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SOME INEQUALITIES FOR SUBMARKOVIAN GENERATORS AND THEIR APPLICATIONS TO THE PERTURBATION THEORY

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ABSTRACT. We characterize the domain in L^p-space of generators of submarkovian semigroups in terms of the form domain in L^2 and give the corresponding inequality. Using this inequality we obtain a criterion for the formal difference $A - B$ of such generators to be a generator of a contraction semigroup in L^p. The conditions on perturbation are expressed in terms of forms, i.e., in L^2-terms.

The following elementary inequality with application to the generators of submarkovian semigroups was found by Stroock [S, CKuS] (see also [V]).

For any $p \geq 1$ and $0 \leq s, t < \infty$

$$4\frac{p-1}{p^2}(s^{p/2} - t^{p/2})^2 \leq (s-t)(s^{p-1} - t^{p-1}) \leq (s^{p/2} - t^{p/2})^2 .$$

The proof is quite simple:

$$\begin{aligned}(s-t)(s^{p-1} - t^{p-1}) &= s^p + t^p - st(s^{p-2} + t^{p-2}) \\ &\leq s^p + t^p - 2ts\sqrt{s^{p-2}t^{p-2}} = (s^{p/2} - t^{p/2})^2 .\end{aligned}$$

On the other hand,

$$\frac{4}{p^2}(s^{p/2} - t^{p/2})^2 = \left(\int_t^s z^{p/2-1} dz\right)^2 \leq \left|\int_t^s dz\right| \cdot \left|\int_t^s z^{p-2} dz\right| \cdot (p-1)^{-1} .$$

In this paper we give two generalizations of this inequality and apply them to the generators of submarkovian semigroups. We get a characterization for the domain of the generator in the L^p-space in terms of the corresponding quadratic form in the L^2-space. We use the two-sided inequality obtained for the development of the perturbation theory for the generators in the L^p-space. The conditions on perturbation are expressed in terms of form-bounded perturbations, i.e., in L^2-terms. So in a sense, in our Theorem 2 below, we give an extension of the KLMN-theorem (see [RSi, K]) to the L^p-spaces.

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Lemma 1. *For any* $p \geq 1$ *and* $s, t \in \mathbb{R}^1$ *the following inequality holds true*:

$$4\frac{p-1}{p^2}(s|s|^{p/2-1} - t|t|^{p/2-1})^2 \leq (s-t)(s|s|^{p-2} - t|t|^{p-2}) \leq a(p)(s|s|^{p/2-1} - t|t|^{p/2-1})^2$$

where

$$a(p) = \sup_{x\in[0,1]} \frac{(x^{1/p}+1)(x^{1/p'}+1)}{(x^{1/2}+1)^2}, \qquad \frac{1}{p} + \frac{1}{p'} = 1, \tag{1}$$

$a(1) = 2$, $a(2) = 1$, $1 \leq a(p) \leq 2$, $\forall p > 1$. (*Moreover, if* s *and* t *are of the same sign, then* $a(p)$ *can be changed by* 1.)

The proof of the left-hand side of the inequality is the same as that of Stroock's. The right-hand side is equivalent to

$$\frac{(s-t)(s|s|^{p-2} - t|t|^{p-2})}{(s|s|^{p/2-1} - t|t|^{p/2-1})^2} \leq a(p).$$

It is easy to see that

$$\sup_{s,t\in\mathbb{R}^1} \frac{(s-t)(s|s|^{p-2} - t|t|^{p-2})}{(s|s|^{p/2-1} - t|t|^{p/2-1})^2} = \sup_{x\in[0,1]} \frac{(x^{1/p}+1)(x^{1/p'}+1)}{(x^{1/2}+1)^2}.$$

Let us point out some properties of the function $a(p)$ which are needed further. Firstly, $a(p) = a(p')$; secondly, $a(p)$ decreases on $(1, 2)$ and increases on $(2, \infty)$.

Now we give some definitions and explanations.

Let (M, μ) be a measurable space with the σ-finite measure μ. We use the following notation: $L^p \equiv L^p(M, \mu)$, $\|\cdot\|_p$ is the norm in L^p, and

$$\langle f, g\rangle \equiv \int_M f(x)\overline{g(x)}\,d\mu(x).$$

We say that A is a generator of a submarkovian semigroup (submarkovian generator) if the following conditions are satisfied:

(i) A is a nonnegative selfadjoint operator in L^2.
(ii) $\|e^{-tA}f\|_\infty \leq \|f\|_\infty$, $\forall f \in L^1 \cap L^\infty$.
(iii) $0 \leq f \in L^2 \Rightarrow e^{-tA}f \geq 0$ almost everywhere.

