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GROUP C^*-ALGEBRAS OF REAL RANK ZERO OR ONE

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ABSTRACT. Let G be a locally compact group and $C^*(G)$ its group C^*-algebra, and denote by $\mathrm{RR}(C^*(G))$ the real rank of $C^*(G)$. This note is a first step towards relating $\mathrm{RR}(C^*(G))$ to the structure of G. We identify the connected groups G with $\mathrm{RR}(C^*(G)) = 0$ as precisely the compact connected ones and characterize the nilpotent groups whose C^*-algebras have real rank zero or one.

1. INTRODUCTION AND RESULTS

The concept of real rank of a C^*-algebra A has been introduced by Brown and Pedersen [3], based on the notion of topological stable rank of Rieffel [14]. For unital A the *real rank*, $\mathrm{RR}(A)$, is defined to be the smallest $n \in \mathbb{N}_0$ with the property that for every $(n+1)$-tuple $x_0, \dots, x_n$ of selfadjoint elements in A and every $\varepsilon > 0$ there are selfadjoint elements $y_0, \dots, y_n$ in A which generate A as a left ideal and satisfy $\|x_k - y_k\| < \varepsilon$ for $k = 0, \dots, n$, provided that such an n exists. Otherwise, $\mathrm{RR}(A) = \infty$. If A is nonunital, then $\mathrm{RR}(A) = \mathrm{RR}(\widetilde{A})$, where $\widetilde{A}$ denotes the C^*-algebra obtained from A by adjoining a unit. This is Rieffel's definition for the topological stable rank, $\mathrm{tsr}(A)$, if one does not demand all the elements to be selfadjoint. By [3, Proposition 1.2] $\mathrm{RR}(A) \le 2\,\mathrm{tsr}(A) - 1$. If A is commutative and unital, then $\mathrm{RR}(A) = \dim \widehat{A}$, the covering dimension of the dual space $\widehat{A}$ of A [3, Proposition 1.1]. Thus the real rank may be viewed as a noncommutative analogue of dimension.

Recently there were some profound contributions to determining the real rank (compare [1–4]). Many C^*-algebras, notably the irrational rotation algebras, turned out to have real rank zero. In any case, it is difficult to compute the real rank. One reason for it is that, unlike the topological stable rank [14, §4], there is no estimate so far for $\mathrm{RR}(A)$ in terms of $\mathrm{RR}(I)$ and $\mathrm{RR}(A/I)$ for closed ideals I in A.

Now group C^*-algebras provide some of the most interesting and important examples in the theory of operator algebras. It is therefore fascinating to relate the real rank of the C^*-algebra $C^*(G)$ of a locally compact group G to the structure of G. At the moment, however, this problem seems to be accessible only for either small real rank or very special classes of groups. Our first result is as follows.

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Theorem 1. *For a connected locally compact group G the following conditions are equivalent.*

(i) *$C^*(G)$ has real rank zero.*

(ii) *The reduced group C^*-algebra, $C_r^*(G)$, has real rank zero.*

(iii) *G is compact.*

The proof of Theorem 1 will be given in §1. It might well be true that for much more general G, $\mathrm{RR}(C^*(G)) = 0$ amounts to G being an inductive limit of compact groups. Such a conjecture is also supported by Theorem 2. It is worthwhile to mention here that $\mathrm{RR}(C_r^*(G)) = 1$ for any noncompact complex connected semisimple Lie group G which is acceptable in the sense of Harish-Chandra [17, §8.1]. Indeed, on the one hand, by [13, Proposition 4.1] $C_r^*(G)$ is isomorphic to $C_0(\widehat{G}_r, \mathscr{K}) = C_0(\widehat{G}_r) \otimes \mathscr{K}$, where $\widehat{G}_r$ denotes the reduced dual of G and $\mathscr{K}$ the algebra of compact operators on an infinite-dimensional separable Hilbert space. On the other hand, by [1, Proposition 3.3] $\mathrm{RR}(C_0(\widehat{G}_0) \otimes \mathscr{K}) = 1$ since $\mathrm{RR}(C_0(\widehat{G}_r)) \geq 1$.

Recall that $x \in G$ is said to be *compact* if the closed subgroup generated by x is compact. Let G^c denote the set of all compact elements in G. In general G^c will neither be a group nor be closed. However, it is a closed normal subgroup if G is nilpotent.

Theorem 2. *Let G be a nilpotent locally compact group. Then:*

(i) $\mathrm{RR}(C^*(G)) = 0$ *if and only if* $G = G^c$.

