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COMPLETE MINIMAL SURFACES IN $\mathbf{R}^3$ OF GENUS ONE AND FOUR PLANAR EMBEDDED ENDS

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ABSTRACT. By using elliptic functions and Weierstrass representation we construct a one-parameter family of complete minimal surfaces in $\mathbf{R}^3$ with genus one and four planar embedded ends. These surfaces are critical points of the Willmore functional.

INTRODUCTION

Complete minimal surfaces M immersed in $\mathbf{R}^3$, $X: M \to \mathbf{R}^3$, with finite total curvature and embedded planar ends are interesting objects. Some authors have shown the existence of such immersions when M has the conformal structure of a punctured sphere S^2 or of a punctured real projective plane $\mathbf{R}P^2$. Bryant [1] and Peng [12] constructed examples when M is S^2 punctured at $N \geq 4$ points, Rosenberg and Toubiana [13] constructed a deformable family of these immersions with M conformally equivalent to S^2 minus four points, and Kusner [8] proved that for each odd $p \geq 3$ there exists an example M_p of these immersions where M_p has conformal structure of a $\mathbf{R}P^2$ punctured at p points.

In this work by using Weierstrass representation by elliptic functions we construct a one-parameter family of complete minimal surfaces in $\mathbf{R}^3$ of genus one, finite total curvature, and four planar embedded ends.

Let us consider in $\mathbf{C}$ the global coordinate $z = u + iv$, the holomorphic differential dz, and for each $y \geq 1$ the lattice $L(iy) = \{m + niy \in \mathbf{C};\ m, n \in \mathbf{Z}\}$. Then $\mathbf{C}/L(iy)$ are compact Riemann surfaces of genus one. If $P(z)$ is the Weierstrass function of $L(iy)$, we put

$$w_1 = \frac{1}{2}, \qquad w_2 = -\frac{1+iy}{2}, \qquad w_3 = \frac{iy}{2},$$

and $e_j = P(w_j)$, $j = 1, 2, 3$.

We will prove the following theorem:

Theorem A. *There exists a one-parameter family of complete minimal surfaces, $I_y: M_y \to \mathbf{R}^3$, $y \geq 1$, with finite total curvature and four planar embedded ends,*

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where M_y *is conformally equivalent to* $\mathbf{C}/L(iy)-\{\pi(0), \pi(w_1), \pi(w_2), \pi(w_3)\}$ *and* $\pi\colon \mathbf{C}\to\mathbf{C}/L(iy)$ *is the canonical projection. Furthermore, there exist* C^∞-*functions* $\alpha(y), \beta(y), c(y)\colon [1, \infty)\to\mathbf{R}-\{0\}$ *such that for each* $y\geq 1$

$$g = g_y = a(y)P(z)+b(y)P(z-w_3)+e_3c(y) \tag{1}$$

and

$$f\,dz = f_y\,dz = [P(z-w_1)+P(z-w_2)+e_3]\,dz \tag{2}$$

is a Weierstrass representation (g_y, f_y) *of* I_y, *where*

$$a(y)=\overline{b(y)}=\alpha(y)+i\beta(y). \tag{3}$$

The main difficulty to prove Theorem A is to show the existence of functions α, β, c such that for each $y\geq 1$ the pair $(g=g_y, f=f_y)$ is a Weierstrass representation of an immersion of M_y in $\mathbf{R}^3$. That is, it is necessary to show that

$$\operatorname{Re}\int \phi_i\,dz = 0, \qquad i=1, 2, 3, \tag{4}$$

for every closed curve γ in M_y, where $\phi_1=(1-g^2)f$, $\phi_2=i(1+g^2)f$, and $\phi_3=2gf$. As $\mathbf{C}/L(iy)$ is a genus one surface, to kill the real period of the integrals (4) by an appropriate choice of parameters α, β, c is a hard problem of a global nature. This is not the case when the genus of the compact surface involved is zero, as in all examples in the first paragraph. For these genus zero surfaces the Weierstrass data are pairs of rational functions depending on complex parameters and (4) is a local problem equivalent to a computation of the residues of the differentials ϕ_i.

