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MAPPING GALOIS EXTENSIONS INTO DIVISION ALGEBRAS

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ABSTRACT. Let A be a ring with a finite group of automorphisms G, and let f_1 and f_2 be homomorphisms from A into some division algebra D such that f_1 and f_2 agree on the fixed ring A^G. Assuming certain additional assumptions, it is shown that f_1 and f_2 differ only by an automorphism in G and an inner automorphism of D.

1. INTRODUCTION

Let A be a commutative ring with a finite group G of automorphisms, and let f_1 and f_2 be maps from A into some field D such that $f_1|_{A^G} = f_2|_{A^G}$. Then there is some $\sigma \in G$ such that $f_2 = f_1 \circ \sigma$ [B, Chapter V, §2, no. 2, Corollary to Theorem 2]. We generalize this result to the setting of noncommutative rings and division algebras, proving—under certain additional assumptions—the existence of some $\sigma \in G$ and some inner automorphism ψ of D such that $f_2 = \psi \circ f_1 \circ \sigma$ (Theorems 1 and 2 below).

We begin with a simple but instructive example due to Montgomery, which shows why inner automorphisms are needed in the noncommutative case. Let $A = \mathbf{C}$ denote the complex numbers, and let G be the group of automorphisms of A generated by complex conjugation. Then $A^G = \mathbf{R}$. Let f_1 be the natural inclusion of A into the quaternions $D = \mathbf{H}$, and let $f_2: A \to D$ be the $\mathbf{R}$-linear map defined by $f_2(i) = j$. Then $f_1|_{A^G} = f_2|_{A^G}$, but $f_1(A) \neq f_2(A)$. Hence for all $\sigma \in G$, $f_2 \neq f_1 \circ \sigma$. Montgomery noted, however, that the Skolem-Noether theorem implies the existence of some inner automorphism ψ of D such that $f_2 = \psi \circ f_1$. (E.g., let ψ be conjugation by $i+j$.) This led her to conjecture that in generalizing the commutative result, one should aim to prove $f_2 = \psi \circ f_1 \circ \sigma$. In this direction, we obtained the following two results:

Theorem 1. *Let A be a ring with a finite group of automorphisms G such that $|G|^{-1} \in A$. Let D be a division algebra, and let f_1 and f_2 be homomorphisms*

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from A into D such that $f_1|_{A^G} = f_2|_{A^G}$. Suppose that $f_1(A)$ is a ring satisfying a polynomial identity. Then there is some $\sigma \in G$ and some inner automorphism ψ of D such that $f_2 = \psi \circ f_1 \circ \sigma$.

A few remarks are in order. First of all, there are two interesting cases where the hypothesis that $f_1(A)$ be a polynomial identity ring is trivially satisfied: If either A itself is a PI-ring, in particular, if A is actually commutative, or if D is finite dimensional over its center. I do not know if the PI-hypothesis is necessary. It is certainly needed in the proof in an essential way; cf. Remark 5 below.

Secondly, note that the map ψ in the theorem, when restricted to $f_1(A^G)$, is the identity. Given any map $f: A \to D$, call f equivalent to f_1 if $f = \psi \circ f_1 \circ \sigma$, where $\sigma \in G$ and where ψ is any inner automorphism of D with $\psi|_{f_1(A^G)} = \mathrm{id}_{f_1(A^G)}$. Trivially, equivalent maps agree on the fixed ring A^G. The theorem says that all maps which agree on A^G are equivalent! This is quite surprising. For example, another way to construct maps which agree on A^G is as follows: Given any automorphism τ of A which induces the identity on A^G, the maps f_1 and $f_1 \circ \tau$ agree on A^G. So by the theorem, f_1 and $f_1 \circ \tau$ are equivalent, i.e., $f_1 \circ \tau = \psi \circ f_1 \circ \sigma$, although $\mathrm{Aut}_{A^G}(A)$ is in general much bigger than G. (Of course, this situation occurs already in the commutative setting, but only if f_1 is not injective. In the noncommutative setting, it appears even if A is a division algebra. And we will see that—given a theorem of Montgomery—it is fairly easy to reduce the proof of Theorem 1 to the latter case; see §3.)

