

## JACOBSON RADICALS OF NEST ALGEBRAS IN FACTORS

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**ABSTRACT. Definition.** Let  $\beta$  be a nest in a separably acting type  $\text{II}_\infty$  factor  $\mathcal{M}$ . An element  $P \in \beta \setminus \{0, I\}$  is said to be a singular point of  $\beta$  if it satisfies either of the following conditions:

(1) There is a strictly increasing sequence  $\{Q_n\} \subseteq \beta$ ,  $\lim_{n \rightarrow \infty} Q_n = P$ , and  $P - Q_n$  is infinite for each  $n \in \mathbb{N}$ . Also, there is a projection  $Q \in \beta$  such that  $Q > P$  and  $Q - P$  is finite.

(2) There is a strictly decreasing sequence  $\{Q_n\} \subseteq \beta$ ,  $\lim_{n \rightarrow \infty} Q_n = P$ , and  $Q_n - P$  is infinite for each  $n \in \mathbb{N}$ . Also, there is a projection  $Q \in \beta$  such that  $Q < P$  and  $P - Q$  is finite.

**Main Theorem.** Let  $\beta$  be a nest in a separably acting factor  $\mathcal{M}$ .

(1) If  $\mathcal{M}$  is of type  $\text{II}_\infty$ , then a necessary and sufficient condition for the Jacobson radical  $\mathcal{R}_\beta$  of  $\text{alg } \beta$  to be a norm-closed singly generated ideal of  $\text{alg } \beta$  is that the nest  $\beta$  is countable and it does not contain a singular point.

(2) If  $\mathcal{M}$  is of type  $\text{II}_1$  or type  $\text{III}$ , then a necessary and sufficient condition for the Jacobson radical  $\mathcal{R}_\beta$  of  $\text{alg } \beta$  to be a norm-closed singly generated ideal of  $\text{alg } \beta$  is that the nest  $\beta$  is countable.

(3) In (1) and (2) the single generation is equivalent to countable generation.

### 1. INTRODUCTION

The nest algebra in a von Neumann algebra was introduced by Gilfeather and Larson [5]. They extend the Ringrose Criterion [7] of the membership for the Jacobson radical of a nest algebra into the von Neumann algebra setting.

In the  $\mathcal{B}(H)$  case Orr [6] characterized necessary and sufficient conditions for a general nest to have the property that the Jacobson radical is norm-closed singly generated. In this paper we will establish conditions for the Jacobson radicals  $\mathcal{R}_\beta$  of  $\text{alg } \beta$  in separably acting type II and type III factors.

### 2. PRELIMINARIES

Let  $\mathcal{M}$  be a factor acting on separable Hilbert space  $\mathcal{H}$ . By a *nest* in  $\mathcal{M}$  we mean a totally ordered family of (selfadjoint) projections containing  $\{0, I\}$

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which is closed in the strong operator topology. A nest  $\beta$  is *countable* if it is a countable set. The nest algebra related to a nest  $\beta$  in a factor  $\mathcal{M}$  is the set

$$\text{alg } \beta = \{T \in \mathcal{M} : TP = PTP \text{ for each } P \in \beta\}.$$

**Definition 1.** Let  $\beta$  be a nest in a separably acting type  $\text{II}_\infty$  factor  $\mathcal{M}$ . An element  $P \in \beta \setminus \{0, I\}$  is said to be a singular point of  $\beta$  if it satisfies either of the following conditions:

(1) There is a strictly increasing sequence  $\{Q_n\} \subseteq \beta$ ,  $\lim_{n \rightarrow \infty} Q_n = P$ , and  $P - Q_n$  is infinite for each  $n \in \mathbb{N}$ . Also, there is a projection  $Q \in \beta$  such that  $Q > P$  and  $Q - P$  is finite.

(2) There is a strictly decreasing sequence  $\{Q_n\} \subseteq \beta$ ,  $\lim_{n \rightarrow \infty} Q_n = P$ , and  $Q_n - P$  is infinite for each  $n \in \mathbb{N}$ . Also, there is a projection  $Q \in \beta$  such that  $Q < P$  and  $P - Q$  is finite.

The following theorem is our main result in this paper.

**Theorem 1.** Let  $\beta$  be a nest in a separably acting factor  $\mathcal{M}$ .

(1) If  $\mathcal{M}$  is of type  $\text{II}_\infty$ , then a necessary and sufficient condition for the Jacobson radical  $\mathcal{R}_\beta$  of  $\text{alg } \beta$  to be a norm-closed singly generated ideal of  $\text{alg } \beta$  is that the nest  $\beta$  is countable and it does not contain a singular point.