(ii) $\mathrm{RR}(C^*(G)) = 1$ *if and only if* G/G^c *is isomorphic to either* $\mathbb{R}$ *or* $\mathbb{Z}$.

The conditions on G^c in (i) and (ii) of the preceding theorem mean that the nilpotent group G/G^c has rank zero and one, respectively (for the notion of rank of G compare §3). We point out, however, that a generalization of Theorem 2—to the extent that for nilpotent groups G, $\mathrm{RR}(C^*(G))$ coincides with the rank of G/G^c—is false. This can be seen by appealing to a remarkable result due to Sheu [16, Theorem 3.18] which states that for every simply connected nilpotent Lie group G, which is a semidirect product of the form $G = \mathbb{R} \ltimes \mathbb{R}^n$, $n \in \mathbb{N}$, $\mathrm{tsr}(C^*(G)) = \mathrm{rank}(G/[G, G])$. Now choose for G the 4-dimensional simply connected nilpotent Lie group, usually denoted G_4, which does not decompose as a direct product of lower-dimensional groups. Then we observe that $G/[G, G]$ has rank 2 and obtain

$$\mathrm{RR}(C^*(G_4)) \leq 2\,\mathrm{tsr}(C^*(G_4)) - 1 = 3.$$

We also take this opportunity to mention that it seems to be unknown whether the real rank of the Heisenberg group C^*-algebra is actually 2 or 3.

The proof of Theorem 2 is postponed to §3.

2. Proof of Theorem 1

We start by recalling some basic features of real rank that will be used to establish Theorem 1. If A is a C^*-algebra and I a closed ideal in A, then $\mathrm{RR}(A) \geq \mathrm{RR}(A/I)$. This can be verified by very much the same reasoning as in [14, Theorem 4.3], where the same inequality has been proved for the topological stable rank. Moreover, by [3, Theorem 2.6] $\mathrm{RR}(A) = 0$ if and only if every hereditary C^*-subalgebra of A has an approximate unit consisting of projections.

As to unitary group representation theory we refer to [5, 6]. Now, associated to a nonzero projection p in A is the nonempty compact open subset

$$S(p) = \{\pi \in \widehat{A};\, \pi(p) \neq 0\}$$

of the dual space $\widehat{A}$ of A [5, (3.3.2) and (3.3.7)]. It follows that if there exists a closed ideal I in A such that $(A/I)^\frown$ is a noncompact connected Hausdorff space, then A/I cannot contain a nonzero projection, and hence

$$\mathrm{RR}(A) \geq \mathrm{RR}(A/I) \geq 1.$$

The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) of Theorem 1 are obvious. Indeed, $C_r^*(G)$ is a quotient of $C^*(G)$, and for G compact $C^*(G)$ is a c_0-direct sum of matrix algebras and hence has real rank zero (see [3]). The remaining implication, (ii) $\Rightarrow$ (iii), will follow by fairly straightforward group structural arguments once we deal with the semisimple and the motion group cases, and this will be done in the two lemmas below.

In the sequel we will frequently employ the following argument. If N is an amenable closed normal subgroup of G, then $(G/N)_r^\frown \subseteq \widehat{G}_r$ and hence $C_r^*(G/N)$ is a quotient of $C_r^*(G)$, whence $\mathrm{RR}(C_r^*(G/N)) \leq \mathrm{RR}(C_r^*(G))$.

Lemma 1. *Let G be a noncompact connected semisimple Lie group. Then* $\mathrm{RR}(C_r^*(G)) \geq 1$.

Proof. Suppose first that G has finite center and is acceptable in the sense of Harish-Chandra [17, §8.1.1]. Fix an Iwasawa decomposition $G = KAN$ of G, and let M be the centralizer of A in K and $P = MAN$ the corresponding minimal parabolic subgroup. As G is noncompact, the vector group A has dimension at least one. Moreover, let W be the associated Weyl group, that is, the quotient of the normalizer of A in K by M. W is finite and acts on $\widehat{A}$ in the obvious way. For $\alpha \in \widehat{A}$, let π_α denote the representation of G induced from the character

$$man \to \alpha(a), \qquad m \in M,\ a \in A,\ n \in N,$$

of P. By a theorem of Kostant [17, Theorem 5.5.2.3] π_α is irreducible, and $\pi_\alpha = \pi_\beta$ if and only if α and β are conjugate under W. The set E of all these so-called class one principal series representations π_α, $\alpha \in \widehat{A}$, is a closed subset of $\widehat{G}_r$, and by [10, Theorem 6.2] the mapping

$$\widehat{A}/W \to E, \qquad W(\alpha) \to \pi_\alpha$$

is a homeomorphism. In particular, E is a noncompact connected Hausdorff space. Now let $A = C_r^*(G)/k(E)$, where $k(E)$ denotes the kernel of E in A. Then $\widehat{A} = E$, and hence A cannot contain any nonzero projection. This shows $\mathrm{RR}(C_r^*(G)) \geq \mathrm{RR}(A) \geq 1$.

