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QUADRATIC AND QUASI-QUADRATIC FUNCTIONALS

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ABSTRACT. In this note we show how Jordan $*$-derivations arise as a "measure" of the representability of quasi-quadratic functionals by sesquilinear ones. Our main result can be considered as an extension of the Jordan-von Neumann characterization of pre-Hilbert space.

1. INTRODUCTION

Let M be a module over a $*$-ring R. A mapping $S: M \times M \to R$ is called a sesquilinear functional if it is linear in the first argument and antilinear in the second argument:

$$(1)\quad S(ax+by, z) = aS(x, z) + bS(y, z), \qquad x, y, z \in M, \ a, b \in R,$$

$$(2)\quad S(x, ay+bz) = S(x, y)a^* + S(x, z)b^*, \qquad x, y, z \in M, \ a, b \in R.$$

In the special case when R is a commutative ring with the trivial involution $a^* = a$, the relation (2) can be rewritten as $S(x, ay+bz) = aS(x, y) + bS(x, z)$. In this case the mapping S is called bilinear.

A quadratic functional Q on M is defined as the composition of some sesquilinear functional from $M \times M$ to R with the diagonal injection of M into $M \times M$; that is, $Q(x) = S(x, x)$, where S is sesquilinear. There is something inappropriate about defining a quadratic functional which is a function of one variable in terms of a sesquilinear functional which involves two variables. This raises the question of what requirements can be imposed on a mapping from M to R to define the set of all quadratic functionals. The best-known identities satisfied by quadratic functionals are the parallelogram law

$$(3)\qquad Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y), \qquad x, y \in M,$$

and the homogeneity equation

$$(4)\qquad Q(ax) = aQ(x)a^*, \qquad x \in M, \ a \in R.$$

A mapping $Q: M \to R$ satisfying these two identities is called a quasi-quadratic functional. In the special case that R is a commutative ring with the trivial involution the relation (4) can be rewritten as $Q(ax) = a^2Q(x)$.

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It seems natural to ask when quasi-quadratic functionals are in fact quadratic functionals. In other words, given a quasi-quadratic functional Q, does there exist a sesquilinear functional S such that $Q(x) = S(x, x)$? In 1963, Halperin in his lectures on Hilbert spaces posed this problem for the special case that M is a vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Here, $\mathbb{R}$ and $\mathbb{C}$ denote the field of real numbers and the field of complex numbers respectively, while $\mathbb{H}$ denotes the skew-field of quaternions. In 1964, Kurepa [4] obtained the general form of quasi-quadratic functionals defined on a vector space over $\mathbb{R}$. In particular, he showed that there exist quasi-quadratic functionals which cannot be represented by bilinear functionals. In 1966, Gleason [2] generalized this result to vector spaces V, $\dim V \geq 2$, over an arbitrary field F, not of characteristic 2, and with the trivial involution. He proved that all quasi-quadratic functionals on V are quadratic if and only if all additive derivations on F are zero. The same result holds for quasi-quadratic functionals defined on a module over a commutative ring R with the trivial involution in which 2 is a unit. This result follows from [1, Theorem 3]. It should be mentioned that in this commutative case with the trivial involution the result of Jordan and von Neumann [3] implies that for each quasi-quadratic functional Q the mapping S defined by

$$4S(x, y) = Q(x+y) - Q(x-y) \tag{5}$$

is symmetric and biadditive and $Q(x) = S(x, x)$ (see [2]). Thus, the above-mentioned results imply that S is homogeneous in both variables if and only if all additive derivations on R are zero.

In 1965, Kurepa [5] gave a positive answer to Halperin's problem for quasi-quadratic functionals defined on a vector space V over $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$. In 1984, Vukman [9] posed the problem of representability of quasi-quadratic functionals by sesquilinear ones on modules over complex $*$-algebras. This problem was treated in [6–11]. The complete solution was given in [7]. It was proved that if Q is a quasi-quadratic functional on a module over a complex $*$-algebra with an identity element, then the mapping S defined by

$$S(x, y) = \tfrac{1}{4}(Q(x+y) - Q(x-y)) + \tfrac{i}{4}(Q(x+iy) - Q(x-iy)) \tag{6}$$

is the unique sesquilinear functional satisfying $Q(x) = S(x, x)$. This result is an extension of the Jordan-von Neumann theorem [3] which characterises pre-Hilbert space among all normed spaces.

