PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 119, Number 4, December 1993

ON ALGEBRAS WITH GELFAND-KIRILLOV DIMENSION ONE

SHIGERU KOBAYASHI AND YUJI KOBAYASHI

(Communicated by Maurice Auslander)

ABSTRACT. We study the growth of finitely generated algebras by means of the Ufnarovskij graph. We improve a result of Bergman about the jumps of the growth function and give a characterization of algebras with GKdim = 1 in terms of the Ufnarovskij graph. For monomial algebras the behavior of the growth functions is studied in more detail.

INTRODUCTION

Let A be an associative algebra over a field K generated by a finite set X. Let V be the K-subspace spanned by X. We define a function d_X^A as

$$d_X^A(n) = \dim\left(\sum_{i=0}^{n} V^i\right)$$

for $n \geq 0$, where V^i is the subspace of A spanned by all the products of i elements of V, in particular, $V^0 = K$. The asymptotic behavior of the sequence $\{d_X^A(n)\}$ is expressed by the *Gelfand-Kirillov dimension* of A, which is defined by

$$\operatorname{GKdim} A = \limsup_{n\to\infty} \frac{\log d_X^A(n)}{\log n}.$$

This number does not depend on the set X of generators of A.

Recall that A is said to have *polynomial growth* (resp. *exponential growth*) if there exists a polynomial f such that $d_V^A(n) \leq f(n)$ (resp. there exists a real number $\varepsilon > 1$ such that $d_V^A(n) \geq \varepsilon^n$) for all sufficiently large n.

A finitely generated algebra $A = K\langle x_1, x_2, \dots, x_d\rangle/J$ is called a *monomial algebra* if the ideal J is generated by a set of monomials in the x_i's. Ufnarovskij [9] associated a graph $G(A)$ with a finitely presented monomial algebra A and gave a criterion for determining the growth rate of A in terms of the graph $G(A)$.

In this paper we investigate the growth of finitely generated (but not necessarily finitely presented) algebras by means of the Ufnarovskij graph. First we improve a result of Bergman [2] (cf. [6]) concerning the jumps $\alpha_n = d_V^A(n) - d_V^A(n-1)$ as follows: If $\alpha_n \leq n$ for some $n > 1$, then $\alpha_{n+h} \leq \lceil\frac{1}{4}(n+1)^2\rceil$ for

Received by the editors July 26, 1991 and, in revised form, April 14, 1992.
1991 *Mathematics Subject Classification*. Primary 16A55; Secondary 16A45.

0002-9939/93 \$1.00 + \$.25 per page

any $h \geq 0$ (Theorem 1). The proof we give is very combinatorial. Next we give a characterization of finitely generated algebras with GKdim $= 1$ in terms of the Ufnarovskij graph (Corollary 2 to Theorem 1). We study the behavior of the sequence $\{\alpha_n\}$ in detail by exhibiting several examples. When A is a monomial algebra of GKdim $= 1$, we can make the sequence $\{\alpha_n\}$ monotonous by dividing A by some nilpotent ideal (Theorem 2).

Small, Stafford, and Warfield [8] showed that finitely generated algebras with GKdim $= 1$ are PI. Their results together with the results in this paper show that the algebras of GK-dimension one form a very special class of algebras.

1. Preliminaries

Let K be a field, $X = \{x_1, x_2, \dots, x_d\}$ a set of indeterminates, and X^* denote the free monoid over X. An element x of X^* is called a word on X and $|x|$ denotes the length of x. Let $<$ be a total order on X. The order $<$ is extended to a total order (first length, next lexicographic) on X^* denoted by the same symbol $<$ as follows:

Let x, y be in X^*. First, if $|x| < |y|$, then $x < y$. Next suppose $|x| = |y|$ and $x = x_1 x'$, $y = y_1 y'$ with $x_1, y_1 \in X$ and $x', y' \in X^*$. Then if $x_1 < y_1$ or both $x_1 = y_1$ and $x' < y'$, then $x < y$.