Next, let H be a connected semisimple Lie group with finite center. Then H possesses a finite covering group G which is acceptable [17, §8.1.1]. Let C denote the kernel of this covering homomorphism. Then C is a finite central subgroup of G and hence $C \subseteq M$; therefore, $E \subseteq (G/C)^\frown$. Thus E is a connected closed noncompact, Hausdorff subset of $\widehat{H}_r$, and this implies

$$\mathrm{RR}(C_r^*(H)) \geq \mathrm{RR}(C_r^*(H)/k(E)) \geq 1.$$

Finally, consider an arbitrary noncompact connected semisimple Lie group F. Then $F/Z(F)$ has a trivial center and is noncompact since otherwise, by the Freudenthal-Weil theorem, F is a direct product of a vector group and a compact group. By the preceding paragraph, and since $(F/Z(F))_r^{\wedge} \subseteq \widehat{F}_r$,

$$\mathrm{RR}(C_r^*(F)) \geq \mathrm{RR}(C_r^*(F/Z(F))) \geq 1. \quad \square$$

Lemma 2. *Suppose G is a locally compact group containing an abelian normal subgroup $N \neq \{e\}$ with $N^c = \{e\}$ and such that G/N is compact. Then* $\mathrm{RR}(C^*(G)) \geq 1$.

Proof. For any representation π of G, let $\pi \mid N$ denote the restriction of π to N and $\mathrm{supp}(\pi \mid N)$ the support of $\pi \mid N$, that is, the set of all characters in $\widehat{N}$ that are weakly contained in $\pi \mid N$ [5, §18]. Also, let $\widehat{N}/G$ denote the quotient space of $\widehat{N}$ with respect to the action of G on $\widehat{N}$. $\widehat{N}/G$ is a locally compact Hausdorff space and, by the duality theory of abelian locally compact groups, is connected since $N^c = \{e\}$. Now

$$r\colon \pi \to \mathrm{supp}(\pi \mid N)$$

defines a continuous and open mapping from $\widehat{G}$ onto $\widehat{N}/G$ (see [8, Lemma 1.2]). Hence, if C is a compact open subset of $\widehat{G}$, then $r(C)$ is compact and open in $\widehat{N}/G$. Yet, $\widehat{N}/G$ being a noncompact connected Hausdorff space, this forces $C = \varnothing$. Therefore, $C^*(G)$ cannot contain any nonzero projection. This shows $\mathrm{RR}(C^*(G)) > 0$. $\square$

Now let G be any connected group, and suppose that $\mathrm{RR}(C^*(G)) = 0$. Let K be the maximal compact normal subgroup of G. Since $\mathrm{RR}(C^*(G/K)) = 0$, we can and will assume that G has no nontrivial compact normal subgroup. In particular, G is a Lie group. Denote by R the radical, that is, the maximal connected solvable normal subgroup, of G. G/R is semisimple and $\mathrm{RR}(C_r^*(G/R)) = 0$, and hence G/R has to be compact by Lemma 1. Thus, if G is noncompact, then there exists a normal subgroup H in G such that $H \subseteq R$ and R/H is a nontrivial vector group. Then Lemma 2 applies to G/H and yields

$$\mathrm{RR}(C_r^*(G)) \geq \mathrm{RR}(C_r^*(G/H)) \geq 1.$$

This contradiction proves that G is compact.

3. Proof of Theorem 2

If G is an arbitrary locally compact group, we call G *compact free* if $G^c = \{e\}$; that is, G contains no nontrivial compact element. In case G is discrete G^c is usually denoted G^t (the set of torsion elements in G), and G is called *torsion free* if $G^t = \{e\}$.