A mapping J defined on a $*$-ring R is called a Jordan $*$-derivation if it is additive and satisfies

$$J(a^2) = aJ(a) + J(a)a^*. \tag{7}$$

We shall denote by $\mathscr{J}$ the set of all Jordan $*$-derivations on R. Over a commutative ring with the trivial involution in which 2 is not a zero divisor, the set of all Jordan $*$-derivations is equal to the set of all additive derivations [1]. A mapping $J_a \colon R \to R$, $a \in R$, defined by $J_a(b) = ba - ab^*$ will be called an inner Jordan $*$-derivation. In [8] it was proved that the representability of quasi-quadratic functionals by sesquilinear functionals on modules over a real Banach $*$-algebra A with an identity element depends on the existence of Jordan $*$-derivations on A which are not inner. The proof of this result given in [8] uses the fact that Banach algebras have enough invertible elements. It is the purpose of this note to extend this result to quasi-quadratic functionals

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defined on modules over arbitrary $*$-rings. In this general setting it is impossible to find a relation (similar to (5) in the commutative case) telling us how to recover from a quadratic functional Q a sesquilinear functional S satisfying $Q(x) = S(x, x)$.

2. Statement of the results

Main Theorem. *Let R be a $*$-ring with identity* 1 *such that* 2 *is a unit in R. Assume that for every Jordan $*$-derivation $J: R \to R$ there exists a unique $a \in R$ such that $J(b) = J_a(b) = ba - ab^*$, $b \in R$. Then every quasi-quadratic functional Q defined on an arbitrary unitary R-module M is a quadratic functional.*

Note that the uniqueness of a in the above theorem is equivalent to the statement that $ba - ab^* = 0$ for all $b \in R$ implies $a = 0$. For the proof of the Main Theorem we shall need the following simple lemma.

Lemma 1. *Let R be a $*$-ring with identity* 1 *such that $ba - ab^* = 0$ for all $b \in R$ implies $a = 0$. If e_i, $i = 1, 2, 3, 4$, are elements from R such that*

$$ae_1a^* + ae_2b^* + be_3a^* + be_4b^* = 0$$

for all $a, b \in R$ then $e_i = 0$, $i = 1, 2, 3, 4$.

The next theorem shows that the existence of noninner Jordan $*$-derivations yields the existence of quasi-quadratic functionals that cannot be represented by sesquilinear ones.

Theorem 2. *Let R be a $*$-ring with identity* 1 *such that* 2 *is not a zero divisor. If $J: R \to R$ is a Jordan $*$-derivation then the mapping $Q: R \times R \to R$ given by $Q((a, b)) = J(ba) - bJ(a) - J(a)b^*$ is a quasi-quadratic functional. If J is not inner then Q is not a quadratic functional.*

A ring R is said to be a prime ring if $aRb = \{0\}$ implies $a = 0$ or $b = 0$. We shall prove that the mapping $F: R \to \mathcal{J}$, $F(a) = J_a$, is one-to-one if R is a noncommutative prime ring. Thus, we shall prove the following result.

Corollary 3. *Let R be a noncommutative prime $*$-ring with identity* 1 *such that* 2 *is a unit in R. Then all Jordan $*$-derivations on R are inner if and only if every quasi-quadratic functional Q defined on an arbitrary unitary R-module M is a quadratic functional.*

Next, we shall show that all the assumptions of the Main Theorem are satisfied if R is a complex $*$-algebra with an identity element. This together with the Main Theorem implies the following extension of the Jordan-von Neumann characterization of pre-Hilbert spaces (see [7]).

Corollary 4. *Let R be a complex $*$-algebra with identity* 1 *and let M be a unitary R-module. Assume that $Q: M \to R$ is a quasi-quadratic functional. Under these conditions the mapping $S: M \times M \to R$ defined by the relation* (6) *is the unique sesquilinear functional satisfying $Q(x) = S(x, x)$.*

We shall conclude by giving an example of a Jordan $*$-derivation which is not inner.

Example 5. There exists a Jordan $*$-derivation on a finite-dimensional noncommutative real $*$-algebra with an identity element which is not inner.