We wish to mention a few papers that are related to Theorem A. Bryant [1] proved a beautiful result which shows that complete minimal surfaces in $\mathbf{R}^3$ with finite total curvature and embedded planar ends are critical points of the Willmore functional. That is, he proved that if $X\colon M=\overline{M}-\{q_1, \dots, q_N\}\to\mathbf{R}^3$ is a complete minimal immersion with finite total curvature and embedded planar ends, where $\overline{M}$ is a compact surface and $0\notin X(M)$, then the differentiable immersion $\widetilde{X}\colon\overline{M}\to\mathbf{R}^3$ given by $\widetilde{X}(q)=X(q)/|X(q)|^2$ if $q\in M$ and by $\widetilde{X}(q_j)=0$, $j=1, \dots, N$, is a critical point of the Willmore functional $W\colon\mathscr{F}(\overline{M})\to\mathbf{R}$, $W(Z)=\int_{\overline{M}}H^2\,dM$. Here $\mathscr{F}(\overline{M})$ is the set of differentiable immersions $Z\colon\overline{M}\to\mathbf{R}^3$ and H is the mean curvature of Z. A critical point of W is called a Willmore immersion. Then I_y, $y\geq 1$, given by Theorem A, produces a one-parameter family $\widetilde{I}_y$ of Willmore immersions.

On the other hand, Montiel and Ros [10] obtained a representation of branched complete minimal surfaces in $\mathbf{R}^3$ with finite total curvature and embedded planar ends by the kernel of the Jacobi operator $L=\Delta+|\nabla\phi|$ for a holomorphic map $\phi\colon\overline{M}\to S^2$, where $\overline{M}$ is a compact Riemann surface, with a metric compatible with its complex structure, and Δ, ∇ are the Laplacian and the gradient, respectively. Nevertheless, apparently their method does not allow neither to construct explicit examples nor to control the number of branch points. Then Weierstrass representations remain the principal method to construct explicit examples of surfaces of these type.

Remark 1. There does not exist a genus one complete minimal surface in $\mathbf{R}^3$ with finite total curvature and N embedded planar ends with $N=1$ or $N=2$.

In fact, suppose there exists such a surface. If $N = 1$, the surface would lie in a half-space and, by the stronger half-space theorem of Hoffman and Meeks [6], it is a plane. On the other hand, if $N = 2$ then Theorem 3 of Schoen [14] shows that the surface is a catenoid. In both cases we have a contradiction. Nevertheless, it is an open question if there exist examples when $N = 3$ or $N > 4$. We observe that in the case $N = 3$ Kusner [9] proved that the existence of a surface with the desired properties implies that the normal vectors at the ends of the surface lie in a plane.

1. Proof of Theorem A

In order to prove Theorem A we need a lemma, a proposition, and some notation. Related to the P function of the lattice $L(iy) = \{m + niy \in \mathbf{C};\ m, n \in Z\}$ we have the complex numbers $\eta_j = \eta_j(y)$,

$$2\eta_j = -\int_{l_j} P(z)\,dz, \qquad j = 1, 3, \tag{5}$$

where $l_j\colon [0, 1] \to \mathbf{C}$ are the paths

$$l_1(t) = t + \frac{iy}{3} \quad \text{and} \quad l_3(t) = \frac{1}{3} + iyt. \tag{6}$$

Also, we write for each $y \geq 1$

$$S_j = 2\eta_1 + e_j, \quad j = 1, 2, 3, \qquad \text{and} \quad S = S_1 + S_2. \tag{7}$$

We observe that $e_j = e_j(y)$, $S_j = S_j(y)$, and $2\eta_1 = 2\eta_1(y)$ are real numbers for every $y \geq 1$. Also, we define real functions $\alpha_j = \alpha_j(y)$, $\beta_j = \beta_j(y)$, and $\gamma_j = \gamma_j(y)$, where $y \geq 1$ and $j = 1, 2$, by

$$\begin{aligned} &\alpha_1 = 4(S_1^2 + S_2^2 + S_1S_2), \qquad \alpha_2 = 2(e_1S_1^2 + e_2S_2^2 - 2e_3S_1S_2), \\ &\beta_1 = -4S_1S_2, \qquad \beta_2 = 2[(e_3 - e_2)S_1^2 + (e_3 - e_1)S_2^2 + 2e_3S_1S_2], \end{aligned} \tag{8}$$

and

$$\gamma_1 = yS/\pi - 2, \qquad \gamma_2 = -S + 2\eta_1(yS/\pi - 2). \tag{9}$$

Proposition 1. *For every $y \geq 1$ we have that*

(a) $\alpha_1\beta_2 - \alpha_2\beta_1 < 0$,
(b) $\beta_2\gamma_1 - \beta_1\gamma_2 < 0$, *and*
(c) $\beta_1\gamma_2 - \beta_2\gamma_1 < \gamma_1\alpha_2 - \gamma_2\alpha_1$.