Suppose that G is abelian. Then G/G^c is compact free and hence $G/G^c = \mathbb{R}^m \times D$, $m \in \mathbb{N}_0$, where D is discrete and torsion free. The *rank* of G is defined to be $m + n$ if $D = \mathbb{Z}^n$, and $\mathrm{rank}(G) = \infty$ if D fails to be finitely generated. Taking into account that $\widehat{G^c}$ is zero dimensional and that $\dim \widehat{D} = \mathrm{rank}(D)$ [9, Theorem 24.28], we conclude from [11, Remark 3 and §4]

$$\begin{aligned} \mathrm{RR}(C^*(G)) &= \mathrm{RR}(C_0(\widehat{G})) = \dim \widehat{G} = \dim(G/G^c)^{\wedge} \\ &= \dim(\mathbb{R}^m + \widehat{D}) = m + \dim \widehat{D} = m + \mathrm{rank}(D) = \mathrm{rank}(G/G^c). \end{aligned}$$

As a reference to dimension theory of topological spaces we mention [12]. Turning to nilpotent groups G, we first collect some facts involving G^c. We denote by G_0 the connected component of e and by $\{e\} = Z_0(G) \subseteq Z_1(G) \subseteq \cdots$ the ascending central series of G. Then:

(1) G^c is a closed normal subgroup of G [7, Corollary 3.5.1 and Lemma 3.8].

(2) If G is compactly generated, then G^c is compact [7, Theorem 9.7].

(3) G_0G^c is open in G, and G/G_0G^c is torsion free [7, Theorem 8.3].

(4) If G is discrete and torsion free, then the factor groups

$$Z_j(G)/Z_{j-1}(G), \qquad j = 1, 2, \ldots,$$

are torsion free as well [15, Theorems 2.25 and 4.37].

Remark 1. It follows from (2) that G/G^c is compact free. For that notice first that if H is a compactly generated nilpotent group, then both H^c and $(H/H^c)^c$ are compact, and therefore H/H^c is compact free. Hence, for any compactly generated open subgroup H of G, $HG^c/G^c = H/H \cap G^c = H/H^c$ is compact free. This implies that G/G^c, being the union of such HG^c/G^c, is compact free.

Remark 2. Remark 1 and the above properties lead to the notion of *rank* for a compact-free nilpotent group H. Indeed, the factor groups

$$Z_k(H/H_0)/Z_{k-1}(H/H_0), \qquad k = 1, 2, \ldots,$$

are torsion free, and H_0 is a simply connected nilpotent Lie group, so that all $Z_j(H_0)/Z_{j-1}(H_0)$, $j = 1, 2, \ldots$, are vector groups. Therefore the rank of H can be defined by

$$\begin{aligned}\operatorname{rank}(H) = &\sum_{j\geq 1} \dim(Z_j(H_0)/Z_{j-1}(H_0)) \\ &+ \sum_{k\geq 1} \operatorname{rank}(Z_k(H/H_0)/Z_{k-1}(H/H_0)).\end{aligned}$$

As mentioned in §1, unlike the abelian case, it is not true in general that $\mathrm{RR}(C^*(H))$ equals $\operatorname{rank}(H)$.

Lemma 3. *For a nilpotent locally compact group* G, $\mathrm{RR}(C^*(G)) \leq 1$ *implies* $\operatorname{rank}(G/G^c) \leq 1$.

Proof. Assume $G \neq G^c$, and let $H = G/G^c$ and $D = G/G_0G^c$. Suppose first that D is trivial. Then H is a simply connected nilpotent Lie group, so that either $H = \mathbb{R}$ or H contains a normal subgroup N such that $H/N = \mathbb{R}^2$. However, $H/N = \mathbb{R}^2$ contradicts

$$\mathrm{RR}(C^*(H/N)) \leq \mathrm{RR}(C^*(G)) \leq 1.$$

It remains to deal with the case $D \neq \{e\}$. Let $r \in \mathbb{N}_0$ be maximal such that $Z_{r-1}(D) \neq D$. Then

$$1 \geq \mathrm{RR}(C^*(D/Z_r)) = \operatorname{rank}(D/Z_r(D))$$

implies $D/Z_r(D) = \mathbb{Z}$. If, moreover, $Z_r(D) \neq \{e\}$, then

$$D/Z_{r-1}(D) = \mathbb{Z} \ltimes Z_r(D)/Z_{r-1}(D)$$

is a torsion free abelian group of rank at least two. Again this is impossible. Hence we have seen that $D = \mathbb{Z}$. It remains to show that H_0 is trivial. $Z_1(H_0)$ is a normal vector subgroup of H. Now continuous automorphisms of vector groups are linear. It follows that if $x \in Z_1(H) \cap H_0 \subseteq Z_1(H_0)$, then $Z_1(H) \cap H_0$ contains the real line through x. This implies that $Z_1(H) \cap H_0$ is a vector group. Continuing this way we obtain that all factor groups

