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UNBOUNDED COMMUTING OPERATORS AND MULTIVARIATE ORTHOGONAL POLYNOMIALS

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ABSTRACT. The multivariate orthogonal polynomials are related to a family of operators whose matrix representations are block Jacobi matrices. A sufficient condition is given so that these operators, in general unbounded, are commuting and selfadjoint. The spectral theorem for these operators is used to establish the existence of the measure of orthogonality in Favard's theorem.

1. INTRODUCTION

Let $\mathbb{N}_0$ be the set of nonnegative integers. For $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ and $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ we use the standard notation $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. The number $|\alpha| = \alpha_1 + \cdots + \alpha_d \in \mathbb{N}_0$ is called the total degree of $\mathbf{x}^\alpha$. For $n \in \mathbb{N}_0$ we denote by Π_n^d the set of polynomials of total degree at most n in d variables, and Π^d the set of all polynomials in d variables.

Let $\mathscr{L}$ be a linear functional defined on Π^d such that $\mathscr{L}(g^2) > 0$ whenever $g \neq 0$. Such an $\mathscr{L}$ is called square positive, it induces an inner product on Π^d. Thus, the Gram-Schmidt orthogonalization process can be applied to $\{x^\alpha\}$ to obtain a system of multivariate orthonormal polynomials. These polynomials share many properties of the univariant orthogonal polynomials. In particular, they satisfy three-term relations that now take vector-matrix form. Let $r_k^d = \dim \Pi_k^d - \dim \Pi_{k-1}^d = \binom{k+d-1}{k}$. For a sequence of polynomials $\{P_j^k\}_{j=1}^{r_k^d}$, where the superscript k means that P_j^k is of total degree k, we use vector notation

$$\mathbb{P}_k(\mathbf{x}) = [P_1^k(\mathbf{x}), P_2^k(\mathbf{x}), \dots, P_{r_k}^k(\mathbf{x})]^T, \tag{1}$$

where $r_k = r_k^d$. That $\{P_j^k\}_{j=1}^{r_k}{}_{k=0}^{\infty}$ is orthonormal with respect to $\mathscr{L}$ is equivalent to $\mathscr{L}(\mathbb{P}_n\mathbb{P}_m^T) = \delta_{m,n} I$, where $I\colon r_n^d \times r_n^d$ is the identity matrix. The notation $A\colon i \times j$ means that A is a matrix of size $i \times j$. We shall call $\{\mathbb{P}_n\}$ orthonormal polynomials for convenience. In [14, 15], we prove the following theorem.

Theorem 1. *Let* $\{\mathbb{P}_k\}_{k=0}^\infty$, $\mathbb{P}_0 \neq 0$, *be a sequence in* Π^d. *Then the following statements are equivalent*:

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(1) *There exists a linear functional which is square positive on* Π^d *and makes* $\{\mathbb{P}_k\}_{k=0}^{\infty}$ *an orthonormal basis in* Π^d.

(2) *For* $k \geq 0$, $1 \leq i \leq d$, *there exist matrices* $A_{k,i}\colon r_k^d \times r_{k+1}^d$ *and* $B_{k,i}\colon r_k^d \times r_k^d$, *such that*

(a) $x_i\mathbb{P}_k = A_{k,i}\mathbb{P}_{k+1} + B_{k,i}\mathbb{P}_k + A_{k-1,i}^T\mathbb{P}_{k-1}$, $1 \leq i \leq d$, *and*