Let $K\langle X\rangle = KX^*$ be the free algebra generated by X over K, and let I be an ideal of $K\langle X\rangle$. Set $A = K\langle X\rangle/I$, and let ψ denote the canonical homomorphism of $K\langle X\rangle$ onto A. Let $M = M^A$ be the set of words $v \in X^*$ such that $\psi(v)$ cannot be a K-linear combination of monomials $\psi(y)$ in A with $y \in X^*$ such that $y < v$. The following lemma is basic.

Lemma 1 [1, 3]. *The mapping ψ is injective on M, and $\psi(M)$ forms a K-linear base of A.*

We set $M_n = M_n^A = M^A \cap X^n$ and define $\alpha_n = \alpha_n^A = |M_n|$, where $|\cdot|$ denotes the cardinal number of a set.

Corollary 1. *We have $d_X^A(n) = \sum_{i=0}^n \alpha_i$.*

We define *the monomial algebra $\widetilde{A}$ associated with A* (more precisely, associated with X, I, and $<$) by

$$\widetilde{A} = K\langle X\rangle/(X^* - M),$$

where $(X^* - M)$ denotes the ideal of $K\langle X\rangle$ generated by $X^* - M$. Moreover, we define *the monomial algebra of level n associated with A* by

$$\widetilde{A}_n = K\langle X\rangle/(X^{\leq n} - M),$$

where $X^{\leq n} = \bigcup_{i \leq n} X^i$.

Corollary 2. *We have*

$$\alpha_n^A = \alpha_n^{\widetilde{A}} \quad \textit{and} \quad d_X^A(n) = d_X^{\widetilde{A}}(n)$$

for $n \geq 0$ and

$$\alpha_{n+h}^A \leq \alpha_{n+h}^{\widetilde{A}_n} \quad \textit{and} \quad d_{n+h}^A \leq d_{n+h}^{\widetilde{A}_n}$$

for $n \geq 0$ and $h \geq 0$.

For $n \geq 1$, we consider a labeled directed graph $G_n(A)$ defined as follows (cf. [9]). The set of vertices of $G_n(A)$ is M_{n-1}; for $u, v \in M_{n-1}$, there is an edge $u \xrightarrow{a} v$ with label $a \in X$ if $ua \in M_n$ and $ua = bv$ for some $b \in X$. We call $G_n(A)$ *the graph of* A *of level* n.

In a graph a cycle (a closed path) is called *simple* if it is not a union of two different cycles. Two cycles are *overlapped* if they have some vertices in common. A set $\{C_1, \dots, C_k\}$ of cycles is called *cycles in a line* if no two of C_i are overlapped and there is a path from C_i to C_{i+1} for every $i = 1, \dots, k-1$.

Lemma 2 (Govorov [5], Ufnarovskij [9]). *The algebra $\widetilde{A}_n$ has either an exponential growth or a polynomial growth. More precisely*

(1) *If $G_n(A)$ has overlapping simple cycles, then $\widetilde{A}_n$ has an exponential growth.*

(2) *If $G_n(A)$ has no overlapping simple cycles and the maximal number of cycles in a line is r, then $\widetilde{A}_n$ has a polynomial growth and* $\operatorname{GKdim} \widetilde{A}_n = r$.

Corollary. *If $G_n(A)$ has no overlapping simple cycles and the maximal number of cycles in a line is r for some n, then A has a polynomial growth and* $\operatorname{GKdim} A \leq r$.

Proof. The results follow from Lemma 2, because $\operatorname{GKdim} A \leq \operatorname{GKdim} \widetilde{A}_n$.

2. Exact bound

Bergman [2] proved that if $\alpha_n \leq n$ for some $n > 0$, then $\alpha_{n+h} \leq n^3$ for all $h \geq 1$. In this section we improve this result by using the Ufnarovskij graph.

Let A be as in §1 and $G = G_{n+1}(A)$ be the graph of A of level $n+1$. By *the word of a path* $v\colon v_0 \xrightarrow{a_1} v_1 \xrightarrow{a_2} \cdots \xrightarrow{a_k} v_k$ in $G_{n+1}(A)$, we mean the word $a_1 a_2 \cdots a_k$.