(2) If  $\mathcal{M}$  is of type  $\text{II}_1$  or type  $\text{III}$ , then a necessary and sufficient condition for the Jacobson radical  $\mathcal{R}_\beta$  of  $\text{alg } \beta$  to be a norm-closed singly generated ideal of  $\text{alg } \beta$  is that the nest  $\beta$  is countable.

(3) In (1) and (2) single generation is equivalent to countable generation.

Let  $\mathcal{M}$  be a factor in  $\mathcal{B}(H)$  for a separable Hilbert space  $\mathcal{H}$  and  $\beta$  be a nest in  $\mathcal{M}$ . We can write  $\beta = \{N_\lambda : \lambda \in \Lambda\}$  for some closed set  $\Lambda \subseteq [0, 1]$  which contains 0 and 1. A net  $\{N_{\lambda_i}\}$  in  $\beta$  converges to  $N_{t_0} \in \beta$  in the strong operator topology if and only if  $\lambda_i$  converges to  $t_0$  in the regular topology on  $[0, 1]$ . A projection  $E = M - N$ ,  $M, N \in \beta$ ,  $M > N$ , is called a  $\beta$ -interval. The projections  $M, N$  are called the upper and lower end points of  $E$ , respectively.

The Jacobson radical of an arbitrary algebra is defined to be the intersection of the kernels of all strictly transitive representations of the algebra. The radical of a Banach algebra is a closed 2-sided topologically nil-ideal which contains every topologically nil left or right ideal in the algebra. If  $\mathcal{A}$  is a unital Banach algebra and  $\mathcal{R}$  is its radical, then

$$\begin{aligned} \mathcal{R} &= \{B \in \mathcal{A} : AB \text{ is quasinilpotent, } A \in \mathcal{A}\} \\ &= \{B \in \mathcal{A} : BA \text{ is quasinilpotent, } A \in \mathcal{A}\}. \end{aligned}$$

Also, if  $\sigma(A)$  denotes the spectrum of  $A$  in  $\mathcal{A}$ , then  $B \in \mathcal{R}$  if and only if  $\sigma(A + B) = \sigma(A)$  for all  $A \in \mathcal{A}$ .

In [5], Gilfeather and Larson generalized the Ringrose Criterion [7] to the radical  $\mathcal{R}_\beta$  of  $\text{alg } \beta$  into the von Neumann algebra setting. They proved:

**Theorem 2** (Gilfeather, Larson, Ringrose [5, 7]). *If  $A \in \text{alg } \beta$ , then  $A \in \mathcal{R}_\beta$  if and only if for each  $\varepsilon > 0$  there exists a finite set  $\{E_i\}$  of mutually orthogonal  $\beta$ -intervals with  $\sum E_i = I$  such that  $\|E_i A E_i\| < \varepsilon$  for all  $i$ .*

### 3. SUFFICIENCY

Let  $\beta$  be a nest in a factor  $\mathcal{M}$ . We will say that an element  $N$  in a nest  $\beta$  is a *left limit point* of  $\beta$  if there is a strictly increasing sequence  $\{N_n\}$  in  $\beta$  such that  $N$  is the sot(strong operator)-limit of  $\{N_n\}$ .

We will use the following notation:

$$E_\lambda(\beta) = \bigvee \{N \in \beta : N \text{ is finite}\},$$

$$E_\rho(\beta) = \bigvee \{N^\perp : N \in \beta \text{ and } N^\perp \text{ is finite}\}.$$

**Definition 2.** Let  $\beta$  be a nest in a type  $\text{II}_\infty$  factor  $\mathcal{M}$ .

(1) The nest is said to be of order type  $I_1$  if it is an infinite set, and either  $\beta$  or its dual  $\beta^\perp$  satisfies the following conditions:

- (a) Both  $E_\lambda$  and  $E_\rho^\perp$  of the nest are left limit points of the nest.
- (b) The projection  $E_\lambda$  is infinite and the projection  $E_\rho$  is finite.

(2) The nest is said to be of order type  $I_2$  if  $\beta$  or its dual nest  $\beta^\perp$  satisfies the following condition:

The identity  $I$  is a limit point of the nest. If  $Q$  is in the nest,  $Q < I$ ,  $Q^\perp$  is infinite. There exists a finite nonzero projection  $P$  in the nest.

(3)  $\beta$  is said to be of order type I if it is of order type  $I_1$  or of order type  $I_2$ .

(4)  $\beta$  is said to be of order type II if it is not of order type I.