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3. Proofs

Proof of Main Theorem. Let Q be a quasi-quadratic functional defined on a unitary R-module M. We shall divide our proof into two steps. First, we shall prove that if the restriction of Q to each submodule of M generated by two elements is a quadratic functional, then Q is a quadratic functional on M. Our second step will be to prove that under the assumptions of the Main Theorem every quasi-quadratic functional defined on an arbitrary unitary R-module M generated by two elements is a quadratic functional.

Step 1. Assume that the restriction of Q to each submodule of M generated by two elements is a quadratic functional. Let us choose arbitrary elements $x, y \in M$. We denote by $M_{x,y} = \{ax + by : a, b \in R\}$ the submodule of M generated by x and y. According to our assumption there exists a sesquilinear functional $S_{x,y}\colon M_{x,y} \times M_{x,y} \to R$ such that

$$\begin{aligned} Q(ax+by) &= S_{x,y}(ax+by, ax+by) \\ &= aS_{x,y}(x, x)a^* + aS_{x,y}(x, y)b^* \\ &\quad + bS_{x,y}(y, x)a^* + bS_{x,y}(y, y)b^*, \qquad a, b \in R. \end{aligned} \tag{8}$$

Let us define a functional $S\colon M \times M \to R$ by $S(x, y) = S_{x,y}(x, y)$ for all $x, y \in M$.

In order to see that the mapping S is well defined we assume that there exists another sesquilinear functional $T_{x,y}\colon M_{x,y} \times M_{x,y} \to R$ satisfying

$$\begin{aligned} Q(ax+by) &= T_{x,y}(ax+by, ax+by) \\ &= aT_{x,y}(x, x)a^* + aT_{x,y}(x, y)b^* \\ &\quad + bT_{x,y}(y, x)a^* + bT_{x,y}(y, y)b^*, \qquad a, b \in R. \end{aligned}$$

Comparing this with (8) and using Lemma 1 we get that $S_{x,y}(x, y) = T_{x,y}(x, y)$. Thus, S is well defined. Moreover, we have proved that

$$S_{y,x}(x, y) = S_{x,y}(x, y) \tag{9}$$

holds for all $x, y \in M$. Let x, y, and z be elements from M. Then we have $S_{x,y}(x, x) = Q(1x + 0y) = Q(1x + 0z) = S_{x,z}(x, x)$. In particular, we obtain $S_{x,x}(x, x) = S_{x,y}(x, x)$. This last relation implies together with (9) that (8) can be rewritten as

$$\begin{aligned} Q(ax+by) &= aS(x, x)a^* + aS(x, y)b^* \\ &\quad + bS(y, x)a^* + bS(y, y)b^*, \qquad a, b \in R, \end{aligned} \tag{10}$$

where x, y are arbitrary elements from M. It follows that $Q(x) = S(x, x)$ is valid for all $x \in M$. In order to complete the first step of our proof we must show that S is a sesquilinear functional.

For arbitrary $x, y \in M$ and $a, b, c, d \in R$ we have

$$\begin{aligned} caS(x, x)a^*c^* &+ caS(x, y)b^*d^* + dbS(y, x)a^*c^* + dbS(y, y)b^*d^* \\ &= Q(cax + dby) = cS(ax, ax)c^* + cS(ax, by)d^* \\ &\qquad + dS(by, ax)c^* + dS(by, by)d^*. \end{aligned}$$

Applying Lemma 1 we get $S(ax, by) = aS(x, y)b^*$. It remains to prove that S is biadditive. Define

$$b_1 = Q(a_1x_1 + a_2x_2 + a_3x_3), \qquad b_2 = Q(a_1x_1 + a_2x_2 - a_3x_3),$$

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and

$$b_3 = Q(a_1x_1 - a_2x_2 - a_3x_3).$$

The parallelogram law (3) gives us

$$\begin{aligned} b_1 + b_2 &= 2Q(a_1x_1 + a_2x_2) + 2Q(a_3x_3), \\ -b_2 - b_3 &= -2Q(a_1x_1 - a_3x_3) - 2Q(a_2x_2), \\ b_1 + b_3 &= 2Q(a_1x_1) + 2Q(a_2x_2 + a_3x_3). \end{aligned}$$