Lemma 3. *For two paths $v\colon v_0 \xrightarrow{a_1} v_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} v_n$ and $w\colon w_0 \xrightarrow{b_1} w_1 \xrightarrow{b_2} \cdots \xrightarrow{b_n} w_n$ in $G_{n+1}(A)$, $v_n = w_n$ holds if and only if their words coincide.*

Proof. Clear from the definition of the graph $G_{n+1}(A)$.

Theorem 1. *Let A be a finitely generated algebra over K as above. If $\alpha_n = |M_n| \leq n$ for some $n \geq 1$, then*

$$\alpha_{n+h} \leq \begin{cases} \frac{1}{4}n(n+2) & \text{if } n \text{ is even}, \\ \frac{1}{4}(n+1)^2 & \text{if } n \text{ is odd} \end{cases}$$

for any $h > 0$.

Proof. The number of vertices of the graph $G = G_{n+1}(A)$ is α_n and does not exceed n by hypothesis. First we claim that

(a) G has no overlapping simple cycles.

On the contrary assume that G has overlapping simple cycles C_1 and C_2. Suppose C_i contains m_i vertices $(i = 1, 2)$ and

$$C_1\colon v_1 \to v_2 \to \cdots \to v_{m_1} \to v_1,$$
$$C_2\colon w_1 \to w_2 \to \cdots \to w_{m_2} \to w_1.$$

We may suppose $v_1 = w_1$ and $v_{m_1} \neq w_{m_2}$ by renumbering the vertices of C_1 and C_2 if necessary. Let k (resp. l) be the maximal number such that v_k (resp. w_l) is in $C_1 \cap C_2$. Then $w_l = v_m$ for some $m \leq k$. Consider another cycle:

$$C_2': v_1 \to v_2 \to \cdots \to (v_m = w_l) \to w_{l+1} \to \cdots \to w_{m_2} \to (w_1 = v_1).$$

Then C_1 and C_2' are overlapping simple cycles, $C_1 \cap C_2'$ consists of m vertices $v_1, \ldots, v_m$, and the paths in C_1 and C_2' from v_1 to v_m coincide. So we can suppose these conditions for the original C_1 and C_2. Since $|G| \leq n$, we have

$$m_1 + m_2 - m \leq n. \tag{2.1}$$

By Lemma 3 the words of the paths to v_1 through C_1 and C_2 of length n coincide. Since the paths from v_1 to v_m belonging to C_1 and C_2 are common, the words of the paths to v_m through C_1 and C_2 of length $n+m-1$ coincide. Let u_i be the word of C_i (ending with v_m), that is, the words of the paths to v_m through C_i of length m_i $(i = 1, 2)$. Then, the words $(u_1)^p$ and $(u_2)^p$ have a common suffix of length $n + m - 1$ for a sufficiently large p. By (2.1) we see that

$$|u_1| + |u_2| - 1 \leq n + m - 1.$$

Now by a well-known theorem (Lothaire [7, Proposition 1.3.5]), we find that u_1 and u_2 are powers of a common word. By Lemma 3, the cycles C_1 and C_2 must coincide, but this is a contradiction.

We say that two subgraphs in a directed graph are *connected* if they are connected by a path in the undirected graph obtained from the original graph by forgetting the direction of edges. Our next objective is to prove that

(b) G does not have two cycles which are connected.

On the contrary suppose G has two (simple) cycles C_1 and C_2 which are connected by an undirected path, and let l be the length of the connecting path. Since C_1 and C_2 do not overlap, we see $l > 0$. Here we need the following lemma.