The main results in [2] are concerned with the extension of the results in [1] into the von Neumann algebra setting. The author proved the following

**Theorem 3** [1]. *Let  $\beta$  be a nest in a separably acting factor  $\mathcal{M}$ .*

(1) *Assume that  $\mathcal{M}$  is of type  $\text{II}_\infty$ . Then the factor  $\mathcal{M}$  is a norm-principal (or norm-closed singly generated) bimodule of  $\text{alg } \beta$  if and only if the nest  $\beta$  is of order type II.*

(2) *Assume that  $\mathcal{M}$  is of type  $\text{II}_1$  or type III. Then the factor  $\mathcal{M}$  is always a norm-principal bimodule of  $\text{alg } \beta$ .*

Let  $\beta$  be a nest in  $\mathcal{M}$  and let  $P \in \beta \setminus \{0, I\}$ . We define a set of projections  $\beta_P$  in  $\mathcal{M}$  by

$$\beta_P = \{P^\perp + N : N \leq P, N \in \beta\} \cup \{N - P : N > P, N \in \beta\} \cup \{0, I\}.$$

It is clear  $\beta_P$  is also a nest.

The nest  $\beta_P$  has some properties which are important to the purpose in this paper.

**Lemma 1.** *Let  $\beta$  be a nest in  $\mathcal{M}$  and let  $P \in \beta \setminus \{0, I\}$ . Then*

- (1)  $(\beta_P)_{P^\perp} = \beta$ ;
- (2)  $P(\text{alg } \beta)P = P(\text{alg } \beta_P)P$ ;
- (3)  $P^\perp(\text{alg } \beta)P^\perp = P^\perp(\text{alg } \beta_P)P^\perp$ .

*Proof.* Statement (1) is obvious.

Let  $T \in P(\text{alg } \beta)P$  and  $N \in \beta_P$ . If  $N \geq P^\perp$ , then there is  $M \in \beta$ ,  $M < P$ , and  $N = P^\perp + M$ . So  $TN = PTPN = PTP(P^\perp + M) = PTMN =$

$PMTMN = NTN$ . If  $N < P^\perp$ , then  $N = M - P$  for some  $M > P$  and  $M \in \beta$ . So  $TN = PTPN = 0 = NTN$ . So we have  $T \in \text{alg } \beta_P$ . Hence

$$(1) \quad P(\text{alg } \beta)P \subseteq P(\text{alg } \beta_P)P.$$

By statement (1) and (1) we have

$$(2) \quad P^\perp(\text{alg } \beta_P)P^\perp \subseteq P^\perp(\text{alg } \beta)P^\perp.$$

Let  $T \in P(\text{alg } \beta_P)P$  and  $M \in \beta$ . If  $M < P$ , then  $P^\perp + M \in \beta_P$ . So  $TM = PTPM = PT(P^\perp + M)P = P(P^\perp + M)T(P^\perp + M)P = MTM$ . If  $M \geq P$ , then  $MP = PM = P$ , so  $MTM = T = TM$ . So we have  $T \in \text{alg } \beta$ , and

$$(3) \quad P(\text{alg } \beta)P \supseteq P(\text{alg } \beta_P)P.$$

By statement (1) and (3) we have

$$(4) \quad P^\perp(\text{alg } \beta_P)P^\perp \supseteq P^\perp(\text{alg } \beta)P^\perp.$$

Lemma 1 is proven.  $\square$

**Lemma 2.** Let  $\beta$  be a nest in a type  $\text{II}_\infty$  factor  $\mathcal{M}$  and let  $P \in \beta \setminus \{0, I\}$  which is not a singular point.

(1) If there is a projection  $Q \in \beta$  such that  $Q - P$  is finite and, for each projection  $R < P$ ,  $R \in \beta$  and the projection  $P - R$  is infinite, then  $P$  has an immediate predecessor  $P_-$  in  $\beta$  and  $P - P_-$  is an infinite atom of  $\beta$ .

(2) If there is a projection  $Q \in \beta$  such that  $P - Q$  is finite and, for each projection  $R > P$ ,  $R \in \beta$  and the projection  $R - P$  is infinite, then  $P$  has an immediate successor  $P_+$  in  $\beta$  and  $P_+ - P$  is an infinite atom of  $\beta$ .

*Proof.* By the definition of the singular point, it is clear.