We say that the operator B is form-bounded relative to A and write $B \in PK_\beta(A)$ if B is a selfadjoint operator in L^2, $\mathscr{D}(|B|^{1/2}) \supset \mathscr{D}(A^{1/2})$, and

$$\|\,|B|^{1/2}\varphi\|_2^2 \leq \beta\|A^{1/2}\varphi\|_2^2 + C(\beta)\|\varphi\|_2^2 \quad \forall \varphi \in \mathscr{D}(A^{1/2})$$

for some $\beta \in (0, 1)$, $C(\beta) \geq 0$.

Now let A be a submarkovian generator. We can define the operator A_p as a generator of the contraction semigroup in L^p:

$$(e^{-tA} \restriction [L^2 \cap L^p])^{\sim}_{L^p\to L^p} = e^{-tA_p} \quad \text{(the closure in } L^p);$$

$$T^t_\infty =: (e^{-tA_1})' \quad (' \text{ is the sign of the adjoint operator}).$$

By representation of a resolvent in terms of a semigroup, we have

$$(\lambda + A_p)^{-1}[L^2 \cap L^p] = (\lambda + A)^{-1}[L^2 \cap L^p] \quad \forall \lambda > 0.$$

Now we are ready to formulate the main result of the paper:

Theorem 1. *Let A be a submarkovian generator. If $f = \operatorname{Re} f \in \mathscr{D}(A_p)$ for some $p \in (1, +\infty)$ then $f|f|^{p/2-1} \equiv g_p \in \mathscr{D}(A^{1/2})$ and the following inequality holds true*:

$$4\frac{p-1}{p^2}\|A^{1/2}g_p\|_2^2 \leq \langle A_p f, f|f|^{p-2}\rangle \leq a(p)\|A^{1/2}g_p\|_2^2 \tag{2}$$

where $a(p)$ is from (1) *and if $f \geq 0$, then $a(p) = 1$.*

Proof. Let $T^t =: T^t_\infty$. Set $P(t, \cdot, G) = T^t\mathbf{1}_G$, $G \in \mathscr{B}$, where $\mathscr{B}$ is the σ-algebra on M and $\mathbf{1}_G$ is the characteristic function of the set G. $P(t, \cdot, G)$ is a finitely additive set function on $\mathscr{B}$ and $P(t, \cdot, M) \leq 1$.

For any simple function $f = \sum_{i=1}^k c_i\mathbf{1}_{G_i}$, where $\{G_i\}$ are disjoint sets of finite measure, $c_i \in \mathbb{R}^1$, let us define

$$\int_M P(t, \cdot, dy)f(y) = T^t f = \sum_{i=1}^k c_i T^t\mathbf{1}_{G_i},$$

$f \in \mathscr{N}$, $\mathscr{N}$ is the set of simple functions. Then

$$T^t v_p = \sum_{i=1}^k c_i|c_i|^{p-2}T^t\mathbf{1}_{G_i}, \qquad T^t g_p = \sum_{i=1}^k c_i|c_i|^{p/2-1}T^t\mathbf{1}_{G_i},$$

where $v_p = f|f|^{p-2}$, $g_p = f|f|^{p/2-1}$.

Let $\varepsilon_t(u, v) = \frac{1}{t}\langle(1 - T^t)u, v\rangle$. Because of selfadjointness of A and consequently of symmetricity on (x, y) of the finite additive product-measure $d\mu_t(x, y) = P(t, x, dy)d\mu(x)$, the following equalities are valid:

$$\langle T^t f, v_p\rangle = \langle f, T^t v_p\rangle = \frac{1}{2}\int d\mu(x)\int P(t, x, dy)(f(x)v_p(y) + f(y)v_p(x)),$$

$$\langle (T^t\mathbf{1}_E)f, v_p\rangle = \langle \mathbf{1}_E, T^t|f|^p\rangle = \frac{1}{2}\int d\mu(x)\int P(t, x, dy)(|f(x)|^p + |f(y)|^p)$$

where E is the support of f. Hence we obtain

$$\begin{aligned}\varepsilon_t(f, v_p) =& \frac{1}{2t}\int d\mu(x)\int P(t, x, dy)(f(x) - f(y))(v_p(x) - v_p(y))\\ &+ \frac{1}{t}\langle(1 - T^t\mathbf{1}_E), |f|^p\rangle,\\ \varepsilon_t(g_p, g_p) =& \frac{1}{2t}\int d\mu(x)\int P(t, x, dy)(g_p(x) - g_p(y))^2\\ &+ \frac{1}{t}\langle(1 - T^t\mathbf{1}_E), |f|^p\rangle.\end{aligned}$$