Lemma 4. *If vertices v_1 and v_2 in G are connected by an undirected path of length $l \geq 0$, then v_1 and v_2 (considered to be words) have a common subword of length $\geq n - l$.*

Proof. We proceed by induction on l. If $l = 0$, then $v_1 = v_2$ and the assertion is trivial. Suppose $l > 0$, and $v_1 = w_0 - w_1 - \cdots - w_l = v_2$ is an undirected path of length l. By the induction hypothesis w_1 and v_2 have a common subword u of length $n - l + 1$. If there is an edge $v_1 \to w_1$, then the suffix of v_1 of length $n - 1$ is equal to the prefix of w_1 of length $n - 1$. Thus v_1 contains at least a prefix of u of length $n - l$; hence, v_1 and v_2 have a common subword of length $n - l$. Similarly, if there is an edge $w_1 \to v_1$, we can get the same conclusion.

Now, by Lemma 4 there exist vertices v_1 in C_1 and v_2 in C_2 which have a common subword u of length $\geq n - l$. Then there exist vertices w_1 in C_1 and w_2 in C_2 such that the words of paths to w_1 in C_1 and to w_2 in C_2 of length $|u|$ are equal to u. Let $m_i = |C_i|$. Then by the assumption of the theorem, we have

$$m_1 + m_2 + l - 1 \leq n. \tag{2.2}$$

Let u_i be the word of C_i ending with w_i $(i = 1, 2)$. Then, for sufficiently large p, $(u_1)^p$ and $(u_2)^p$ have the common suffix u of length $\geq n - l$. By (2.2) we get

$$|u_1| + |u_2| - 1 \leq n - l. \tag{2.3}$$

Therefore, u_1 and u_2 are powers of a common word, and consequently C_1 and C_2 must coincide. This contradiction proves our claim (b).

Next we claim that for any vertices v_1 and v_2 in G there do not exist two different paths from v_1 to v_2 of the same length $l \leq n$. In fact, if $v_1 = w_0 \xrightarrow{a_1} w_1 \xrightarrow{a_2} \cdots \xrightarrow{a_l} w_l = v_2$ and $v_1 = w_0' \xrightarrow{b_1} w_1' \xrightarrow{b_2} \cdots \xrightarrow{b_l} w_l' = v_2$ are paths from v_1 to v_2 of length $l \leq n$, then by Lemma 3, $a_i = b_i$ for $i = 1, \dots, l$. Thus the paths must coincide.

Let C be a simple cycle in G and v be a vertex outside C. By a path from C to v we mean a path $w_0 \to w_1 \to \cdots \to w_p = v$ such that $w_0 \in C$ and $w_i \notin C$ for $i \geq 1$. We claim that

(c) There are not two different paths from C to v, and there are not two different paths from v to C either.

Let

$$C\colon u_1 \xrightarrow{a_1} u_2 \xrightarrow{a_2} \cdots \xrightarrow{a_m} u_m \xrightarrow{a_1} u_1.$$

Suppose that

$$u_1 = w_0 \xrightarrow{b_1} w_1 \xrightarrow{b_2} \cdots \xrightarrow{b_p} w_p = v$$

and

$$u_k = w_0' \xrightarrow{c_1} w_1' \xrightarrow{c_2} \cdots \xrightarrow{c_q} w_q' = v$$

are different paths from C to v, where $w_i, w_i' \notin \cdot C$ for $i = 1 \geq 1$. If $p = q + k - 1$, then there are two paths from u_1 to v of the same length $\leq n$, contradicting the above claim. If $q \leq p < q + k - 1$, then $ub_1 \cdots b_p$ and $\tilde{u}a_{q+k-p} \cdots a_{p-q+1}c_1 \cdots c_q$ are the words of paths to v of length $\leq n$, where u (resp. $\tilde{u}$) is the word of C ending with u_1 (resp. u_{q+k-p}). By Lemma 3 they coincide and so $u = \tilde{u}$. This is impossible, because C is simple. In the other cases we can similarly show the impossibility. This proves the first assertion. To see the second assertion, consider the dual graph $\widehat{G}$ of G defined as follows: The set of vertices is the same as G. There is an edge $u \xrightarrow{a} v$ with label $a \in X$ in $\widehat{G}$ if $au \in M_{n+1}$ and $au = vb$ for $b \in \Sigma$. Then $u \to v$ in G if and only if $v \to u$ in $\widehat{G}$. Since any graph theoretical property satisfied on G is also satisfied on $\widehat{G}$, the second assertion follows from the first.