$$Z_j(H) \cap H_0 / Z_{j-1}(H) \cap H_0, \qquad j = 1, 2, \dots,$$

are vector groups. Consequently, if H_0 is nontrivial, we find a normal subgroup N of H such that $N \subseteq H_0$ and $V = H_0/N$ is a nontrivial vector group and contained in the center of H/N. This is impossible because

$$\mathrm{RR}(C^*(H/N)) = \mathrm{RR}(C_0(\widehat{\mathbb{Z}} \times \widehat{V})) = 1 + \dim V. \quad \square$$

Since $G/G^c = \mathbb{R}$ or $G/G^c = \mathbb{Z}$ clearly yields $\mathrm{RR}(C^*(G)) \geq 1$, owing to Lemma 3 the proof of Theorem 2 will be complete once we have verified that conversely $G/G^c = \mathbb{R}$ or $G/G^c = \mathbb{Z}$ implies $\mathrm{RR}(C^*(G)) \leq 1$, as well as $G = G^c$ gives $\mathrm{RR}(C^*(G)) = 0$.

For this it will be convenient to have some properties of real rank for general C^*-algebras available which are well known for the topological stable rank [14] and can be shown similarly. First, for any C^*-algebra A, $\mathrm{RR}(A) = \mathrm{RR}(\widetilde{A})$ regardless of whether A has a unit or not. Indeed, as explained in [11, §4], this follows by the same discussion as given for the topological stable rank in [14, Proposition 4.2]. Next, if A is a direct sum of C^*-subalgebras B and C, then $\mathrm{RR}(A) \leq \max(\mathrm{RR}(B), \mathrm{RR}(C))$ (in fact, equality holds). This is obvious for unital algebras and can be verified for nonunital ones using the preceding remark. These facts essentially yield the following lemma (compare [14, Theorems 5.1 and 5.2; 3, Proposition 3.1]).

Lemma 4. (i) *If A is a C^*-algebra and $(A_\lambda)_{\lambda\in\Lambda}$ is a nested system of C^*-subalgebras of A such that $A = \overline{\bigcup_{\lambda\in\Lambda} A_\lambda}$, then*

$$\mathrm{RR}(A) \leq \sup_{\lambda\in\Lambda} \mathrm{RR}(A_\lambda).$$

(ii) *If A is a c_0-direct sum of C^*-subalgebras A_σ, $\sigma \in \Sigma$, then*

$$\mathrm{RR}(A) \leq \sup_{\sigma\in\Sigma} \mathrm{RR}(A_\sigma).$$

Proof. (i) We may assume that A has a unit 1 and that $1 \in A_\lambda$ for all λ. Let $n = \sup_{\lambda\in\Lambda} \mathrm{RR}(A_\lambda) < \infty$, and let selfadjoint elements $a_0, \dots, a_n$ in A and $\varepsilon > 0$ be given. Then there exist $\lambda \in \Lambda$ and selfadjoint elements $b_0, \dots, b_n$ in A_λ such that $\|b_j - a_j\| < \varepsilon$ for $0 \leq j \leq n$. Since $\mathrm{RR}(A_\lambda) \leq n$, there are selfadjoint elements $u_0, \dots, u_n$ in A_λ satisfying $A_\lambda = \sum_{j=0}^n A_\lambda u_j$ and $\|u_j - b_j\| < \varepsilon$, $0 \leq j \leq n$. This readily shows $\mathrm{RR}(A) \leq n$.

(ii) follows from (i) since for $\sigma_1, \dots, \sigma_m \in \Sigma$,

$$\mathrm{RR}\left(\bigoplus_{j=1}^m A_{\sigma_j}\right) \leq \max_{1\leq j\leq m} \mathrm{RR}(A_{\sigma_j}),$$

and these finite direct sums form a nested system as required in (i). $\square$

Suppose now that $G = G^c$. Then, according to property (2) above, every compact subset of G generates a compact subgroup of G. As a result the union of all subalgebras $C^*(K)$, where K is a compact open subgroup of G, is dense in $C^*(G)$. Since $\mathrm{RR}(C^*(K)) = 0$, Lemma 4(i) gives $\mathrm{RR}(C^*(G)) = 0$. Therefore, to finish the proof of Theorem 2, it suffices to show:

Lemma 5. *Let G be a nilpotent group such that either $G/G^c = \mathbb{R}$ or $G/G^c = \mathbb{Z}$. Then* $\mathrm{RR}(C^*(G)) \leq 1$.