**Lemma 3.** Let  $\beta$  be a nest in a type  $\text{II}_\infty$  factor  $\mathcal{M}$  and let  $P \in \beta \setminus \{0, I\}$  with both  $P$  and  $P^\perp$  infinite. Assume that  $\beta$  has no singular point. Then  $\beta_P$  is an order type II nest in  $\mathcal{M}$ .  $\square$

*Proof.* Define

$$\lambda_P^- = \bigwedge \{Q \in \beta : P - Q \text{ is finite}\},$$

$$\lambda_P^+ = \bigvee \{Q \in \beta : Q - P \text{ is finite}\}.$$

For the projections just defined, we have the following cases.

For  $\lambda_P^-$ , we have:

- (1)  $\lambda_P^- = 0$  and  $P - \lambda_P^-$  is infinite.
- (2)  $\lambda_P^- > 0$  and  $P - \lambda_P^-$  is finite.
- (3)  $\lambda_P^- > 0$  and  $P - \lambda_P^-$  is infinite.

For  $\lambda_P^+$ , we have:

- (1)  $\lambda_P^+ = I$  and  $\lambda_P^+ - P$  is infinite.
- (2)  $\lambda_P^+ < I$  and  $\lambda_P^+ - P$  is finite.
- (3)  $\lambda_P^+ < I$  and  $\lambda_P^+ - P$  is infinite.

It is clear that the case (1)–(2) is the dual of (2)–(1); the case (1)–(3) is the dual of (3)–(1); and the case (2)–(3) is the dual of (3)–(2). Notice that a nest is of order type II if and only if its dual has order type II and a nest has no

singular point if and only if its dual has no singular point. It suffices to discuss the following six cases: (1)–(1), (1)–(2), (1)–(3), (2)–(2), (2)–(3), and (3)–(3).

Case (1)–(1). It is clear that  $\beta_P$  is of order type II.

Case (1)–(2). Since  $\lambda_P^+ - P$  is finite and  $P^\perp$  is infinite, by Lemma 2,  $(\lambda_P^+)_+ - \lambda_P^+$  must be an infinite atom. So,  $\beta_P$  is of order type II.

Case (1)–(3). It is clear.

Case (2)–(2). In this case  $\lambda_P^+ - \lambda_P^-$  is finite and  $P$  and  $P^\perp$  are infinite. So by Lemma 2 again  $(\lambda_P^+)_+ - \lambda_P^+$  and  $\lambda_P^- - (\lambda_P^-)_-$  are infinite atoms. So,  $\beta_P$  is of order type II.

Case (2)–(3). Similar to Case (2)–(2).

Case (3)–(3). It is clear.

Lemma 3 is proven.  $\square$

Let  $\mathcal{I}_c$  be the (only) norm-closed nontrivial ideal in a type  $\text{II}_\infty$  factor  $\mathcal{M}$ . In [2] the author proved

**Lemma 4** [2]. *Let  $\beta$  be a nest in a type  $\text{II}_\infty$  factor  $\mathcal{M}$ . A necessary and sufficient condition for  $\mathcal{I}_c$  to be a norm-closed singly generated bimodule of  $\text{alg } \beta$  is that the nest does not satisfy either one of the following conditions:*

- *The 0 projection is a limit point of the nest and any nonzero projection in  $\beta$  is infinite.*
- *The identity  $I$  is a limit point of the nest and, for any projection  $Q < I$ ,  $Q \in \beta$ ,  $Q^\perp$  is infinite.*

**Proposition 1.** *Let  $\beta$  be a nest in a type  $\text{II}_\infty$  factor  $\mathcal{M}$  and let  $P \in \beta \setminus \{0, I\}$ . Assume that  $\beta$  has no singular point. Then there exists an operator  $G_P \in P(\text{alg } \beta)P^\perp \subseteq \mathcal{R}_\beta$  such that*

$$P\mathcal{M}P^\perp = P(\text{alg } \beta)P^\perp = P(\mathcal{R}_\beta)P^\perp = [( \text{alg } \beta)G_P(\text{alg } \beta)].$$

*Proof.* (1) Assume both  $P$  and  $P^\perp$  are infinite. By Lemma 3 and Theorem 3, there is an operator  $R \in \mathcal{M}$  such that  $\mathcal{M} = [( \text{alg } \beta_P)R(\text{alg } \beta_P)]$ .