By Lemma 1 we have

$$4\frac{p-1}{p^2}\varepsilon_t(g_p, g_p) \leq \varepsilon_t(f, v_p) + \frac{1}{t}\left(4\frac{p-1}{p^2} - 1\right)\langle(1 - T^t\mathbf{1}_E), |f|^p\rangle$$

and

$$\varepsilon_t(f, v_p) \leq a(p)\varepsilon_t(g_p, g_p).$$

So we get

$$4\frac{p-1}{p^2}\varepsilon_t(g_p, g_p) \leq \varepsilon_t(f, v_p) \leq a(p)\varepsilon_t(g_p, g_p). \tag{3}$$

Since the set $\mathscr{N}$ is dense in $\operatorname{Re} L^p$, $1 < p < \infty$, $v_p \in \operatorname{Re} L^{p'}$, (3) holds true for all $f = \operatorname{Re} f \in L^p$.

Now let $f = \operatorname{Re} f \in \mathscr{D}(A_p)$. Then (3) and the equality

$$\mathscr{D}(A^{1/2}) = \left\{ \psi \in L^2 : \sup_{t>0} \varepsilon_t(\psi, \psi) < \infty \right\},$$

which follows from the spectral theorem, yield $g_p \in \mathscr{D}(A^{1/2})$ and the left-hand side of (2) if we set $t \downarrow 0$. The right-hand side of (2) now follows from (3) and the left-hand side. □

Remark. Theorem 1 is the generalization of the corresponding results of Stroock [S, CKuS] and Varopoulos [V] in the sense of our assumptions on the measurable space M. Besides, the main inequality has been proved on the natural domain.

Let $a(p)$ be defined by (1). For a fixed $\beta \in (0, 1)$ the equation $\beta a(p) = 4(p-1)/p^2$ has exactly two solutions $t_1 \in (1, 2)$ and $t_2 = t_1' \in (2, \infty)$, where $t_1' = t_1/(t_1 - 1)$. This is a direct consequence of the above-mentioned properties of the function $a(p)$.

Theorem 1 and different consequences are discussed in more detail in [LPSe]. Here we give only the application to the perturbation theory.

Theorem 2. *Let A and B be generators of submarkovian semigroups and $B \in PK_\beta(A)$. Then the form-difference $A \dot{-} B = C$ is well defined and the following inequality is valid*:

$$\|e^{-tC} f\|_p \le e^{a(p)C(\beta)t} \|f\|_p \quad \forall p \in [t(\beta), t'(\beta)] \tag{4}$$

$\forall f \in L^2 \cap L^p$, where $C(\beta)$ is from the condition $PK_\beta(A)$, and $t(\beta) = t_1$ and $t'(\beta) = t_2$ are the corresponding roots of the equation $\beta a(q) = 4(q-1)/q^2$, $1 < q < \infty$.

Proof. Let

$$B_n = nB(B+n)^{-1} = n - n^2(B+n)^{-1}$$

(Yosida approximation), where B_n is a bounded selfadjoint operator in L^2. It is easy to check that B_n is a generator of a submarkovian semigroup. Besides, $B_n \le B$ and $B_n \in PK_\beta(A)$ with the same β and $C(\beta)$. So the operator $C_{p,n} = A_p - B_{n,p}$ with $\mathscr{D}(C_{p,n}) = \mathscr{D}(A_p)$ is the generator of the quasi-contractive semigroup T_n^t in L^p, $1 \le p < \infty$, $\forall n = 1, 2, \ldots$. Due to Stein [St, p. 67] these semigroups are holomorphic on L^p, $1 < p < \infty$.