(b) $\operatorname{rank} A_k = r_{k+1}^d$,

where $A_{-1,i}$ *is taken to be zero.*

This theorem extends Favard's theorem for the univariant case (cf. [3, p. 21]). The vector-matrix equation in (2) is the analogue of the three-term relation. We shall write $A_{n,1} = a_n$ for $d = 1$. For practical purposes, for example in the Gaussian cubature formula, it is very important to know whether $\mathscr{L}$ in Theorem 1 has an integral representation with respect to a nonnegative Borel measure—measure of orthogonality. For the univariant case it is well known that such a measure always exists. Moreover, if Carleman's condition $\sum(1/a_n) = \infty$ holds, the measure is unique. However, since it is known that $\mathscr{L}(g^2) > 0$ is not equivalent to $\mathscr{L}(g) > 0$ for $g > 0$, unless $d = 1$, it follows from the theory of the moment problem (cf. [2, 5]) that there may not exist such a measure of orthogonality for $d > 1$. In [15] we proved that if the coefficient matrices in the three-term relation are uniformly bounded in spectral norm, then there is a unique measure. The purpose of this paper is to study the unbounded case. Our result shows that an analogy of Carleman's condition is sufficient for the existence and uniqueness of the measure of orthogonality.

We use an approach that is based on the operator theory. The connection between univariant orthogonal polynomials and the Jacobi matrix as an operator on l^2 is well known (cf. [1, 13]). For multivariate orthonormal polynomials, we define associated operators as follows. Let $\mathscr{L}$ be a square positive linear functional, and let $\{\mathbb{P}_n\}_{n=0}^{\infty}$ be a sequence of orthonormal polynomials satisfying the three-term relation in Theorem 1. Then linear operators T_i, $1 \leq i \leq d$, are defined to be matrix operators.

$$T_i = \begin{bmatrix} B_{0,i} & A_{0,i} & & \mathbf{0} \\ A_{0,i}^T & B_{1,i} & A_{1,i} & \\ & A_{1,i}^T & B_{2,i} & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix}, \qquad 1 \leq i \leq d, \tag{2}$$

which act via matrix multiplication on l^2. The domain of T_i consists of all sequences in l^2 for which matrix multiplication yields a sequence in l^2. The connection between these operators and $\{\mathbb{P}_n\}$ can be seen as follows. Let $\{\psi_n\}_{n=0}^{\infty}$ be the canonical orthonormal basis for l^2. We rewrite this basis as $\{\psi_n\}_{n=0}^{\infty} = \{\phi_j^k\}_{j=1}^{r_k}{}_{k=0}^{\infty}$ according to the lexicographical order, and introduce the formal vector notation

$$\Phi_k = [\phi_1^k, \ldots, \phi_{r_k}^k]^T, \qquad k \in \mathbb{N}_0.$$

The orthogonality of $\{\psi_n\}_{n=0}^{\infty}$ can be described as

$$\langle \Phi_k \Phi_m^T \rangle = (\langle \phi_i^k, \phi_j^m \rangle)_{i=1}^{r_k}{}_{j=1}^{r_m} = \delta_{ij}\delta_{km}I.$$

We shall say that $\{\Phi_n\}_{n=0}^{\infty}$ is orthonormal. Let $\mathscr{P}$ be the space of Π^d equipped with the inner product induced by $\mathscr{L}$ and the orthonormal basis $\{\mathbb{P}_n\}$. We define multiplication operators $\Lambda_1, \dots, \Lambda_d$ on $\mathscr{P}$ by $\Lambda_i P = x_i P$. Let U be a map from $\mathscr{P}$ into l^2 defined by $U: \mathbb{P}_n \mapsto \Phi_n$. Then U is a unitary map. The three-term relation implies that the multiplication operators $\Lambda_1, \dots, \Lambda_d$ are transformed into operators given by the matrices $T_1, \dots, T_d$.

For $d = 1$ we have only one operator, whose matrix representation is the classical Jacobi matrix. We call the matrices T_i block Jacobi matrices. The elements of T_i are matrices whose sizes increase when moving down the main diagonal. If these operators are selfadjoint and commuting, then the spectral theorem for a commuting family of operators can be applied to establish the existence of the measure. In [15] we considered the bounded case. Since the fact that operators commute means their spectral measures commute, there is a significant difference between bounded and unbounded cases (cf. [7]). We describe our results in §2 and prove them in §3.