Thirdly, in general one cannot drop the assumption that $|G|^{-1} \in A$, even if A is an affine prime Noetherian PI-ring finite over its center and if D is a (commutative) field (Example 7). However, we have the following positive result:

Theorem 2. *Let A be a ring with a finite group of automorphisms G. Let D be a division algebra, and let f_1 and f_2 be homomorphisms from A into D such that $f_1|_{A^G} = f_2|_{A^G}$. Suppose that either*

(a) *f_1 is injective and A satisfies a polynomial identity, or*
(b) *A is commutative and D is finite over its center.*

Then there is some $\sigma \in G$ and some inner automorphism ψ of D such that $f_2 = \psi \circ f_1 \circ \sigma$.

In generalizing the commutative result, one could also try to replace the field D by a finite-dimensional central simple algebra. We will see later in Example 8 that this approach does not work, even if A is commutative.

I should mention a related result on equivalence of maps by F. Pop and H. Pop [PP]. They proved among other things the following: Let A be a semisimple algebra over a field k, and let D be a separable k-algebra. Then up to composition by inner automorphisms of D, there are only finitely many algebra homomorphisms from A to D.

This paper is organized as follows. Section 2, the heart of this paper, deals with the special case that A is a division algebra. Section 3 contains the proofs of the main theorems, and §4 consists of two examples.

Finally, I would like to thank Susan Montgomery for bringing this problem to my attention.

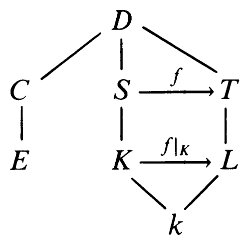
2. The division algebra case

The following proposition is the basic technical result we will need. It is the special case that $A = S$ is a division algebra. Note that here we make no assumptions on the order of G.

Proposition 3. *Let S be a division algebra finite over its center K. Let G be a finite group of automorphisms of S. Let f_1 and f_2 be embeddings of S into a division algebra D such that $f_1|_{K^G} = f_2|_{K^G}$. Then there is some $\sigma \in G$ and some inner automorphism ψ of D such that $f_2 = \psi \circ f_1 \circ \sigma$.*

Let k be some subfield of K^G such that K^G is a finite purely inseparable extension of k. The proof will show that it is actually sufficient to require only that $f_1|_k = f_2|_k$. Moreover, if D is finite over its center, K^G may be an infinite purely inseparable extension of k. We will prove the proposition in this slightly greater generality. This will be important in the proof of Theorem 2(b).

Proof. We first show that we may assume that $S \subseteq D$, and that f_1 is the natural embedding. Namely, if the proposition is true in the latter case, then $f_2 \circ f_1^{-1} = \psi \circ \sigma'$ for some $\sigma' \in (f_1 \circ G \circ f_1^{-1})$ and some $\psi \in \mathrm{Inn}(D)$. Say $\sigma' = f_1 \circ \sigma \circ f_1^{-1}$ for $\sigma \in G$. Then $f_2 = \psi \circ (\sigma' \circ f_1) = \psi \circ f_1 \circ \sigma$, as desired. So from now on we will assume that $S \subseteq D$, and that f_1 is the natural embedding. To simplify notation, set $f = f_2$, $T = f(S)$, and $L = f(K)$. Then $f|_k = \mathrm{id}_k$, and L is the center of T. We have to find $\sigma \in G$ and $\psi \in \mathrm{Inn}(D)$ such that $f = \psi \circ \sigma$. The following figure illustrates the situation:



Here C denotes the center of D, and E always denotes a subfield of C.

In the sequel, we will work with various division subalgebras of D generated by some subfield E of C and either S or T. For example, ES denotes the division algebra generated by E and S inside D. If EK denotes the compositum of the fields E and K inside D, the $EK \otimes_K S$ is a simple ring with center EK. The natural map from $EK \otimes_K S$ into ES is thus injective; hence $EK \otimes_K S$ is a domain. Since S is finite over K, $EK \otimes_K S$ is finite over the field EK. Thus it is a division algebra and therefore isomorphic to ES. To summarize, $ES \approx EK \otimes_K S$ is a division algebra which is finite dimensional over its center EK. (See Remark 5 below for a discussion of what would happen if S were not finite over K.)