Finally, we claim that

(d) For any vertices u and v in G there do not exist two different paths from u to v of the same length l.

When $l \leq n$, we already proved the result. If $l > n$, a path of length l must intersect with some cycle C. If u and v are in C, there is only one path because G has no overlapping cycles. If u or v is not in C, the assertion follows from the claim (c).

Now we are ready to complete the proof. Note that $\alpha_{n+h}^{\widetilde{A}_n}$ is the number of paths in G of length h and $\alpha_{n+h} \leq \alpha_{n+h}^{\widetilde{A}_n}$. We shall prove that in an (abstract) graph G with n vertices satisfying (a)–(d), the number of paths of length h is bounded by $\frac{1}{4}n(n+2)$ if n is even and by $\frac{1}{4}(n+1)^2$ if n is odd. We prove this by induction on n. We may suppose that G is connected. Since the assertion is trivial when $n = 1$, let $n > 0$. A vertex with no edge coming in (resp. going out) is called a *source* (resp. a *sink*). If G has no source nor sink, then G is a cycle itself. Then the number of paths of length h is constantly n. If G has no source or no sink, the number of paths of length h in G is again n by the claim (c). Now suppose that G has both a source u and a sink v. Then, the number of paths going out from u of length h or the number of paths coming into v of length h is not greater than $n/2$, because, if not, there were two different paths from u to v of length $2h$, but this is impossible by (d). If there are at most $\lceil n/2 \rceil$ paths from u (resp. into v) of length h, then consider the graph G' obtained from G removing the vertex u (resp. v). By the induction hypothesis, the number of paths of length h in G' is at most $n^2/4$ if n is even, and $\frac{1}{4}(n-1)(n+1)$ if n is odd. Since the paths in G' of length h together with the paths of length h starting from (resp. ending with) the removed vertex form the set of paths of length h in G, we get the desired bound for the number of paths in G.

In the following example we see that the bound in Theorem 1 is attained, and thus our result is sharp.

Example 1. Let n be a positive integer. Set $d = n/2 + 1$ if n is even, and $d = (n+1)/2$ if n is odd. Let $X = \{x_1, x_2, \dots, x_d\}$ and I be the ideal of $K\langle X \rangle$ generated by

$$\{x_1 x_i \mid i = 2, \dots, d\} \cup \{x_i x_1 \mid i = d+1, \dots, n\}$$
$$\cup \{x_i x_1^k x_j \mid i, j = 2, \dots, d;\ k = 0, \dots, n-2\}.$$

Let $A = K\langle X \rangle / I$. Then the graph $G = G_{n+1}(A)$ is given as in Figure 1. Easily we can check that $\alpha_n^A = n$ and

$$\alpha_{n+h}^A = (d-1)(n-d) + n = \begin{cases} \frac{1}{4}n(n+2) & \text{if } n \text{ is even}, \\ \frac{1}{4}(n+1)^2 & \text{if } n \text{ is odd} \end{cases}$$

for $h \geq 1$.

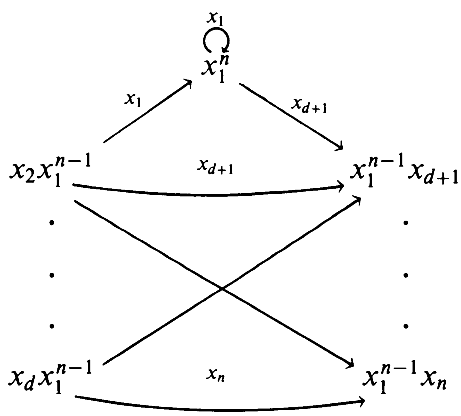

FIGURE 1

Corollary 1. *For a finitely generated algebra A over K, the following statements are equivalent.*

(1) $\operatorname{GKdim} A \leq 1$.
(2) $\alpha_n^A \leq n$ *for some* n.
(3) *For some* n, *every connected component of* $G_n(A)$ *has at most one cycle.*
(4) $\operatorname{GKdim} \widetilde{A}_n \leq 1$ *for some* n.