2. Main results

Let $\mathscr{M} = \mathscr{M}(\mathbb{R}^d)$ denote the set of nonnegative Borel measures μ on $\mathbb{R}^d$, such that

$$\int_{\mathbb{R}^d} |\mathbf{x}^\alpha| \, d\mu(\mathbf{x}) < +\infty, \quad \forall \alpha \in \mathbb{N}_0^d.$$

For $\mu \in \mathscr{M}$ the numbers $\mu_\alpha = \int \mathbf{x}^\alpha \, d\mu(\mathbf{x})$, $\alpha \in \mathbb{N}_0^d$, are called the moments of μ. We are interested in the case for which $\mathscr{L}$ has a unique integral representation

$$\mathscr{L}(f) = \int_{\mathbb{R}^d} f(\mathbf{x}) \, d\mu(\mathbf{x}), \qquad \mu \in \mathscr{M}.$$

The uniqueness of such a representation is in terms of the determinacy of the measure. Two measures are called equivalent if they have the same moments. The measure μ is called determinate if the equivalent class of measures having the same moments as μ consists of μ only.

Let $\|\cdot\|_2$ be the spectral norm for matrices that is induced by the Euclidean norm for vectors:

$$\|A\|_2 = \max\{\sqrt{\lambda} : \lambda \text{ is an eigenvalue of } A^T A\}.$$

In [15], we proved that T_i is bounded if and only if $\{\|A_{k,i}\|_2\}$ and $\{\|B_{k,i}\|_2\}$ are bounded. We now state our main result.

Theorem 2. *Let $\{\mathbb{P}_n\}_{n=0}^{\infty}$, $\mathbb{P}_0 \neq 0$, be a sequence in Π^d that satisfies the three-term relation and rank condition in* (2). *If*

$$(*) \qquad \sum_{k=0}^{\infty} \frac{1}{\|A_{n,i}\|_2} = \infty, \qquad 1 \le i \le d,$$

then there exists a determinate measure $\mu \in \mathscr{M}$ such that $\{\mathbb{P}_n\}$ is orthonormal with respect to μ.

This theorem extends the result in [15] by allowing the unbounded cases. A typical example is the product of Hermite polynomials. In the univariant case the condition $(*)$ is well known (cf. [1, p. 24; 4]). The classical result of Carleman on the determinacy in moment problem [1, p. 86] follows from this condition. For $n \in \mathbb{N}_0$ let $\mu_n = \mathscr{L}(x^n)$ for $\mathscr{L}$ defined on Π^1. Carleman's result says that $\{\mu_n\}$ is determinate if it satisfies Carleman's condition $\sum_{n=0}^{\infty}(\mu_{2n})^{-1/2n} = \infty$. This result was extended by Nussbaum [8] to the multivariate moment problem. Let $\mu_\alpha = \mathscr{L}(\mathbf{x}^\alpha)$ for $\mathscr{L}$ on Π^d and denote by $\mu_{n,i}$ the marginal sequences $\mu_{n,i} = \mathscr{L}(x_i^n)$, $1 \le i \le d$. Then Naussbaum's result says that $\{\mu_\alpha\}$ is determinate if $\{\mu_{n,i}\}$ verifies Carleman's condition. The proof of this result in [8] is based on the theory of quasi-analytic vectors. A natural question is then whether Nussbaum's result follows from condition $(*)$ in Theorem 2. This is discussed in the following.