Let $D' \subseteq D$ be the division algebra generated by CS and CT. Then Ck is central in D'. (This follows immediately from the construction of D': Let R_1 be the subring of D generated by CS and CT, and for any integer $i > 1$, let R_i be the subring of D generated by R_{i-1} and the inverses of all nonzero elements in R_{i-1}. Then $D' = \bigcup_i R_i$ is a division algebra, and Ck is central in D' since it is central in all R_i.) Since every inner automorphism of D' extends to D, we may therefore replace D by D' and thus assume that k is contained in the center C of D.

We will see in Lemma 4 below that for some $\sigma \in G$, $f \circ \sigma$ extends to an isomorphism $g: CS \to CT$ such that $g|_C = \mathrm{id}_C$. If K^G is a finite extension of k, then CK is finite over $Ck = C$. Since S is finite over K, $CS \approx CK \otimes_K S$ is finite over CK. Thus CS is finite over C, and so is $CT = g(CS)$. If K^G is not finite over k, D is finite over C by assumption. So in any event, now both CS and CT are finite dimensional over C. It follows by the Skolem-Noether theorem that g is the restriction of some inner automorphism ψ of D. Since $\psi|_S = g|_S = f \circ \sigma$, we conclude that $f = \psi \circ \sigma^{-1}$, proving the proposition modulo the following lemma.

Lemma 4. *There is a $\sigma \in G$ such that $f \circ \sigma$ extends to an isomorphism $g: CS \to CT$ such that $g|_C = \mathrm{id}_C$.*

Proof. Let $\mathcal{M}$ be the set of all tuples (E, h) such that E is a subfield of C containing k and h is an isomorphism $ES \to ET$ extending f and inducing the identity on E. Note that $\mathcal{M} \neq \varnothing$ since $(k, f) \in \mathcal{M}$. We introduce a partial ordering on $\mathcal{M}$ by setting $(E, h) \leq (E', h')$ iff $E \subseteq E'$ and $h'|_{ES} = h$. We next show that $\mathcal{M}$ is inductively ordered. Let $\{(E_i, h_i)\}$ be a chain in $\mathcal{M}$. Then $E = \bigcup_i E_i$ is a subfield of C. Since $ES \approx EK \otimes_K S$, every element α of ES involves only a finite number of elements of E. Thus $\alpha \in E_iS$ for some i. It follows easily that the h_i extend to an isomorphism $h: ES \to ET$ inducing the identity on E. Hence $\mathcal{M}$ is inductively ordered.

Thus Zorn's lemma implies the existence of a maximal element (E_1, g_1). Assume that there is a subfield E_2 of C properly containing E_1 and an isomorphism $g_2: E_2S \to E_2T$ inducing the identity on E_2 and extending $f \circ \sigma_2$ for some $\sigma_2 \in G$. Choose (E_2, g_2) maximal with respect to these properties, and continue. Since G is finite, we eventually obtain a subfield E of C maximal with respect to the following property $(*)$: There is an isomorphism $g: ES \to ET$ inducing the identity on E and extending $f \circ \sigma$ for some $\sigma \in G$. Since we do not mind replacing f by $f \circ \sigma$, we can as well assume that g extends f.

Let α be an element of C. We will show that $\alpha \in E$. Let $ES[x]$ and $ET[x]$ be the polynomial rings in one central indeterminate over the division algebras ES and ET. Consider the homomorphisms $\Phi: ES[x] \to ES[\alpha]$ and $\Psi: ET[x] \to ET[\alpha]$ sending x to α. The kernel of Φ is generated by a polynomial $q(x)$ with central coefficients [MR, 9.6.3]. That is, $q(x) \in EK[x]$. It follows that $q(x) = 0$ iff $\operatorname{Ker}\Phi = 0$ iff α is transcendent over EK iff α is transcendent over E iff α is transcendent over EL iff $\operatorname{Ker}\Psi = 0$. Here we used that both EK and EL are algebraic over E. Assume that $\operatorname{Ker}\Phi = 0$. Then it follows by the above remarks that α is a central indeterminate over both ES and ET. Set $E' = E(\alpha)$. Then g extends to an isomorphism $g': E'S \to E'T$ by defining $g'(\alpha) = \alpha$. Clearly g' extends f, and $g'|_{E'} = \mathrm{id}_{E'}$. This is a contradiction to the maximality of E. Hence $\operatorname{Ker}\Phi \neq 0$, and α is algebraic over EK with irreducible polynomial $q(x)$.