Let $u_n(t) =: e^{-tC_{2,n}} f$, $f \in L^2 \cap L^p$. Then $u_n(t) \in \mathscr{D}(C_{p,n})$ for any $t > 0$ and $-du_n(t)/dt = C_{p,n} u_n(t)$. Note that $e^{-tC_{2,n}}[\operatorname{Re} L^2] \subset \operatorname{Re} L^2$ (it is a consequence of $B_n[\operatorname{Re} L^2] \subset \operatorname{Re} L^2$ and the Trotter product formula). Now without loss of generality, we can assume $f = \operatorname{Re} f$; then $u_n = \operatorname{Re} u_n$.

Multiplying both sides of the equation $-\frac{d}{dt} u_n = (A_p - B_{n,p}) u_n$ by $u_n |u_n|^{p-2}$, integrating over M, and using Theorem 1 twice (for the operator A_p and for the operators $B_{n,p}$) and condition $B_{n,2} \le B \in PK_\beta(A)$ we obtain

$$\begin{aligned} -\frac{1}{p}\frac{d}{dt}\|u_n\|_p^p &= \langle (A_p - B_{n,p})u_n\,,\, u_n|u_n|^{p-2}\rangle \\ &= \langle A_p u_n\,,\, u_n|u_n|^{p-2}\rangle - \langle B_{n,p}u_n\,,\, u_n|u_n|^{p-2}\rangle \\ &\ge 4\frac{p-1}{p^2}\|A^{1/2}(u_n|u_n|^{p/2-1})\|_2^2 - a(p)\|B_{n,p}^{1/2}(u_n|u_n|^{p/2-1})\|_2^2 \\ &\ge \left(4\frac{p-1}{p^2} - \beta a(p)\right)\|A^{1/2}(u_n|u_n|^{p/2-1})\|_2^2 - C(\beta)a(p)\|u_n\|_p^p. \end{aligned}$$

Consequently, for any $p \in [t(\beta)\,,\, t'(\beta)]$

$$\frac{d}{dt}\|u_n\|_p^p \le pC(\beta)a(p)\|u_n\|_p^p.$$

Thus $\|u_n(t)\|_p \le e^{C(\beta)a(p)t}\|u_n(0)\|_p$ or

$$\|e^{-t(A-B_n)}f\|_p \le e^{C(\beta)a(p)t}\|f\|_p \quad \forall f \in L^2 \cap L^p. \tag{5}$$

Since $A \dot{-} B \le C_{2,n} \le C_{2,m}$ provided $n \ge m$, then $e^{-tC_{2,n}} \to e^{-tC}$ strongly in L^2 [K, Chapter 6]. So that by (5) and by Fatou's lemma we can pass to the limit in (5):

$$\|e^{-t(A \dot{-} B)}f\|_p \le e^{C(\beta)a(p)t}\|f\|_p \quad \forall f \in L^2 \cap L^p. \quad \square$$

Thus we have defined the operator $(A-B)_p$ in terms of form-boundedness and showed that this operator is a generator of quasi-contraction semigroups in L^p.

Remarks. 1. The sharp constant in the right-hand side of the inequality (2) can be less than $a(p)$ for certain submarkovian generators, for example, for potentials. Therefore the constant $a(p)$ in (4) could be replaced by $\lim_n \inf a_{B_n}(p)$.

2. The semigroup $e^{-t(A \dot{-} B)}$ need not be positivity preserving. However, if $e^{-t(A-B_{n,2})}$ are positivity preserving for sufficiently large n then setting $u_n(t) = e^{-tC_{2,n}}|f|$, $f \in L^2 \cap L^p$, we can repeat the proof of Theorem 2 using inequality (2) with $f \ge 0$ and $a(p) = 1$. Instead of (5) we obtain

$$\|e^{-t(A-B_{n,2})}|f|\|_p \le e^{C(\beta)t}\|f\|_p\,, \qquad f \in L^2 \cap L^p.$$

Thus taking into account the inequality $|e^{-tC_{2,n}}f| \le e^{-tC_{2,n}}|f|$, we get

$$\|e^{-t(A \dot{-} B)_p}\|_p \le e^{C(\beta)t} \quad \forall p \in [t_+(\beta)\,,\, t'_+(\beta)],$$

where $t_+(\beta) = 2/1 + \sqrt{1-\beta}$, $t'_+(\beta) = 2/1 - \sqrt{1-\beta}$.

3. Theorem 2 for the case of the Schrödinger operator $-\Delta - V$ and the sharpness of the dependence $t(\beta)$ as a function of β was proved in [KoSe].