Recall that  $P^\perp \in \beta_P$ ; we have

$$\begin{aligned} P\mathcal{M}P^\perp &= P(\text{alg } \beta)P^\perp = P[( \text{alg } \beta_P)R(\text{alg } \beta_P)]P^\perp \\ &= [P(\text{alg } \beta_P)PRP^\perp(\text{alg } \beta_P)P^\perp] \\ &= [P(\text{alg } \beta_P)P( PRP^\perp)P^\perp(\text{alg } \beta_P)P^\perp]. \end{aligned}$$

Let  $G_P = PRP^\perp$ . By Lemma 1 we have

$$\begin{aligned} P\mathcal{M}P^\perp &= P(\text{alg } \beta)P^\perp = P(\mathcal{R}_\beta)P^\perp \\ &= [P(\text{alg } \beta)PG_PP^\perp(\text{alg } \beta)P^\perp] = [( \text{alg } \beta)G_P(\text{alg } \beta)]. \end{aligned}$$

(2) Assume that  $P$  is finite. (The case when  $P^\perp$  is finite is a dual case to this. We omit it.) Then  $P\mathcal{M}P^\perp = P\mathcal{I}_cP^\perp$ . Since  $\beta$  has no singular point, we have only two possible cases:

- (a) There is a projection  $Q \in \beta$ ,  $P < Q$ , and  $Q - P$  is finite.

(b) For each projection  $Q > P, Q \in \beta$ , the projection  $Q - P$  is infinite. Then by Lemma 2  $P$  is a left end point of an infinite atom of  $\beta$ .

It is clear that in either case the nest  $\beta_P$  satisfies the condition in Lemma 4. By Lemma 4 there exists an operator  $R \in \mathcal{M}$  such that

$$\mathcal{F}_c = [(\text{alg } \beta_P)R(\text{alg } \beta_P)].$$

Therefore,

$$P\mathcal{M}P^\perp = P\mathcal{F}_cP^\perp = P[(\text{alg } \beta_P)R(\text{alg } \beta_P)]P^\perp.$$

Now the same technique we used in part (1) applies. Proposition 1 is proven.  $\square$

**Proposition 2.** *Let  $\beta$  be a countable nest in a separably acting factor  $\mathcal{M}$ . If for each  $P \in \beta \setminus \{0, I\}$  we have  $P(\text{alg } \beta)P^\perp = [(\text{alg } \beta)G_P(\text{alg } \beta)]$  for some operator  $G_P \in \text{alg } \beta$ , then the Jacobson radical  $\mathcal{R}_\beta$  is a norm-principal ideal of  $\text{alg } \beta$ .*

*Proof.* The proof is the same as that of Proposition 4 in [1]. The reader is referred to the proof in [1].

By Theorem 3 and Propositions 1 and 2, the sufficient part of Theorem 1 is proven.  $\square$

#### 4. NECESSITY

In this section we will first show that if the Jacobson radical is a norm-principal ideal of  $\text{alg } \beta$ , then the nest must be countable.

**Proposition 3.** *Let  $\beta$  be a nest in a separably acting factor  $\mathcal{M}$ . If  $\mathcal{R}_\beta$  is a norm-closed countably generated ideal of  $\text{alg } \beta$ , then the nest  $\beta$  must be countable.*

*Proof.* Let  $\beta = \{N_\lambda : \lambda \in \Lambda \subset [0, 1]\}$ . Let  $\phi : (\text{alg } \beta) \times [0, 1] \rightarrow \mathbb{R}^+$  be Erdos's diagonal function of  $\beta$ , defined as follows. Given  $A \in \text{alg } \beta$  and  $t \in (0, 1)$ ,

$$\begin{aligned} \phi(A, t) &= \inf\{\|N[\lambda_1, \lambda_2]AN[\lambda_1, \lambda_2]\| : \lambda_1, \lambda_2 \in \Lambda, \lambda_1 < t < \lambda_2\}, \\ \phi(A, 0) &= \inf\{\|N_\lambda AN_\lambda\| : \lambda \in \Lambda, 0 < \lambda\}, \\ \phi(A, 1) &= \inf\{\|(I_\mathcal{M} - N_\lambda)A(I_\mathcal{M} - N_\lambda)\| : \lambda \in \Lambda, \lambda < 1\}. \end{aligned}$$

Assume that  $\mathcal{G} = \{G_n\}$  is a countable subset in  $\mathcal{R}_\beta$  and let  $G \in \mathcal{G}$ . Employing the diagonal function  $\phi$  and the same method in the proof of Proposition 6 in [1], we prove that  $S_G = \{\lambda \in \Lambda : \phi(G, \lambda) \neq 0\}$  is a countable set. So  $S_\mathcal{G} = \bigcup S_{G_n}$  is a countable subset of  $\Lambda$ . Let  $\mathcal{I}_\mathcal{G}$  denote the norm-closed ideal generated by  $\mathcal{G}$  and let  $A \in \mathcal{I}_\mathcal{G}$ . Applying the same method in the proof of Proposition 6 in [1] or in [6], we prove that  $S_A \subseteq S_\mathcal{G}$ .