Solving this system of equations and using (10) we obtain

$$b_1 = \sum_{i,j=1}^{3} a_i S(x_i, x_j) a_j^*.$$

In particular, for arbitrary $x, y, z \in M$ and $a, b \in R$ we have the relation

$$\begin{aligned} Q(ax + ay + bz) = {} & a(S(x, x) + S(y, x) + S(x, y) + S(y, y))a^* \\ & + a(S(x, z) + S(y, z))b^* \\ & + b(S(z, x) + S(z, y))a^* + bS(z, z)b^*. \end{aligned}$$

On the other hand, using (10) we get that

$$\begin{aligned} Q(a(x + y) + bz) = {} & aS(x + y, x + y)a^* + aS(x + y, z)b^* \\ & + bS(z, x + y)a^* + bS(z, z)b^*. \end{aligned}$$

Comparing the two expressions for $Q(ax + ay + bz)$ we obtain, using Lemma 1, the biadditivity of S. Thus, under the assumptions of the Main Theorem, a quasi-quadratic functional Q on M is a quadratic functional if and only if its restriction to each submodule generated by two elements is a quadratic functional.

Step 2. Let $M = \{ax + by : a, b \in R\}$ be a unitary R-module generated by x and y. We have to prove that for a given quasi-quadratic functional $Q \colon M \to R$ there exists a sesquilinear functional S from $M \times M$ to R such that $Q(z) = S(z, z)$ for all $z \in M$.

Let us define a functional $D \colon R \times R \to R$ by

$$D(a, b) = Q(ax + by) - aQ(x)a^* - bQ(y)b^* - 2^{-1}(afb^* + bfa^*), \tag{11}$$

where $f = Q(x + y) - Q(x) - Q(y)$. We shall first prove that D is biadditive. Clearly, it is enough to prove that the functional E given by $E(a, b) = Q(ax + by) - aQ(x)a^* - bQ(y)b^*$ is biadditive. Applying the parallelogram law (3) we get

$$\begin{aligned} & 2E(a, b) + 2E(c, b) \\ & \quad = 2Q(ax + by) + 2Q(cx + by) - 2Q(ax) - 2Q(cx) - 4Q(by) \\ & \quad = Q((a + c)x + 2by) + Q((a - c)x) - 2Q(ax) - 2Q(cx) - Q(2by) \\ & \quad = Q((a + c)x + 2by) - Q((a + c)x) - Q(2by) = E(a + c, 2b). \end{aligned} \tag{12}$$

Substituting $c = 0$ and using the obvious relation $E(0, b) = 0$ we obtain

$$2E(a, b) = E(a, 2b). \tag{13}$$

It follows from (12) and (13) that the mapping E is additive in the first argument. The same must be true for the functional D. In the same way we prove that D is additive in the second argument.

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It is not difficult to verify that (4) and (11) imply

$$D(a, a) = 0, \qquad a \in R,$$

and

$$D(ca, cb) = cD(a, b)c^*, \qquad a, b, c \in R.$$

Using these two relations and biadditivity of D we shall prove that the mapping $J: R \to R$ given by $J(a) = D(a, 1)$ is a Jordan $*$-derivation satisfying

$$D(a, b) = J(ab) - aJ(b) - J(b)a^*, \qquad a, b \in R. \tag{14}$$

Clearly, J is additive. For arbitrary $a, b, c, d \in R$ we have

$$\begin{aligned} aD(b, c)a^* &+ D(db, ac) + D(ab, dc) + dD(b, c)d^* \\ &= D((a+d)b, (a+d)c) = (a+d)D(b, c)(a+d)^* \\ &= aD(b, c)a^* + dD(b, c)a^* + aD(b, c)d^* + dD(b, c)d^*, \end{aligned}$$

which yields

$$D(db, ac) + D(ab, dc) = dD(b, c)a^* + aD(b, c)d^*.$$

Putting $c = d = 1$ we get $D(b, a) + J(ab) = J(b)a^* + aJ(b)$. As $D(a, a) = 0$ implies $D(a, b) = -D(b, a)$, we have proved that (14) is valid. Replacing a in this relation by ba we see that

$$bJ(a)b^* = J(bab) - baJ(b) - J(b)a^*b^*$$

holds for all $a, b \in R$. Putting $a = 1$ and using $J(1) = 0$ we finally get $J(b^2) = bJ(b) + J(b)b^*$ for all $b \in R$.