Proof. We first reduce to the case that G is compactly generated. To this end, let $\mathscr{H}$ denote the set of all open compactly generated subgroups of G. Obviously, the C^*-subalgebras $C^*(H)$, $H \in \mathscr{H}$, satisfy the assumptions of Lemma 4(i). Hence it is enough to show that $\mathrm{RR}(C^*(H)) \leq 1$ for each $H \in \mathscr{H}$. Now

$$H/H^c = H/H \cap G^c = HG^c/G^c$$

is an open subgroup of G/G^c. Thus, if H is noncompact, then $H/H^c = \mathbb{R}$ or H/H^c is infinite cyclic. Since H^c is compact for any such H, to prove Lemma 5 we can henceforth assume that $K = G^c$ is compact.

Assume that G contains a compact normal subgroup K such that $G/K = \mathbb{R}$. It is known that in this case G is isomorphic to $\mathbb{R} \times K$. Here is a sketch of the proof. It follows from Mackey's theory (see [6, Chapter XI]) and from the fact that every multiplier on $\mathbb{R}$ is trivial that all the irreducible representations of G are finite dimensional. Since G is almost connected, the Freudenthal-Weil theorem [5, (16.5.3)] shows that G is a semidirect product $G = C \ltimes V$ where C is compact and V is a vector group. Obviously, $C = K$, $V = \mathbb{R}$, and hence $G = K \times \mathbb{R}$. Recall that $C^*(K)$ is a c_0-direct sum of matrix algebras M_i, $i \in I$. Consequently, we obtain from Lemma 4(ii) and [1, Corollary 3.2] that

$$\mathrm{RR}(C^*(G)) = \mathrm{RR}(C_0(\mathbb{R}) \otimes C^*(K)) \leq \sup_{i \in I} \mathrm{RR}(C_0(\mathbb{R}) \otimes M_i) \leq 1.$$

We now turn to the case $G/K = \mathbb{Z}$. For $\sigma \in \widehat{K}$, let S_σ denote the stability group of σ in G and $\mathrm{ind}_K^G \sigma$ the representation of G induced by σ, and set $A_\sigma = \mathrm{ind}_K^G \sigma(C^*(G))$. Then $C^*(G)$ is isomorphic to a c_0-direct sum of A_σ, $\sigma \in \Sigma$, where $\Sigma \subseteq \widehat{K}$ contains exactly one element from each G-orbit in $\widehat{K}$. By Lemma 4(ii) it therefore suffices to show $\mathrm{RR}(A_\sigma) \leq 1$ for every σ. If $S_\sigma = K$, then $\mathrm{ind}_K^G \sigma$ is a CCR-representation and hence $\mathrm{RR}(A_\sigma) = 0$.

We are left with the case that S_σ has finite index in G. Choose $\pi \in \widehat{G}$ such that $\pi | K \geq \sigma$, and let $\widehat{G}_\sigma = \{\rho \in \widehat{G};\ \rho | K \geq \sigma\}$. Then π, being induced from some finite dimensional representation of S_σ, is finite dimensional, and $\lambda \to \pi \otimes \lambda$ is a continuous and open mapping from $(G/K)^\wedge = \widehat{\mathbb{Z}}$ onto $\widehat{G}_\sigma = \widehat{A}_\sigma$. Moreover,

$$\lambda \to \mathrm{trace}(\pi \otimes \lambda)(f) = \mathrm{trace}\, \pi(\lambda f)$$

is continuous on $\widehat{\mathbb{Z}}$ for every $f \in C^*(G)$. Thus $\widehat{A}_\sigma$ is homeomorphic to the quotient space $\widehat{\mathbb{Z}}/\sim$, where the equivalence relation $\sim$ is defined by $\lambda_1 \sim \lambda_2$ if and only if $\pi \otimes \lambda_1 = \pi \otimes \lambda_2$. It follows that A_σ is a continuous trace C^*-algebra with dual space homeomorphic to $\widehat{\mathbb{Z}}/\sim\, = \mathbb{T}$. Finally, since $H^3(\mathbb{T}, \mathbb{Z}) = 0$, the Dixmier-Douady class of A_σ is trivial, and hence A_σ is isomorphic to $C(\mathbb{T}, \pi(A_\sigma))$. Again [1, Corollary 3.2] gives $\mathrm{RR}(A_\sigma) = 1$. □

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