Consider the polynomial $p(x) = \prod_\sigma \sigma(q(x))$, where σ runs over all elements in $\mathrm{Gal}(EK/E)$. Since K is normal over k, EK is normal over E. If K is separable over k, then also EK is separable over E, and $p(x)$ belongs to the polynomial ring $E[x]$. Otherwise, some power of $p(x)$ belongs to $E[x]$; we denote that power again by $p(x)$. Since $g(p(x)) = p(x)$, $g(p(\alpha)) = 0$. It follows that for some $\sigma \in \mathrm{Gal}(EK/E)$, $g(\sigma(q(x)))$ is the irreducible polynomial of α over EL. That is, the kernel of Ψ is generated by $g(\sigma(q(x)))$.

We claim that σ, which is an automorphism of EK, extends to an automorphism of ES stabilizing S. Since K is normal over k, and since $k \subseteq E$ is pointwise fixed under σ, $\sigma|_K$ is an automorphism of K/k. Since the group G of automorphisms of S maps via restriction onto $\mathrm{Gal}(K/K^G) = \mathrm{Gal}(K/k)$, there is some $\tau \in G$ such that $\tau|_K = \sigma|_K$. Hence $\sigma \otimes \tau$ induces an automorphism of $ES \approx EK \otimes_K S$ extending both σ and τ. We denote this automorphism again by σ.

Let $E' = E(\alpha)$. As seen above, the irreducible polynomials of α over EK and EL are $q(x)$ and $g(\sigma(q(x)))$, respectively. Hence $g \circ \sigma \colon ES \to ET$ extends to an isomorphism $g' \colon E'S \to E'T$ by setting $g'(\alpha) = \alpha$. Clearly $g'|_{E'} = \mathrm{id}_{E'}$ and g' extends $f \circ \tau$. This is a contradiction to the maximality of E, unless $\alpha \in E$. It follows that $E = C$, completing the proof of the lemma, and thus the proof of Proposition 3. □

As an immediate corollary, we obtain Theorem 2(a):

Proof of Theorem 2(a). Since A is a PI-domain, its division ring of fractions $\mathrm{Q}(A)$ can be obtained by a central localization. One concludes easily that A^G has also a division ring of fractions $\mathrm{Q}(A^G)$, and that $\mathrm{Q}(A^G) = \mathrm{Q}(A)^G$. Now the result follows immediately from Proposition 3. □

Remark 5. Where in the proof of Proposition 3 did we use the fact that S is finite over its center K? Certainly via the Skolem-Noether theorem: There we needed it to show that CS is finite over C. And this is in fact the only place where we needed this assumption: Assume that S is not necessarily finite over its center K. If again E is an arbitrary subfield of C, and if R denotes the image of $EK \otimes_K S$ in ES, then R is now not necessarily equal to ES. But one can check that R is an Ore domain, so that $S \approx \mathrm{Q}(EK \otimes_K S)$. Moreover, the center of $\mathrm{Q}(EK \otimes_K S)$ is still EK (if c belongs to the center of $\mathrm{Q}(R)$, then $\{r \in R | cr \in R\}$ is a nonzero two-sided ideal of R and thus all of R; hence $c \in R$, so that the center of $\mathrm{Q}(R)$ is isomorphic to EK). Using these facts, the proof of Lemma 4 goes through. In order to attempt a generalization of Proposition 3 to the case of an arbitrary division algebra S, we reduced thus to the case that f extends to a map $CS \to CT$ with $f|_C = \mathrm{id}_C$. It is not clear if this is of any help. I should remark that the PI-assumption in Theorem 1 is also used in the proof in the next section: There we need it to show that a certain prime homomorphic image A/P of A is a Goldie ring. Of course, one could assure this by assuming, e.g., that A is Noetherian. □

3. The proofs of the main theorems

These proofs consist now of reduction steps to the case that A is a division algebra. First we need the following lemma which incorporates a general reduction technique which is sometimes useful when studying actions of finite groups on rings. It is most likely known, but included for lack of a reference.

Lemma 6. *Let A be a ring with a finite group of automorphisms G such that $|G|^{-1} \in A$. Let P be a prime ideal of A such that A/P is a right Goldie ring. Set $H = \mathrm{Stab}_G(P)$ and $I = \bigcap_{\sigma \in G} \sigma(P)$. Then*

$$\mathrm{Q}(A^G/P \cap A^G) \approx \mathrm{Q}(A/I)^G \approx \mathrm{Q}(A/P)^H,$$

where the isomorphisms are induced by the quotient map $A \to A/P$.