We now turn to the generalization of one of the inequalities proved in Theorem 1.

Lemma 2. *Let $\varphi : \mathbb{R}_+^1 \to \mathbb{R}_+^1$ be a function such that*

(i) $\varphi(z) = 0 \quad \forall z \in [0\,,\, b]$ *for some* $b \ge 0$;
(ii) $\varphi'(z) > 0 \quad \forall z > b$;

(iii) *the function* $g_\varphi(z)$ *is differentiable for* $z > b$ *where*

$$g_\varphi(z) = \begin{cases} z\phi(z) - \kappa & if\ z \geqslant b\,, \\ 0 & if\ z < b\,, \end{cases}$$

$$\phi(z) = \sqrt{\varphi'(z)}\ , \qquad \kappa \equiv z\phi(z)|_{z=b}\,;$$

(iv) $\sup_{z>b}(1 + z\phi'(z)/\phi(z))^2 = c_\varphi^{-1} < \infty$.

Then for all $t, s \in [b, +\infty)$

$$c_\varphi^{-1}(t-s)(\varphi(t) - \varphi(s)) \geq (g_\varphi(t) - g_\varphi(s))^2\,.$$

Proof.

$$\begin{aligned}(g_\varphi(t) - g_\varphi(s))^2 &= \left(\int_s^t dg_\varphi(z)\right)^2 = \left(\int_s^t (z\phi'(z) + \phi(z))\,dz\right)^2 \\ &\leq \left(\int_s^t \left(1 + \frac{z\phi'(z)}{\phi(z)}\right)^2 dz\right)\left(\int_s^t \phi^2(z)\,dz\right) \\ &\leq c_\varphi^{-1}(t-s)(\varphi(t) - \varphi(s))\,. \quad \square\end{aligned}$$

Theorem 3. *Let* φ *and* g_φ *be the same functions as in Lemma* 2. *Let* A *be a generator of the submarkovian semigroup* e^{-tA}. *If* $f = \operatorname{Re} f \in \mathscr{D}(A_p)$ *for some* $p \in [1, +\infty)$ *and* $\varphi(|f|)A_p f \in L^1(M, \mu)$, *then* $g_\varphi(|f|) \in \mathscr{D}(A^{1/2})$ *and*

$$c_\varphi\|A^{1/2}g_\varphi(|f|)\|_2^2 \leq \langle A_p f\,, (\operatorname{sgn} f)\varphi(|f|)\rangle\,.$$

The proof is almost the same as the proof of Theorem 1 when using the equality $T_\infty^t f(x) = \int P(t, x, dy)f(y)$, $\forall f \in L^\infty$, the evident inequality

$$\langle A_p f\,, (\operatorname{sgn} f)\varphi(|f|)\rangle \geq \lim_{t\downarrow 0}\left\langle \frac{1 - T^t}{t}|f|,\ \varphi(|f|)\right\rangle,$$

and Lemma 2. If we set $\varphi(z) = z^{p-1}$, $b = 0$, then $\phi(z) = \sqrt{p-1}z^{p/2-1}$, $g_\varphi(z) = \sqrt{p-1}z^{p/2}$, $c_\varphi^{-1} = p^2/4$. Hence we obtain the left-hand side of Stroock's inequality. If $\varphi(z) = \ln z$, $b = 1$, then

$$g_\varphi(z) = \begin{cases} \sqrt{z} - 1 & \text{if } z \geqslant 1\,, \\ 0 & \text{if } z < 1 \end{cases}$$

and $c_\varphi = 4$. Moreover, $(s-t)(\ln s - \ln t) \geq 4(\sqrt{s} - \sqrt{t})^2$ $\forall s, t \geq 1$. So from Theorem 3 we get that the conditions $f = \operatorname{Re} f \in \mathscr{D}(A_p)$ for some $p \geq 1$ and $\ln_+|f| \cdot A_p f \in L^1$ are sufficient to conclude $\mathbf{1}_{|f|>1}\sqrt{|f|} \in \mathscr{D}(A^{1/2})$ and the corresponding inequality holds true. This fact was a crucial tool in the investigation of the essential selfadjointness of the Schrödinger operator with negative form-bounded potential in the case of zero-bound [LSe]. It should be pointed out that for this case the analog of the right-hand side of the inequality of Theorem 1 cannot be obtained.

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