Assume  $\beta$  is an uncountable nest. We will show that  $\mathcal{I}_\mathcal{G}$  is a proper subset of  $\mathcal{R}_\beta$ . Since  $\Lambda$  is an uncountable compact subset of  $\Lambda$ , there exists a point  $t_0 \in \Lambda \setminus (S_\mathcal{G} \cup \{0, 1\})$  and  $t_0$  is a two-sided limit point of  $\Lambda$ . Let  $\{\alpha_n\}_{n=1}^\infty$  be a strictly increasing sequence in  $\Lambda$  converging to  $t_0$  and let  $\{\beta_n\}_{n=1}^\infty$  be a strictly decreasing sequence in  $\Lambda$  converging to  $t_0$ .

Denote  $E_n = N_{\alpha_{n+1}} - N_{\alpha_n}$  and  $F_n = N_{\beta_n} - N_{\beta_{n+1}}$ . Then for each  $n \in \mathbb{N}$  there exist non-0 projections  $E'_n \leq E_n$  and  $F'_n \leq F_n$  such that  $E'_n \sim F'_n$  for each  $n \in \mathbb{N}$ . Let  $Q_n$  be a partial isometry with initial projection  $F'_n$  and final projection  $E'_n$ . Define an operator  $R_0 = \sum_{n=1}^\infty Q_n$ . The operator  $R_0$  is in  $\mathcal{R}_\beta$  and  $t_0 \in S_{R_0}$ , so  $t_0 \in S_\mathcal{G}$ . This contradicts the selection of  $t_0$ . Proposition 3 is proven.  $\square$

Next we will prove that if a nest  $\beta$  in a type  $\text{II}_\infty$  factor has a singular point, then the radical  $\mathcal{R}_\beta$  cannot be even countably generated.

Let  $F$  be a fixed nonzero finite projection in the type  $\text{II}_\infty$  factor  $\mathcal{M}$ . Then there exists a faithful normal semifinite tracial weight  $\varrho$  on  $\mathcal{M}$  such that  $\varrho(F) = 1$  (cf. [4, Proposition 8.5.5]). For an operator  $T \in \mathcal{M}$ , if its range projection  $R_T$  is finite, we define a *range measure* of  $T$  by  $\mu(T) = \varrho(R_T)$ . If the range projection is infinite, we define  $\mu(T) = \infty$ . In [2] we proved the following

**Lemma 5** [2]. *Let  $T$  and  $S$  be finite range operators. Then:*

- (1)  $\mu(TS) \leq \min\{\mu(S), \mu(T)\}$ .
- (2)  $\mu(T + S) \leq \mu(S) + \mu(T)$ .
- (3) *If  $Q$  is a partial projection and  $\mu(Q) > \mu(T)$ , then  $\|Q - T\| \geq 1$ .*
- (4) *If  $\{N_n\}$  is a sequence of projections such that  $N_n \rightarrow 0$  in the strong operator topology and  $T \in \mathcal{M}$  has finite range projection  $R_T$ , then  $\lim \mu(N_n T) = 0$ .*

The following theorem together with Proposition 3 completes the proof of the necessary part of Theorem 1. Theorem 4 also provides a crucial step for the proof of Theorem 1(3).

**Theorem 4.** *Let  $\beta$  be a nest in a separably acting type  $\text{II}_\infty$  factor  $\mathcal{M}$ . Assume  $\beta$  has a singular point and  $\mathcal{S}$  be a sequence of operators in  $\mathcal{R}_\beta$ . Then  $[(\text{alg } \beta)\mathcal{S}(\text{alg } \beta)]$  is a proper subset of  $\mathcal{M}$ .*

*Proof.* (1) Let  $\mathcal{S} = \{G_n\}$  be an arbitrary given sequence of operators in  $\mathcal{R}_\beta$ . Assume that  $P$  is a singular point of  $\beta$ . We can assume that  $\beta$  and  $P$  satisfy Definition 1(1). (The other case is the dual to this.)