Note that here G acts on A/I but not necessarily on A/P. This lemma is useful in that it relates—on the quotient ring level—the actions of G on A and A/I with the action of H on A/P, allowing in certain situations the reduction from arbitrary rings to prime rings. We shall use it in the proof of Theorem 1.

One should note that the lemma does not hold "before" localization. Let me be more precise. Recall that if a group G acts on a semiprime right Goldie ring R such that $|G|^{-1} \in R$, then also R^G is a semiprime right Goldie ring, and $\mathrm{Q}(R^G) = \mathrm{Q}(R)^G$ [MR, 10.5.19]. We will use these facts frequently. Thus the lemma says that the quotient rings of $A^G/P \cap A^G$ and $(A/P)^H$ are isomorphic. However, it is in general not true that $A^G/P \cap A^G$ and $(A/P)^H$ are isomorphic, even if A is commutative; see [B, Chapter V, §2, no. 2, Exercise 10].

Proof. Note that $P \cap A^G = I \cap A^G$, so that $A^G/P \cap A^G = A^G/I \cap A^G \approx (A/I)^G$. Thus $\mathrm{Q}(A^G/P \cap A^G) \approx \mathrm{Q}(A/I)^G$, and this isomorphism is induced by the quotient map $A/I \to A/P$. To prove the second isomorphism, we may as well replace A by A/I and thus assume that $I = 0$. It is easy to see that A is now a semiprime right Goldie ring. (Use the embedding $A \hookrightarrow \bigoplus_{\sigma \in G} A/\sigma(P)$, and the fact that the latter ring is right Goldie.) Note that G permutes the minimal prime ideals of A transitively. Denote by B the total ring of fractions of A. Then B is a direct sum of simple Artinian rings, say $B_1, \dots, B_n$. The action of G on A extends to B, and G permutes the B_i transitively. Say $Q = B_2 \oplus \cdots \oplus B_n$ is the prime ideal of B which is the localization of P. Then $B_1 \approx B/Q \approx \mathrm{Q}(A/P)$. Note that $H = \mathrm{Stab}_G(P) = \mathrm{Stab}_G(B_1)$.

The projection from B onto B_1 induces a homomorphism from B^G into B_1^H. Since G permutes the B_i transitively, this map is one-to-one. It is actually also onto, as one can see as follows: Let σ_i be right coset representatives of H in G such that $\sigma_i(B_1) = B_i$. Let $b \in B_1^H$, and set $x = \sum_i \sigma_i(b)$. Then given $\tau \in G$, there is a permutation π and elements $h_i \in H$ such that $\tau\sigma_i = \sigma_{\pi(i)} h_i$. Thus $\tau(x) = \sum_i \sigma_{\pi(i)}(h_i(b)) = x$. Hence $x \in B^G$, and the image of x in B_1^H is b. Thus $B^G \approx B_1^H$, i.e., $\mathrm{Q}(A)^G \approx \mathrm{Q}(A/P)^H$, and this isomorphism is induced by the quotient map $A \to A/P$. $\square$

Proof of Theorem 1. Denote by P_i the kernel of f_i. Then $P_1 \cap A^G = P_2 \cap A^G$. It follows by a theorem of Montgomery [M_1] (see also [M_2]) that there is some $\sigma \in G$ such that $\sigma(P_2) = P_1$. Replacing f_1 by $f_1 \circ \sigma$, we may therefore assume that f_1 and f_2 have the same kernel P. Moreover, A/P is now by assumption a PI-ring and thus a Goldie ring. So the hypotheses of Lemma 6 are satisfied. It follows that the maps f_i induce maps $\overline{f}_i \colon \mathrm{Q}(A/P) \to D$ which agree on $\mathrm{Q}(A/P)^H \approx \mathrm{Q}(A^G/P \cap A^G)$. Hence by Proposition 3, there is some $\sigma \in H$ and some inner automorphism ψ of D such that $\overline{f}_2 = \psi \circ \overline{f}_1 \circ \sigma_{\mathrm{Q}(A/P)}$. (If σ acts on a ring R, we denote here the corresponding automorphism by σ_R.) We conclude that $f_2 = \psi \circ f_1 \circ \sigma_A$, as was to be shown. $\square$

Proof of Theorem 2(b). Using Proposition 3 instead of some elementary facts about commutative fields, this result follows exactly like [B, Chapter V, §2, no. 2, Corollary to Theorem 2]. The proof is only included for completeness.