For $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^d$, we denote by $\mathbf{x}^n$ the vector $\{\mathbf{x}^\alpha\}_{|\alpha|=n}$ where the elements are numbered according to the lexicographical order in $\{\alpha \in \mathbb{N}_0, |\alpha| = n\}$. The orthonormal polynomials $\mathbb{P}_n$ with respect to $\mathscr{L}$ can be written as $\mathbb{P}_n = G_n\mathbf{x}^n + \mathbb{Q}_n$, where $\mathbb{Q}_n$ is a polynomial vector with components in Π_{n-1}^d. In [16] we proved that the matrix G_n is invertible. Let $L_{n,i}$ denote the matrices of size $r_{n-1}^d \times r_n^d$ satisfying $L_{n,i}\mathbf{x}^n = x_i\mathbf{x}^{n-1}$. Since $x_i^n = L_{1,i}\cdots L_{n,i}\mathbf{x}^n$, we have

$$\mathscr{L}(x_i^n\mathbb{P}_n^T) = L_{1,i}\cdots L_{n,i}\mathscr{L}(\mathbf{x}^n\mathbb{P}_n^T) = L_{1,i}\cdots L_{n,i}G_n^{-1}\mathscr{L}(\mathbb{P}_n\mathbb{P}_n^T).$$

Therefore, from equation $A_{n,i}G_{n+1} = G_nL_{n+1,i}$ [16], we have $\mathscr{L}(\mathbb{P}_0) = G_0$ $(= 1$, say) and

$$\mathscr{L}(x_i^n\mathbb{P}_n^T) = A_{0,i}\cdots A_{n-1,i}, \qquad n \ge 1.$$

On the other hand, $\mathscr{L}$ induces an inner product on Π^d by $\langle f, g\rangle = \mathscr{L}(fg)$. Thus from Cauchy's inequality in the inner product space we have

$$\|\mathscr{L}(x_i^n\mathbb{P}_n^T)\|_2^2 \le \mathscr{L}(x_i^{2n})\mathscr{L}(\mathbb{P}_n\mathbb{P}_n^T) = \mu_{2n,i}.$$

It then follows that $\|A_{0,i}\cdots A_{n-1,i}\|_2^2 \le \mu_{2n,i}$. From a general inequality of Carleman (cf. [1, p. 86])

$$\sum_{n=0}^{\infty}(u_1u_2\cdots u_n)^{1/n} < e\sum_{n=0}^{\infty}u_n,$$

where the u_k are nonnegative real numbers not all of which are zero, it then follows that

$$\sum_{n=1}^{\infty}(\mu_{2n,i})^{-1/2n} \le \sum_{n=1}^{\infty}\frac{1}{(\|A_{0,i}\cdots A_{n-1,i}\|_2)^{1/n}} \le e\sum_{n=1}^{\infty}\frac{\|A_{0,i}\cdots A_{n-2,i}\|_2}{\|A_{0,i}\cdots A_{n-1,i}\|_2}.$$

Therefore, Nussbaum's condition implies

$$(**) \qquad \sum_{n=0}^{\infty}\frac{\|A_{0,i}\cdots A_{n-1,i}\|_2}{\|A_{0,i}\cdots A_{n,i}\|_2} = \infty, \qquad 1 \le i \le d.$$

Unfortunately, this condition does not match our condition $(*)$ unless $d = 1$, in which case the matrices become numbers.

Clearly, both conditions $(*)$ and $(**)$ can be viewed as natural extensions of the univariant condition $\sum(1/a_n) = \infty$. This also applies to the condition

$$(\dagger) \qquad \sum_{k=0}^{\infty}\|A_{n,i}^{\dagger}\|_2 = \infty,$$

where $A_{n,i}^{\dagger}$ denotes the generalized inverse of $A_{n,i}$. Since $A_{n,i}$, as the coefficient of the three-term relation, is necessarily of full rank, $\operatorname{rank} A_{n,i} = r_n^d$, the generalized inverse exists. We note that $(*)$ implies $(**)$, and $(**)$ implies $(\dagger)$. An open question is whether Theorem 2 can be strengthened to hold under condition $(**)$, or even $(\dagger)$. If the answer is yes, then the above discussion gives another proof of Nussbaum's result.