Assume that  $\mathcal{R}_\beta = [(\text{alg } \beta)\mathcal{S}(\text{alg } \beta)]$ . According to the assumption we have

$$\begin{aligned} (Q - P_1)(\mathcal{R}_\beta)(Q - P_1) &= [((Q - P_1)(\text{alg } \beta)(Q - P_1))((Q - P_1)\mathcal{S}(Q - P_1))((Q - P_1)(\text{alg } \beta)(Q - P_1))] \\ &= [(\text{alg}((Q - P_1)\beta))((Q - P_1)\mathcal{S}(Q - P_1))(\text{alg}((Q - P_1)\beta))], \end{aligned}$$

where  $\text{alg}((Q - P_1)\beta)$  is the nest algebra in the (type  $\text{II}_\infty$ ) factor  $(Q - P_1) \times \mathcal{M}(Q - P_1)$  related to the nest  $(Q - P_1)\beta$ . It is clear that  $(Q - P_1)(\mathcal{R}_\beta)(Q - P_1)$  is the Jacobson radical of  $\text{alg}((Q - P_1)\beta)$ . So our assumption implies that the countable set  $(Q - P_1)\mathcal{S}(Q - P_1)$  is a generating set for the radical  $\mathcal{R}_{(Q - P_1)\beta}$ . It is also clear that the projection  $(Q - P_1)P$  is a singular point in the nest  $(Q - P_1)\beta$ . Therefore, we can simply assume that  $P_1 = 0$  and  $Q = I$ . We assume this is the case. Under this new assumption  $P^\perp$  is finite and  $P$  is infinite, and

$$\begin{aligned} P\mathcal{M}P^\perp &= P(\text{alg } \beta)P^\perp = P\mathcal{F}_cP^\perp = P\mathcal{R}_\beta P^\perp \\ &= P[(\text{alg } \beta)\mathcal{S}(\text{alg } \beta)]P^\perp = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3, \end{aligned}$$

where

$$\begin{aligned} \mathcal{I}_1 &= [(P \text{ alg } \beta P)(P\mathcal{S}P)(P \text{ alg } \beta P^\perp)], \\ \mathcal{I}_2 &= [(P \text{ alg } \beta P)(P\mathcal{S}P^\perp)(P^\perp \text{ alg } \beta P^\perp)], \\ \mathcal{I}_3 &= [(P \text{ alg } \beta P^\perp)(P^\perp\mathcal{S}P^\perp)(P^\perp \text{ alg } \beta P^\perp)]. \end{aligned}$$

(2) Now we construct an operator (a partial isometry)  $T_0 \in P\mathcal{M}P^\perp$  which is not in  $P[(\text{alg } \beta)\mathcal{E}(\text{alg } \beta)]P^\perp$ . This will complete the proof. We will use the techniques we developed in [2]. We have two cases.

*Case 1.* There is a strictly decreasing sequence of projections  $\{Q_n\} \subset \beta$  such that  $s\text{-}\lim_{n \rightarrow \infty} Q_n = P$ . Let  $\varrho$  be the faithful normal semifinite tracial weight on  $\mathcal{M}$  such that  $\varrho(P^\perp) = 1$ , and let  $\mu$  be the related range measure. Then  $\mu(P^\perp) = 1$ . We can assume that the sequence  $\{P_n\}$  possesses the property  $\mu(P_{n+1} - P_n) \geq 1$ . Denote  $\alpha_n = \mu(Q_n - Q_{n+1})$ , where  $Q_1 = I$ . It is a routine exercise to verify that  $\sum_{j=1}^\infty \alpha_j = 1$ . Notice that  $\mu(PG_nP^\perp Q_k) \leq 1$  for  $n, k \in \mathbb{N}$  and  $\lim P_n = P$ . For fixed  $n, k \in \mathbb{N}$  and  $\varepsilon > 0$ , by Lemma 5 there is an  $n_k \in \mathbb{N}$  such that  $\mu(P_m^\perp PG_nP^\perp Q_k) < \varepsilon$  for each  $m \geq n_k$ . For  $\varepsilon = \frac{1}{2k}(1 - \sum_{j=1}^k \alpha_j)$  we select  $n_k$  such that  $\mu(P_m^\perp PG_jP^\perp Q_k) < \frac{1}{2k}(1 - \sum_{j=1}^k \alpha_j)$  for each  $j \leq k$  and  $m \geq n_k$ . We can select the subsequence  $\{P_{n_k}\}$  to be strictly increasing. Since  $P_{n_{k+1}} - P_{n_k} \succeq P^\perp \succeq Q_k - Q_{k+1}$ , there exists a partial isometry  $V_k$  with initial projection  $Q_k - Q_{k+1}$  and final projection  $F_k < P_{n_{k+1}} - P_{n_k}$ . So we have a sequence of partial isometries  $\{V_k\}$  with initial projections and final projections mutually orthogonal. We define  $T_0 = \sum_{k=1}^\infty V_k$ . The operator  $T_0$  is a partial isometry and  $T_0 \in P\mathcal{M}P^\perp$ . It is clear that  $P_{n_k}^\perp T_0 Q_k = P_{n_k}^\perp T_0$  is also a partial isometry with  $\mu(P_{n_k}^\perp T_0 Q_k) = 1 - \sum_{j=1}^{k-1} \alpha_j$  for  $k \geq 2$ .