Proof. From (8) and by using that $e_1 + e_2 + e_3 = 0$ we find

$$\frac{\alpha_1\beta_2 - \alpha_2\beta_1}{8(e_3 - e_2)} = S_1^4 + \frac{3e_3}{e_3 - e_2}S_1^2S_2^2 + 2S_1^3S_2 + \frac{e_3 - e_1}{e_3 - e_2}(S_2^4 + 2S_1S_2^3).$$

Using (see [15, vol. 3, p. 138])

$$S_j > 0, \quad j = 1, 2, \qquad S_3 < 0, \qquad \text{and} \qquad e_1 > e_2 > e_3 < 0, \tag{10}$$

we conclude the proof of (a).

(b) Using (a) of Proposition 3 and (b) of Proposition 5 that appears in [3] we have

$$\frac{y}{\pi}S_1 - 2 > 0 \quad \text{and} \quad \frac{y}{\pi}S_2 - 1 \leq 0. \tag{11}$$

Then, from (7)–(9) and using (10) and (11) we find

$$\beta_2\gamma_1 - \beta_1\gamma_2 = 2\left(\frac{y}{\pi}S - 2\right)[(e_3 - e_2)S_1^2 + (e_3 - e_1)S_2^2 + 2S_1S_2S_3] - 4S_1S_2S < 0.$$

This completes the proof of (b).

(c) It is sufficient to show that

$$\begin{aligned}&\gamma_2(\beta_1 + \alpha_1) - \gamma_1(\beta_2 + \alpha_2)\\ &\quad = 4\left[-S + 2\eta_1\left(\frac{y}{\pi}S - 2\right)\right](S_1^2 + S_2^2) + 4\left(\frac{y}{\pi}S - 2\right)(e_2S_1^2 + e_1S_2^2)\\ &\quad = 4S\left[\left(\frac{y}{\pi}S - 2\right)S_1S_2 - S_1^2 - S_2^2\right] = 4S^2\left(\frac{y}{\pi}S_1S_2 - S\right)\\ &\quad = 4S^2\left[\frac{y}{\pi}S_1\left(S_2 - \frac{\pi}{y}\right) - S_2\right] < 0.\end{aligned}$$

From (10) and (11) we conclude that the inequality above is verified. This completes the proof of Proposition 1.

Lemma 1. *There exist differentiable functions* $\alpha, \beta\colon [1, \infty) \to (0, \infty)$ *such that* $\alpha = \alpha(y)$ *and* $\beta = \beta(y)$ *are a solution of the equations*

$$\frac{(e_1 - e_2)^2}{S^2}[\alpha_j(\alpha^2 - \beta^2) + \beta_j(\alpha^2 + \beta^2)] = \gamma_j, \qquad j = 1, 2, \tag{12}$$

in the lattice $L(iy)$.

Proof. By using Proposition 1(a) we have that equations (12) are equivalent to equations

$$\frac{(e_1 - e_2)^2}{S^2}(\alpha^2 - \beta^2) = \frac{\beta_2\gamma_1 - \beta_1\gamma_2}{\alpha_1\beta_2 - \alpha_2\beta_1}$$

and

$$\frac{(e_1 - e_2)^2}{S^2}(\alpha^2 + \beta^2) = \frac{\alpha_1\gamma_2 - \alpha_2\gamma_1}{\alpha_1\beta_2 - \alpha_2\beta_1}.$$

Again, by using Proposition 1, we conclude that for each fixed $y \geq 1$ the equations above are, respectively, a hyperbola and a circle in the variables α and β. Furthermore, these paths are transversal. Then there exist differentiable functions $\alpha(y), \beta(y)\colon [1, \infty) \to (0, \infty)$ such that $\alpha = \alpha(y)$, $\beta = \beta(y)$ are a solution of the equations above in $L(iy)$. This completes the proof of Lemma 1.