According to our assumptions, J is an inner Jordan $*$-derivation. Thus, we can find an element $g \in R$ such that $J(a) = ag - ga^*$ is valid for all $a \in R$. It follows from (14) that

$$D(a, b) = agb^* - bga^*, \qquad a, b \in R.$$

Applying (11) one can easily see that

$$Q(ax + by) = ae_{11}a^* + ae_{12}b^* + be_{21}a^* + be_{22}b^*, \qquad a, b \in R,$$

where $e_{11} = Q(x)$, $e_{12} = g + 2^{-1}f$, $e_{21} = 2^{-1}f - g$, and $e_{22} = Q(y)$. We define $S: M \times M \to R$ by

$$S(ax + by, cx + dy) = ae_{11}c^* + ae_{12}d^* + be_{21}c^* + be_{22}d^*, \qquad a, b, c, d \in R.$$

In order to see that S is well defined we choose $a_1, a_2 \in R$ such that $a_1x + a_2y = 0$. For arbitrary elements $b_1, b_2 \in R$ we have

$$\begin{aligned} \sum_{i,j=1}^{2} b_i e_{ij} b_j^* &= Q(b_1x + b_2y) = Q((a_1 + b_1)x + (a_2 + b_2)y) \\ &= \sum_{i,j=1}^{2} (a_i + b_i)e_{ij}(a_j^* + b_j^*) \\ &= \sum_{i,j=1}^{2} a_ie_{ij}a_j^* + \sum_{i,j=1}^{2} a_ie_{ij}b_j^* + \sum_{i,j=1}^{2} b_ie_{ij}a_j^* + \sum_{i,j=1}^{2} b_ie_{ij}b_j^*. \end{aligned}$$

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It follows from $0 = Q(a_1x + a_2y) = \sum_{i,j=1}^{2} a_i e_{ij} a_j^*$ that

$$\sum_{i,j=1}^{2} a_i e_{ij} b_j^* + \sum_{i,j=1}^{2} b_i e_{ij} a_j^* = 0. \tag{15}$$

Putting $b_1 = 1$ and $b_2 = 0$ we get $p + q = 0$, where

$$p = a_1 e_{11} + a_2 e_{21}, \qquad q = e_{11} a_1^* + e_{12} a_2^*.$$

On the other hand, if we set in (15) $b_1 = c$ and $b_2 = 0$, we obtain $pc^* + cq = 0$. Together with $cq + cp = 0$ this implies $cp - pc^* = 0$ for all $c \in A$. It follows that $p = q = 0$, or

$$S(a_1x + a_2y, x) = 0 = S(x, a_1x + a_2y).$$

In a similar way we get

$$S(a_1x + a_2y, y) = 0 = S(y, a_1x + a_2y).$$

Thus, S is well defined. Clearly, it is a sesquilinear functional satisfying $Q(z) = S(z, z)$ for all $z \in M$. This completes the proof.

Proof of Lemma 1. Putting $a = 1$ and $b = 0$ we get $e_1 = 0$. Similarly, we obtain $e_4 = 0$. Substituting $a = b = 1$ we see that $e_2 = -e_3$. Substituting once again $b = 1$ we get that $ae_2 - e_2a^* = 0$ is valid for all $a \in R$. Thus, $e_2 = e_3 = 0$. This completes the proof.

Proof of Theorem 2. It is easy to verify that Q satisfies the parallelogram law (3). In order to see that also the homogeneity law (4) is fulfilled we must show that every Jordan $*$-derivation $J: R \to R$ satisfies

$$J(cbca) = cbJ(ca) + J(ca)b^*c^* + cJ(ba)c^* - cbJ(a)c^* - cJ(a)b^*c^* \tag{16}$$

for all $a, b, c \in R$. For this purpose first replace a by $a + b$ in (7) to get

$$J(ab) + J(ba) = bJ(a) + aJ(b) + J(a)b^* + J(b)a^* \tag{17}$$

for all $a, b \in R$. Consider now $d = J(a(ab + ba) + (ab + ba)a)$. Using (17) we see that

$$\begin{aligned} d &= aJ(ab + ba) + (ab + ba)J(a) + J(ab + ba)a^* + J(a)(b^*a^* + a^*b^*) \\ &= 2abJ(a) + a^2J(b) + aJ(a)b^* + 2aJ(b)a^* + baJ(a) \\ &\quad + bJ(a)a^* + 2J(a)b^*a^* + J(b)a^{*2} + J(a)a^*b^*. \end{aligned}$$