Denote by P_i the kernel of f_i. Then $P_1 \cap A^G = P_2 \cap A^G$. Thus there is some $\sigma \in G$ such that $\sigma(P_2) = P_1$. Replacing f_1 by $f_1 \circ \sigma$, we may therefore assume that f_1 and f_2 have the same kernel P. Denote by K and k the fields of fractions of A/P and $A^G/P \cap A^G$, respectively. Then f_1 and f_2 induce embeddings $\overline{f}_1$ and $\overline{f}_2$ of K into D such that $\overline{f}_1|_k = \overline{f}_2|_k$. By [B, loc. cit., Theorem 2], K/k is normal with finite Galois group H. By Proposition 3 applied to $S = K$, there is some $\overline{\sigma} \in H$ and some inner automorphism ψ of D such that $\overline{f}_2 = \psi \circ \overline{f}_1 \circ \overline{\sigma}$. Every element $\alpha \in \mathrm{Stab}_G(P)$ defines an automorphism $\overline{\alpha}$ of A/P which extends to K and leaves k pointwise fixed. Again by the already cited theorem in [B], the map $\mathrm{Stab}_G(P) \to H$ sending α to $\overline{\alpha}$ is onto. Hence $\overline{\sigma}$ comes from some $\sigma \in \mathrm{Stab}_G(P)$. It follows that $f_2 = \psi \circ f_1 \circ \sigma$. □

4. Examples

First we show that in general it is impossible in the noncommutative case to drop the assumption that $|G|^{-1} \in A$, even if A is an affine prime Noetherian PI-algebra finite over its center, and D is a (commutative) field. It is fairly simple to find such examples where f_1 and f_2 have kernels which are not conjugate under the group action. Essentially, examples of this type are based on the fact that Montgomery's theorem does not hold if $|G|$ is not invertible in A. In the following somewhat more complicated example, f_1 and f_2 have the same kernel and agree on A^G, but are anyhow not equivalent.

Example 7. Let $R = k[x, t]$ be a polynomial ring in two variables over a field of prime characteristic $p \neq 2$. Let

$$A = \begin{pmatrix} R & R \\ tR & k[x^2] + tR \end{pmatrix}.$$

Then A is an affine prime Noetherian PI-algebra finite over its center $(k[x^2] + tR) \cdot I_2$. Let G be the group of automorphisms of A generated by conjugation by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Then $|G| = p$, and $A^G = (k[x^2] + tR) \cdot I_2 + R \cdot e_{12}$. Let P be the prime ideal

$$P = \begin{pmatrix} tR & R \\ tR & k[x^2] + tR \end{pmatrix}.$$

Then $A/P = k[x]$ and $A^G/P \cap A^G = k[x^2]$. (Note that $A^G/P \cap A^G \neq (A/P)^G = k[x]$; if $|G|$ were invertible in A, equality would hold.) Let $D = k(x)$, and let f_1 be the composition $A \to A/P \hookrightarrow D$. Let f_2 be the composition of f_1 with the automorphism of D which sends x to $-x$. Then f_1 and f_2 agree on A^G. But f_1 and f_2 are not equivalent: The field D admits only the trivial inner automorphism $\psi = \mathrm{id}$. And if $\sigma \in G$, then $f_1 = f_1 \circ \sigma$ since σ is inner and $f_1(A) = A/P$ is commutative. Thus $\psi \circ f_1 \circ \sigma = f_1 \neq f_2$. □

Our last example shows that in generalizing the commutative result, one cannot replace the field D by a finite-dimensional central simple algebra, even if A is commutative.

Example 8. Let $A = \mathbf{C}$ be the complex numbers, and let G be the group of automorphisms of A given by complex conjugation. Let D be the (2×2)-matrices over $\mathbf{C}$. Let f_1 be the natural embedding of A as the center of D, and let f_2 be the map sending $a \in \mathbf{C}$ to the matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & \overline{a} \end{smallmatrix}\right)$. Then $f_2(A)$ is not central in D, so that for all automorphisms ψ of D, $f_2(A) \neq \psi(f_1(A))$. □

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