Proof of Theorem A. We will first prove that for each $y \geq 1$ the pair $(g = g_y, f = f_y)$ defined in (1)–(3) is a Weierstrass representation of a complete minimal immersion of M_y in $\mathbf{R}^3$, where M_y is defined in Theorem A, $\alpha(y)$ and $\beta(y)$ are given by Lemma 1, and

$$c = c(y) = -\frac{2(e_1S_2 + e_2S_1)}{e_3S}\alpha(y). \tag{13}$$

In order to do this, it is sufficient that conditions (C_1), (C_2), and (C_3) are satisfied by (g_y, f_y). That is,

(C_1) $z \in M_y$ is a pole of order m of g_y if and only if z is a zero of order $2m$ of f_y;

(C_2) $\int_\mu (1 + |g|^2)|f\,dz| = +\infty$ for every divergent path $\mu \subset M_y$; and

(C_3) $\mathrm{Re}\int_l gf\,dz = 0$ and $\int_l g^2 f\,dz = \overline{\int_l f\,dz}$ for every closed path $l \subset M_y$.

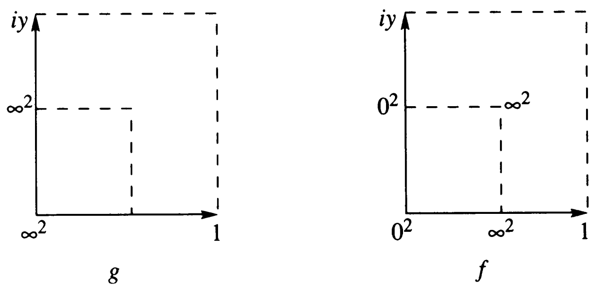

FIGURE 1

The main difficulty here is to show that the choice of α, β, and c determined by Lemma 1 and (13) is such that condition (C_3) is satisfied. We remark that (C_3) is equivalent to (4) in the introduction.

In order to show that (C_j), $j = 1, 2, 3$, are satisfied, observe that g and f are holomorphic functions in M_y and $f(z) \neq 0$ for every $z \in M_y$. Then (C_1) is satisfied. Also, we observe that $g^2 f$ has double poles at $z = 0$ and $z = w_3$ and that f has double poles at $z = w_1$ and $z = w_2$. Then it follows that the metric defined by (g, f) is complete. That is, condition (C_2) is satisfied.

Figure 1 shows the poles and zeros of g and f in the lattice $L(iy)$.

On the other hand, as $g'(w_2) = g'(w_1) = 0$ we conclude that

$$\operatorname{Res}_z gf = \operatorname{Res}_z g^2 f = \operatorname{Res}_z f = 0 \quad \text{for every } z \in M_y. \tag{14}$$

From (14) we conclude that (C_3) is satisfied if and only if

(C_3^*) $\operatorname{Re}\int_{l_k} gf\,dz = 0$, $\int_{l_k} g^2 f\,dz = \overline{\int_{l_k} f\,dz}$, $k = 1, 3$, where l_k are the paths given by (6).

Also, by using (1) and (2) (see also Figure 1), we conclude that there exist $a_j(y) = a_j \in \mathbf{C}$, $j = 0, 1, 2$, and $b_j(y) = b_j \in \mathbf{C}$, $j = 0, 1, 2, 3, 4$, such that

$$gf = a_1 P(z - w_1) + a_2 P(z - w_2) + a_0 \tag{15}$$

and

$$g^2 f = b_0 P(z) + \sum_{j=1}^{3} b_j P(z - w_j) + b_4. \tag{16}$$

Now we will find a_j and b_j as functions of the variables $a = a(y) = \alpha(y) + i\beta(y)$ and $b = b(y) = \alpha(y) - i\beta(y)$. In order to do this, we use (1) and (2) to find the local developments of g and f at $0 \in \mathbf{C}$ (see [15, vol. 1, p. 159]):