*Claim.*  $d(T_0, \mathcal{S}_2) \geq 1$ .

*Proof of the claim.* Let  $n_0$  be an arbitrary number in  $\mathbb{N}$ . Let  $A_j, B_j \in \text{alg } \beta$  and  $T_j \in P\mathcal{E}P^\perp$  for  $j = 1, 2, \dots, n_0$ . Then the operator in the form of  $\sum_{j=1}^{n_0} A_j T_j B_j$  is norm-dense in  $\mathcal{S}_2$ . Take an arbitrary such operator  $\sum_{j=1}^{n_0} A_j T_j B_j$ . Since  $T_j \in P\mathcal{E}P^\perp = \{PG_kP^\perp\}$ , there is a  $k(j) \in \mathbb{N}$  such that  $T_j = PG_{k(j)}P^\perp$ . Let  $k_0 = \max\{\max[k(j) : j = 1, 2, \dots, n_0], n_0\}$ . Then  $\mu(P_{k_0}^\perp T_j) < \frac{1}{2k_0}(1 - \sum_{j=1}^{k_0} \alpha_j)$ ,  $1 \leq j \leq k_0$ .

By Lemma 5 we have

$$\begin{aligned} \mu \left( P_{k_0}^\perp \left( \sum_{j=1}^{n_0} A_j T_j B_j \right) Q_{k_0} \right) &\leq \sum_{j=1}^{n_0} \mu(P_{k_0}^\perp A_j T_j B_j Q_{k_0}) \leq \sum_{j=1}^{n_0} \mu(P_{k_0}^\perp T_j) \\ &\leq n_0 \frac{1}{2k_0} \left( 1 - \sum_{j=1}^{k_0} \alpha_j \right) \leq \frac{1}{2} \left( 1 - \sum_{j=1}^{k_0} \alpha_j \right) < 1 - \sum_{j=1}^{k_0-1} \alpha_j \\ &= \mu(P_{k_0}^\perp T_0) = \mu(P_{k_0}^\perp T_0 Q_{k_0}). \end{aligned}$$

By Lemma 5 again,

$$\left\| P_{k_0}^\perp \left( T_0 - \sum_{j=1}^{n_0} A_j T_j B_j \right) Q_{k_0} \right\| \geq 1.$$

So,  $\|T_0 - \sum_{j=1}^{n_0} A_j T_j B_j\| \geq 1$ . That is,  $d(T_0, \mathcal{S}_2) \geq 1$ .

Next we prove that  $T_0 \notin [(\text{alg } \beta)\mathcal{E}(\text{alg } \beta)]$ . It suffices to prove that  $\lim \|P_{k_0}^\perp R Q_{k_0}\| = 0$  for each element in  $\mathcal{S}_1$  or in  $\mathcal{S}_3$ . This is clear by the Ringrose Criterion.



Case 2. There is a projection  $Q' \in \beta$ ,  $Q' > P$ , and  $Q' - P$  is a finite atom. In this case we can simply assume that  $P^\perp$  is the finite atom. By the Ringrose Criterion (Theorem 2), we have  $P^\perp \mathcal{G} P^\perp = \{0\}$ . Since the factor is of type  $\text{II}_\infty$ , we can refine the nest into a new nest  $\beta'$  by inserting a strictly decreasing sequence of projection  $\{Q_n\}$  with limit point  $P$  into the nest  $\beta$ . The nest  $\beta'$  is in Case 1. Using the same construction (with  $\beta'$ ) as in the proof of Case 1, we prove that there exists an operator  $T_0 \in P \mathcal{F}_c P^\perp$  such that

$$\left\| P_{k_0}^\perp \left( T_0 - \sum_{j=1}^{n_0} A_j T_j B_j \right) \right\| \geq 1.$$

So  $d(T_0, \mathcal{F}_2) \geq 1$ . Since  $\mathcal{F}_3 = \{0\}$  and  $\lim \|P_{k_0}^\perp R\| = 0$  for each element in  $\mathcal{F}_1$  by the Ringrose Criterion again, we have proven Case 2. Theorem 4 is proven.  $\square$

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