Proof. (1) $\Rightarrow$ (2) If $\alpha_n^A > n$ for all n, then $d^A(n) \geq \frac{1}{2}n(n-1)$. This implies $\operatorname{GKdim} A \geq 2$.

(2) $\Rightarrow$ (3) By the proof of Theorem 1.

(3) $\Rightarrow$ (4) By Lemma 2.

(4) $\Rightarrow$ (1) Clear because $\operatorname{GKdim} A \leq \operatorname{GKdim} \widetilde{A}_n$.

Corollary 2. *For a finitely generated algebra A over K, the following statements are equivalent.*

(1) $\operatorname{GKdim} A = 1$.

(2) *For all sufficiently large* n, *every connected component of* $G_n(A)$ *has at most one cycle and at least one component has one cycle.*

(3) $\operatorname{GKdim} \widetilde{A}_n = 1$ *for all sufficiently large* n.

In Corollary 1(2) above n cannot be replaced by $n+1$ as the following example shows.

Example 2. Let $X = \{x, y\}$ and I be the ideal of $K\langle X\rangle$ generated by $\{yx^iy \mid i = 0, 1, 2, \ldots, n-1\}$. Let $A = K\langle X\rangle/I$. Then $\alpha_n^A = n+1$ and the graph $G = G_{n+1}(A)$ is given as in Figure 2. Since G has overlapping cycles, the growth of $A = \widetilde{A}_{n+1}$ is exponential.

In view of the equivalence of Corollary 2(1), (3), it is an interesting question whether for any natural number k the following two statements are equivalent:

(i) $\operatorname{GKdim} A \leq k$,
(ii) $\operatorname{GKdim} \widetilde{A}_n \leq k$ for some n.

Since $\operatorname{GKdim} K \leq \operatorname{GKdim} \widetilde{A}_n$, (ii) implies (i). The next example shows the converse is not true if $k \geq 2$.

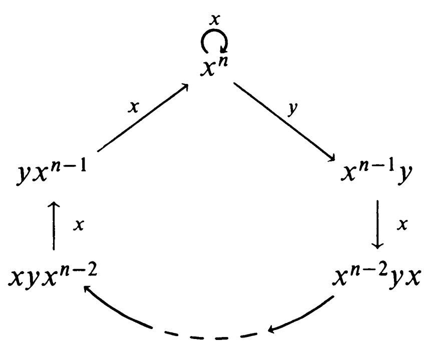

FIGURE 2

Example 3. Let $X = \{x, y\}$ and I be the ideal of $K\langle X\rangle$ generated by $\{yx^iy | i = 0, 1, 2, \dots\}$. Let $A = K\langle X\rangle/I$. Then $M^A = \{x^i, x^iyx^j | i, j = 0, 1, 2, \dots\}$ forms a base of A, and $\mathrm{GKdim}\, A = 2$. However, $\widetilde{A}_n$ has the exponential growth for every $n \geq 1$ (recall Example 2).

In general, the growths of $\widetilde{A}_n$'s do not strictly dominate the growth of A as seen in the following examples. Therefore, the equivalence of Corollary 2(1), (3) is a very special phenomenon in case of GK-dimension one.

Example 4. Let $X = \{x, y\}$ and I be the ideal of $K\langle X\rangle$ generated by $\{xy^2, x^2y\} \cup \{y^2(xy)^ix^2 | i = 0, 1, 2, \dots\}$. Let $A = K\langle X\rangle/I$. Then $\mathrm{GKdim}\, A = 2$, but $\mathrm{GKdim}\, \widetilde{A}_n = 3$ for all $n \geq 3$.

Example 5. Let ε be any real number such that $0 \leq \varepsilon \leq \frac{1}{2}$, and let A be the algebra with Super-GKdim $A = \varepsilon$ given by Borho-Kraft [4, 2.16]. Then $\widetilde{A}_n$ has exponential growth, that is, Super-GKdim $\widetilde{A}_n = 1$ for all n.