$$g(z) = \frac{a}{z^2} + e_3(b + c) + o(z^2) \tag{17}$$

and

$$\begin{aligned} f(z) &= \frac{P''(w_1) + P''(w_2)}{2} z^2 + \frac{P''''(w_1) + P''''(w_2)}{4!} z^4 + o(z^6) \\ &= (L_1 + L_2) z^2 + (e_1 L_1 + e_2 L_2) z^4 + o(z^6) \\ &= (e_1 - e_2)^2 z^2 - 2e_3 (e_1 - e_2)^2 z^4 + o(z^6), \end{aligned} \tag{18}$$

where

$$L_1 = (e_1 - e_2)(e_1 - e_3) \quad \text{and} \quad L_2 = (e_1 - e_2)(e_3 - e_2).$$

Observe that to find (18) we use

$$P'' = 6P^2 + 2(e_1e_2 + e_1e_3 + e_2e_3)$$

and that the notation $o((z - z_0)^n)$, $n \geq 0$, is such that

$$\lim_{z \to z_0} (z - z_0)^{-n+1} o((z - z_0)^n) = 0.$$

From (17) and (18) we find at $0 \in \mathbf{C}$

$$gf(z) = a(e_1 - e_2)^2 + o(z^2) \tag{19}$$

and

$$g^2 f(z) = a^2(e_1 - e_2)^2 \frac{1}{z^2} + 2e_3(e_1 - e_2)^2[a(b + c) - a^2] + o(z^2). \tag{20}$$

Then, from (15), (16), (19), and (20) we find that

$$b_o = (e_1 - e_2)^2 a^2 \tag{21}$$

and

$$e_1a_1 + e_2a_2 + a_o = a(e_1 - e_2)^2. \tag{22}$$

Also, by using (15), (16), and local developments of g and f at w_1 and w_2 we find, respectively,

$$a_1 = e_1a + e_2b + e_3c, \qquad b_1 = a_1^2 \tag{23}$$

and

$$a_2 = e_2a + e_1b + e_3c, \qquad b_2 = a_2^2. \tag{24}$$

Then, from (3) and (22)–(24) we obtain

$$a_o = -2e_1e_2(a + b) + e_3^2 c = -4e_1e_2\alpha + e_3^2 c. \tag{25}$$

Also, from (1) and (2) we find at $z = w_3$

$$g(z) = \frac{b}{(z - w_3)^2} + o((z - w_3)^0), \qquad f(z) = (e_1 - e_2)^2(z - w_3)^2 + o((z - w_3)^4).$$

Then these expressions and (16) imply that

$$b_3 = (e_1 - e_2)^2 b^2. \tag{26}$$

Also, comparing the local development of (16) at $z = 0$ and (20) we have that

$$b_4 + \sum_{j=1}^{3} e_j b_j = 2e_3(e_1 - e_2)^2[a(b + c) - a^2].$$

This result together with (3), (23), (24), (26), and the fact that $e_1 + e_2 + e_3 = 0$ implies that

$$\begin{aligned} b_4 = {} & 2e_3[e_1e_2 - (e_1 - e_2)^2](\alpha^2 - \beta^2) + e_3^3 c^2 \\ & + 2e_3[e_1e_2 + (e_1 - e_2)^2](\alpha^2 + \beta^2) - 8e_1e_2e_3\alpha c. \end{aligned} \tag{27}$$

On the other hand, from (5), (6), and (15) we have

$$\int_{l_k} gf\,dz = -(a_1 + a_2)2\eta_k + 2w_k a_0\,, \qquad k = 1\,,\,3. \tag{28}$$

We observe that $2\eta_1 \in \mathbf{R}$, $2\eta_3 = iy(2\eta_1) - 2\pi i$ (Legendre's relation), and $e_j \in \mathbf{R}$, $j = 1\,,\,2\,,\,3$. Then from (3), (13), (23)–(25), and (28) we find

$$\operatorname{Re}\int_{l_k} gf\,dz = 0\,, \qquad k = 1\,,\,3. \tag{29}$$

Also, by using (2) and (16) we find that the second equality in (C_3^*) is equivalent to

$$-\left(\sum_{j=0}^{3} b_j\right) 2\eta_k + 2w_k b_4 = -\overline{4\eta_k + 2w_k e_3}\,, \qquad k = 1\,,\,3.$$

From Legendre's relation, we conclude that the equations below are equivalent to equations

$$\sum_{j=0}^{3} b_j = \frac{y}{\pi}S - 2 \tag{30}$$

and

$$b_4 = -S + 2\eta_1\left(\frac{y}{\pi}S - 2\right)\,, \tag{31}$$

where S is defined in (7).

