, r$. Thus $\theta'(ghg^{-1}h^{-1}) = \text{identity}$. Since θ' is injective on H_3 and $ghg^{-1}h^{-1} \in H_3$ because G_3/H_3 is abelian, we see $gh = hg$ for all $g \in G_4$, $h \in H_4$. As H_4 was a maximal abelian subgroup we see $H_4 \supseteq G_4$, so H_4 is finite index in G_3. By Theorem 3.1 we see that G_3 is WCHP and so $\theta'(G_3) = \phi(G)$ is finite dimensional.

Corollary 5.2. *Let G be an abelian group. Then G is WCHP^{++}.*

Proof. In the definition of WCHP^{++}, G_1 is abelian and so G_3 is WCHP by Lemma 5.1.

Theorem 5.3. *Let G be a solvable group. Then G is WCHP^{+}.*

Proof. Follows by induction using Corollaries 4.5 and 5.2.

Proposition 5.4. *Let G be a compact topological group. Then G is WCHP^{++}.*

Proof. Suppose we are in the situation of Definition 4.4. Then G_1 is compact. Suppose ϕ is a dense eligible representation of G_3 in $\mathfrak{B}$. Apply the WCHP subgroup construction to ϕ with subgroup H_3. As $\phi(l^1(H_3))$ lies in the centre of $\mathfrak{B}$ the orbits O_i are singletons. Let C be a compact subset of G_3 such that $CH_3 = G_3$, and let $\Gamma = \phi(H_3)^-$. Then Γ and $\phi(C)$ are compact, so $\phi(G_3)^- \subseteq \phi(C)\Gamma$ is compact. Application of [1, Corollary 2.5] to the identity map from the compact group $\phi(G_3)^-$ into $\mathfrak{B}$ shows that $\phi(G_3)^-$ and hence $\phi(G_3)$ is finite dimensional.

We can also prove a theorem like Theorem 4.1 in [4] giving a criterion for WCHP^{++} groups.

Theorem 5.5. *Let G be a locally compact group containing a closed subgroup H such that:*

(i) *H is WCHP^{+}.*

(ii) *G has no proper closed normal subgroups containing H.*

(iii) *The set of equivalence classes of finite-dimensional nontrivial continuous bounded representations of H has no finite orbits under the action of $K = \{k : k \in G,\ kHk^{-1} = H\}$. Here K acts on a representation π of H by $(k \circ \pi)(h) = \pi(k^{-1}hk)$.*

Then G is WCHP^{++} and hence WCHP^{+}.

Proof. First of all we show that G has no nontrivial continuous bounded finite-dimensional representations. If ϕ is one such representation then we use the WCHP subgroup construction ((i) is not needed at this stage as we know *a priori* that $\phi(l^1(H))$ is finite dimensional) with K in place of G. For each ϕ_j the whole orbit $K \circ \phi_j$ lies in $\{\phi_1, \dots, \phi_r\}$ and so ϕ_j is a finite-dimensional bounded continuous representation of H with a finite K-orbit. This shows that ϕ_j is trivial and hence $\phi(h) = 1$ for all $h \in H$. Since $\phi^{-1}(1)$ is a closed normal subgroup of G, we see it is G.

We now set out to show that the conditions of Definition 4.4 are satisfied. The quotient map $G \to l^1(G/G_1)$ is a finite-dimensional bounded continuous representation of G and so is trivial; that is, we need consider only the case $G = G_1$. Suppose ϕ is an eligible representation of G_3 with $\phi(H_3) \subseteq \mathbb{C}1$, and let H' and K' be the inverse images in G_3 of H and K in G. Then $\mathfrak{C} = \phi(l^1(H'))$ is finite dimensional and we can apply the WCHP subgroup construction with subgroup H'. Let $\mathfrak{C}' = \mathfrak{C} \otimes \mathfrak{C}^{\mathrm{op}}$ where $\mathfrak{C}^{\mathrm{op}}$ is the opposite algebra of $\mathfrak{C}$ and put $\Phi(h) = \phi(h) \otimes \phi(h)^{-1} \in \mathfrak{C}'$ for $h \in H'$. Then Φ is a continuous bounded representation of H' which is trivial on H_3 and so induces a representation Φ' of H on $\mathfrak{C}'$. Because $k' \circ \phi$ is equivalent to ϕ for all $k' \in K'$, the same is true of Φ and so $k \circ \Phi'$ is equivalent to Φ' for all $k \in K$. This implies that Φ' is trivial, so $\Phi(h') = 1$ and hence $\phi(h') \in \mathbb{C}1$ for all

$h' \in H'$. Thus $\phi^{-1}(\mathbb{C}1)$ is a closed normal subgroup of G_3 containing H' and hence is G_3. Thus $\phi(G_3)$ is one dimensional.

6. Open questions

(i) Is every locally compact group WCHP? Theorem 3.2 shows that it is enough to settle this question for discrete free groups. If ϕ is a dense eligible homomorphism from G into $\mathfrak{B}$ with $\|\phi(g)\| \leq K$ $(g \in G)$ then $p(b) = \sup\{\|\phi(g)b\| : g \in G\}$ is a seminorm on $\mathfrak{B}$ with $\|b\| \leq p(b) \leq K\|b\|$ and $p(\phi(g)b) = p(b)$. Thus p is equivalent to $\| \ \|$, and if we form the operator norm $|||b||| = \sup\{p(bc); \ p(c) \leq 1\}$ then $||| \ |||$ is a unital Banach algebra norm on $\mathfrak{B}$ with $|||\phi(g)||| = 1$; that is, the elements of $\phi(G)$ are isometric multipliers in $\mathfrak{B}$. For the case in which $\mathfrak{B}$ is reflexive our question is thus equivalent to "Can the group of regular isometric elements of a unital reflexive Banach algebra be infinite dimensional?"

(ii) Is a closed normal subgroup of finite index in a WCHP$^+$ group also a WCHP$^+$ group? This is related to showing that every WCHP$^+$ group is WCHP^{++}.

(iii) Is every amenable group WCHP? More particularly, is the inductive limit of a sequence of finite groups a WCHP group? Even if we take the restricted direct product of a sequence of finite groups, the result does not seem to be known, though, if the orders of the groups in the sequence are bounded, Theorem 3.3 can be used (take the H_i to be products of cyclic subgroups, one from each factor group in such a way that every cyclic subgroup of any of the factors appears in at least one H_i).

(iv) If every irreducible representation of $L^1(G)$ is of dimension $\leq k$ for some $k \in \mathbb{N}$ then $L^1(G)$ satisfies the standard polynomial identity $S_{2k} = 0$ [1, p. 383] and so therefore does $\mathfrak{B}$ if there is a homomorphism of $L^1(G)$ into $\mathfrak{B}$ with dense range. This shows that every irreducible representation of $\mathfrak{B}$ is of dimension $\leq k$ and so G is WCHP by [2, Theorem 1.3]. Is it possible to extend this to show that if every irreducible representation of $L^1(G)$ is finite dimensional then G is WCHP?

(v) In [3] the author described the connection between the WCHP property and the L^p conjecture. If G is a discrete infinite WCHP group, then $l^p(G)$ is not closed under convolution for any $p > 1$ because otherwise the identity map from $l^1(G)$ into $l^p(G)$ would be a weakly compact homomorphism with infinite-dimensional range. In [5], Saeki shows that if $p > 1$ then $l^p(G)$ is never closed under convolution. The author is indebted to the referee for drawing this paper to his attention.

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