3. Monomial algebras

If $\mathrm{GKdim}\, A = 1$, then the sequence $\{\alpha_n^A\}$ is bounded, but its behavior is zigzag in general.

Example 6. Let $X = \{x,y\}$ and I be the ideal of $K\langle X\rangle$ generated by $\{yx^{2i+1}y | i = 0, 1, 2, \dots\} \cup \{xyx, xyy, yyx, yyy\}$. Let $A = K\langle X\rangle/I$. Then $M^A = \{1, x, y\} \cup \{x^i, x^{i-1}y, yx^{i-1} | i = 2, 3, \dots\} \cup \{yx^{2i}y | i = 0, 1, 2, \dots\}$ is a base of A. Thus

$$\alpha_0 = 1, \qquad \alpha_1 = 2,$$

and for $i \geq 3$

$$\alpha_i = \begin{cases} 4 & \text{if } i \text{ is even}, \\ 3 & \text{if } i \text{ is odd}. \end{cases}$$

In the following we show that we can 'regularize' a monomial algebra A of GKdim $= 1$ by dividing it by some nilpotent ideal.

Theorem 2. *Let $A = K\langle X\rangle/I$ be a finitely generated monomial algebra over K with $\mathrm{GKdim}\, A = 1$. Then there exists a nilpotent ideal J of A generated by monomials such that $\mathrm{GKdim}\, A/J = 1$ and the sequence $\{\alpha_n^{A/J}\}$ satisfies the following property. There is a positive integer N such that $\alpha_i^{A/J} \leq \alpha_{i+1}^{A/J}$ for $i = 0, \dots, N-1$ and $\alpha_{N+h}^{A/J} = \alpha_N^{A/J}$ for all $h \geq 0$.*

Proof. By Corollary 2, there exists $n > 0$ such that every connected component of $G = G_{n+1}(A)$ has at most one cycle and some connected component has a cycle. Let $C_1, \dots, C_k$ be all the simple cycles of G, and let $v_1, \dots, v_k$ be the corresponding words. Since C_i are simple, v_i are primitive; that is, v_i are not powers of their proper subwords. Among the words v_i, let $v_1, \dots, v_{k_0}$ $(1 \leq k_0 \leq k)$ be not nilpotent in A. Since a sufficiently long word which is not zero in A is the word of a path which passes through some C_i $(1 \leq i \leq k_0)$ sufficiently many times, there is an integer L such that any word w which is not zero in A contains the power v_i^p of some v_i $(1 \leq i \leq k_0)$ as a subword for p with $|v_i^p| \geq |w| - L$. Let N_i^j $(1 \leq j \leq k_0)$ be the set of subwords of length $i \geq 0$ of the powers v_j^p, $p = 1, 2, 3, \ldots$, and set $N_i = \bigcup_{j=1}^{k_0} N_i^j$ and $N = \bigcup_{i=0}^{\infty} N_i$. It is easily seen that $|N_i^j| \leq |N_{i+1}^j| \leq |v_j|$ and

$$1 = |N_0| \leq |N_1| \leq \cdots \leq |N_l| = |N_{l+1}| = \cdots,$$

where $l = \max_{1 \leq j \leq k_0} |v_j|$.

Now let J be the ideal of A generated by $M^A - N$. Then clearly $M_i^{A/J} = N_i$ for $i \geq 0$ so that $\alpha_i^{A/J} = |N_i|$ and $\operatorname{GKdim} A/J = 1$. Let x be in J^q, where q is sufficiently large. Suppose x is not zero and $c \cdot v$ with $c \in K$ and $v \in \Sigma^*$ is its nonzero term. By the remark mentioned above, some word of $M^A - N$ which is contained in v as a subword must be a subword of some v_i^p $(1 \leq i \leq k_0)$. But this is impossible because N contains all the subwords of v_i^p. This contradiction shows that $x = 0$ and J is nilpotent, completing the proof.