On the other hand from (21), (23), (24), and (26) we find

$$\sum_{j=0}^{3} b_j = [(e_1 - e_2)^2 + e_1^2 + e_2^2](a^2 + b^2) + 4e_1e_2ab$$
$$+ 2(e_1e_3 + e_2e_3)(a + b)c + 2e_3^2c^2.$$

So, by using (3) and (13) we have

$$\sum_{j=0}^{3} b_j = 2[(e_1 - e_2)^2 + e_1^2 + e_2^2](\alpha^2 - \beta^2) + 4e_1e_2(\alpha^2 + \beta^2)$$
$$+ \frac{8}{S^2}[e_3S(e_1S_2 + e_2S_1) + (e_1S_2 + e_2S_1)^2]\alpha^2.$$

Also, as $e_3 = -e_1 - e_2$ and $S = S_1 + S_2$ we obtain

$$S^2\sum_{j=0}^{3} b_j = 4(e_1 - e_2)^2[(S_1^2 + S_2^2)\alpha^2 - S^2\beta^2].$$

This result together with notation (8) shows that equation (30) is equivalent to

$$\frac{(e_1 - e_2)^2}{S^2}[\alpha_1(\alpha^2 - \beta^2) + \beta_1(\alpha^2 + \beta^2)] = \gamma_1. \tag{30*}$$

Also, from (27) and (13) we find

$$\begin{aligned} b_4 &= 2e_3[e_1e_2 - (e_1 - e_2)^2](\alpha^2 - \beta^2) \\ &+ \frac{4e_3}{S^2}(e_1S_2 + e_2S_1)^2\alpha^2 + 2e_3[e_1e_2 + (e_1 - e_2)^2](\alpha^2 + \beta^2) \\ &+ \frac{16e_1e_2}{S}(e_1S_2 + e_2S_1)\alpha^2. \end{aligned}$$

So, as $e_1 + e_2 = -e_3$ and $S = S_1 + S_2$ we have

$$S^2 b_4 = -(e_1 - e_2)^2[4(e_2S_1^2 + e_1S_2^2)\alpha^2 - 4e_3S^2\beta^2].$$

This result together with notation (8) implies that equation (31) is equivalent to

$$(31^*) \qquad \frac{(e_1 - e_2)^2}{S^2}[\alpha_2(\alpha^2 - \beta^2) + \beta_2(\alpha^2 + \beta^2)] = \gamma_2.$$

But (30*) and (31*) are exactly the equations of Lemma 1. Then (30) and (31) are equivalent to equations (12) of this lemma. This proves that (30) and (31) are satisfied when $\alpha(y)$ and $\beta(y)$ are given by Lemma 1. This proves that

$$\int_{l_k} g^2 f\,dz = \overline{\int_{l_k} f\,dz}, \qquad k = 1, 3.$$

These results and (29) show that (C_3^*) is satisfied. Finally, (C_3^*) and (14) show that (C_3) is satisfied. This completes the proof that, for each $y \geq 1$, (g_y, f_y) is a Weierstrass representation of a complete minimal immersion of M_y in $\mathbf{R}^3$. On the other hand, as g_y is the Gauss normal of the immersion and its order is four, the total curvature is -16π. Also, M_y is a genus one surface with four ends. These facts imply that the ends of the immersion are embedded. Furthermore, as g_y and f_y have, respectively, poles of order two and zeros of order two at $z = 0$ and $z = w_3$ and f_y have double poles at $z = w_1$ and $z = w_2$ where $g'(w_1) = g'(w_2) = 0$, we conclude that the ends of the immersion are planar ends. This completes the proof of Theorem A.

Remark 2. We observe that we have chosen a pair (α, β) that is a solution of (30) and (31) with the condition $\alpha > 0$, $\beta > 0$. The choice of the other possible solution of these equations does not produce a new minimal immersion.

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