On the other hand,

$$\begin{aligned} d &= 2J(aba) + J(a^2b) + J(ba^2) \\ &= 2J(aba) + bJ(a^2) + a^2J(b) + J(a^2)b^* + J(b)a^{*2} \\ &= 2J(aba) + baJ(a) + bJ(a)a^* + a^2J(b) + aJ(a)b^* + J(a)a^*b^* + J(b)a^{*2}. \end{aligned}$$

Comparing the two expressions for d we arrive at

$$J(aba) = J(a)b^*a^* + aJ(b)a^* + abJ(a), \qquad a, b \in R. \tag{18}$$

Replacing a in (18) by $a + c$ we obtain

$$\begin{aligned} J(abc + cba) = J(a)b^*c^* + aJ(b)c^* + abJ(c) + J(c)b^*a^* \\ + cJ(b)a^* + cbJ(a), \qquad a, b, c \in R. \end{aligned} \tag{19}$$

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Applying (18) and (19) we get

$$\begin{aligned} J(cbca) &= J(cb(ca) + (ca)bc) - J(c(ab)c) \\ &= cbJ(ca) + J(ca)b^*c^* + c(J(b)a^* + aJ(b) - J(ab))c^* . \end{aligned}$$

Applying (17) we get (16). Thus, we have proved that Q is a quasi-quadratic functional.

Assume now that J is not inner. If there is a sesquilinear functional S which generates Q, then S is of the form $S((a, b), (c, d)) = aed^* + bfc^*$ for some $e, f \in R$. The relation $Q((a, b)) = S((a, b), (a, b))$ with $b = 1$ gives us $J(a) = -ae - fa^*$. Since $J(1) = 0$, we have $e = -f$, so that J is an inner Jordan $*$-derivation. This contradiction completes the proof.

Proof of Corollary 3. Let us first assume that all Jordan $*$-derivations on R are inner. We claim that $J_a = 0$, $a \in R$, implies $a = 0$. Indeed, for such an a we have

$$ba = ab^* \tag{20}$$

for all $b \in R$. Replacing b by bc and applying (20) two times we get

$$(bc - cb)a = 0. \tag{21}$$

Substituting $c = dc$ in (21) we obtain $(bdc-dcb)a = 0$, which can be rewritten as

$$(bd - db)ca + d(bc - cb)a = 0$$

where b, c, d are arbitrary elements from R. The second term is zero by (21). As R is noncommutative and prime, we have necessarily $a = 0$. Using the Main Theorem one can complete the proof of the "if part". Theorem 2 shows that the converse is also true.

Proof of Corollary 4. Substituting $a = ia$ and $b = i$ in (17) we prove that every Jordan $*$-derivation on R is inner. From $J_a(i) = 2ia$ it follows that $a \neq 0$ implies that J_a is nonzero. Using the Main Theorem one can complete the proof.

Verification of Example 5. Let R be a real $*$-algebra consisting of elements $\lambda + u\mu$, where λ and μ are complex numbers. We define the operations by $t(\lambda+u\mu) = t\lambda+u(t\mu)$ for real t, $(\lambda_1+u\mu_1)+(\lambda_2+u\mu_2) = (\lambda_1+\lambda_2)+u(\mu_1+\mu_2)$, $(\lambda_1+u\mu_1)(\lambda_2+u\mu_2) = \lambda_1\lambda_2+u(\mu_1\lambda_2+\overline{\lambda_1}\mu_2)$ and the involution by $(\lambda+u\mu)^* = \overline{\lambda} - u\mu$.

There exists a nontrivial and therefore discontinuous additive derivation on $\mathbb{R}$, that is, an additive function $f: \mathbb{R} \to \mathbb{R}$ satisfying $f(ts) = tf(s) + sf(t)$ for all pairs $t, s \in \mathbb{R}$ (see [12]). Putting $D(s + it) = f(s) - if(t)$ we get a function $D: \mathbb{C} \to \mathbb{C}$ which is additive and satisfies $D(\lambda^2) = 2\overline{\lambda}D(\lambda)$. It is not difficult to verify that the mapping $J: R \to R$ given by $J(\lambda + u\mu) = uD(\lambda)$ is a Jordan $*$-derivation. However, it is discontinuous and therefore noninner.

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