In Theorem 2, we cannot drop the assumption that A is a monomial algebra. In fact we have

Example 7. Let $X = \{x, y\}$ and I be the ideal of $K\langle X\rangle$ generated by $\{x - x^2, xy, yx\}$. Let $A = K\langle X\rangle/I$. Then $M^A = \{1, x, y, y^2, y^3, \ldots\}$ and $\alpha_0 = 1$, $\alpha_1 = 2$, $\alpha_i = 1$ for $i > 1$. Thus in order to regularize $\{\alpha_i\}$, we must divide A by $J = (x)$. But J is not nilpotent in A.

In view of Theorem 2 it is natural to ask: If $\operatorname{GKdim} A/J = 1$ for some nilpotent ideal J, then does $\operatorname{GKdim} A = 1$? If A is finitely presented, that is, I is generated by a finite number of monomials, the answer is affirmative.

Theorem 3. *Let $A = K\langle X\rangle/I$ be a finitely presented monomial algebra. If J is a nil ideal of A and* $\operatorname{GKdim} A/J = 1$, *then* $\operatorname{GKdim} A = 1$.

Proof. Assume on the contrary that $\operatorname{GKdim} A > 1$. Then by Corollary 1 to Theorem 1, one connected component of $G_n(A)$ has at least two cycles for any $n \geq 1$. Since I is a finitely generated monomial ideal, $A = \widetilde{A}_n$ for sufficiently large n. By assumption $G_{n_0}(A/J)$ has no two cycles which are connected for some sufficiently large n_0. Hence some vertex v of a cycle of $G_{n_0}(A)$ is not a vertex of $G_{n_0}(A/J)$. This implies that v is a linear combination of words w such that $w < v$ in A/J. That is, $v - \sum a_i w_i \in J$ in A for some $a_i \in K$ and $w_i < v$. Since $v - \sum a_i w_i$ is nilpotent, v^p is a linear combination of words w such that $w < v^p$ in A for some $p > 0$. Since A is a monomial algebra, it follows that $v^p = 0$ in A; but this is impossible because v is a vertex of a cycle in $G_{n_0}(A)$.

The following example shows that we cannot drop the assumption that A is finitely presented in Theorem 3.

Example 8. Let $X = \{x, y\}$ and I be the ideal of $K\langle X\rangle$ generated by $\{yx^iy \mid i = 0, 1, 2, \ldots\}$. Let $A = K\langle X\rangle/I$ and let J be the ideal of A generated by $y \pmod I$. Then $J^2 = 0$ and $\operatorname{GKdim} A/J = 1$ since $A/J \cong K[x]$; however, we have $\operatorname{GKdim} A = 2$ (Example 3).

References

1. D. J. Anick, *Non-commutative graded algebras and their Hilbert series*, J. Algebra **78** (1982), 120–140.

2. G. M. Bergman, *A note on growth functions of algebras and semigroups*, mimeographed notes, Univ. of California, Berkeley, 1978.

3. ——, *The diamond lemma for ring theory*, Adv. in Math. **29** (1978), 178–218.

4. W. Borho and H. Kraft, *Über die Gelfand-Kirillov dimension*, Math. Ann. **220** (1976), 1–26.

5. V. E. Govorov, *Graded algebras*, Mat. Zametki **12** (1972), 197–204; English transl., Math. Notes **12** (1972), 552–556.

6. G. R. Krause and T. H. Lenagan, *Growth of algebras and Gelfand-Kirillov dimension*, Pitman, London, 1985.

7. M. Lothaire, *Combinatorics on words*, Addison-Wesley, Boston, MA, 1983.

8. W. Small, J. T. Stafford, and R. B. Warfield, *Affine algebras of Gelfand-Kirillov dimension one are PI*, Math. Proc. Cambridge Philos. Soc. **97** (1985), 407–414.

9. V. Ufnarovskij, *A growth criterion for graphs and algebras defined by words*, Mat. Zametki **31** (1982), 465–472; English transl., Math. Notes **37** (1982), 238–241.

Department of Mathematics, Naruto University of Education, Naruto 772, Japan

Department of Information Sciences, Faculty of Science, Toho University, Funabashi 274, Japan