3. Proofs

First we recall the part of spectral theory that will be needed (see [10, 11, 12]). Let $\mathscr{H}$ be a separable Hilbert space and $T_i: \mathscr{H} \mapsto \mathscr{H}$ be selfadjoint operators. Let E_i be the spectral measure of T_i, $T_i = \int x\, dE_i(x)$, which is a projection-valued measure defined for Borel sets of $\mathbb{R}$ such that $E_i(\mathbb{R})$ is the identity operator in $\mathscr{H}$ and $E_i(B \cap C) = E_i(B) \cap E_i(C)$ for Borel sets $B, C \subseteq \mathbb{R}$. The fact that $\{T_1, \dots, T_d\}$ commute means that their spectral measure commutes, i.e., $E_i(B)E_j(C) = E_i(C)E_j(B)$ for any $i, j = 1, \dots, d$ and any two Borel sets $B, C \subseteq \mathbb{R}$. If $T_1, \dots, T_d$ commute, then

$$E = E_1 \otimes \cdots \otimes E_d$$

is a spectral measure on $\mathbb{R}^d$ with values that are selfadjoint projections in $\mathscr{H}$. In particular, E is the unique measure such that

$$E(B_1 \times \cdots \times B_d) = E_1(B_1) \cdots E_d(B_d)$$

for any Borel sets $B_1, \dots, B_d \subseteq \mathbb{R}$. The measure E is called the spectral measure of the commuting family $T_1, \dots, T_d$. A vector $\Phi_0 \in \mathscr{H}$ is a cyclic vector in $\mathscr{H}$ with respect to the commuting family of selfadjoint operators $T_1, \dots, T_d$ in $\mathscr{H}$ if the linear manifold $\{P(T_1, \dots, T_d)\Phi_0,\ P \in \Pi^d\}$ is dense in $\mathscr{H}$. The spectral theorem for $T_1, \dots, T_d$ is as follows.

Theorem 3. *Let $\mathscr{H}$ be a separable Hilbert space and $T_1, \dots, T_d$ be a commuting family of selfadjoint operators in $\mathscr{H}$. If Φ_0 is a cyclic vector in $\mathscr{H}$ with respect to $T_1, \dots, T_d$, then $T_1, \dots, T_d$ are unitarily equivalent to the multiplication operators $X_1, \dots, X_d$,*

$$(X_i f)(\mathbf{x}) = x_i f(\mathbf{x}), \qquad 1 \le i \le d,$$

defined on $L^2(\mathbb{R}^d, \mu)$, where the measure μ is defined by $\mu(B) = \langle E(B)\Phi_0, \Phi_0\rangle$ for the Borel set $B \subset \mathbb{R}^d$.

The unitary equivalence means that there exists a unitary mapping $U: \mathscr{H} \to L^2(\mathbb{R}^d, \mu)$ such that $UT_iU^{-1} = X_i$, $1 \le i \le d$. The unitary equivalence in Theorem 3 associates the cyclic vector Φ_0 with the function $f(\mathbf{x}) = 1$ and $(T_1^{\alpha_1} \cdots T_d^{\alpha_d})\Phi_0$ with $f(\mathbf{x}) = \mathbf{x}^\alpha$.

We need to show that if condition $(*)$ is satisfied, then the operators $T_1, \dots, T_d$ defined in equation (2) form a commuting family of selfadjoint operators that has a cyclic vector.

Let $\mathscr{D}(T)$ denote the domain of the operator T, $\mathscr{D}(T) = \{f: Tf \in \mathscr{H}\}$. From now on we set $\mathscr{H} = l^2$. We denote by $\mathscr{D}$ the set of all finite linear combinations of canonical basis vectors, or in our vector notation Φ_k,

$$\mathscr{D} = \left\{ \sum_{k=0}^{N} \mathbf{a}_k^T \Phi_k : N \in \mathbb{N}_0,\ \mathbf{a}_k \in \mathbb{R}^{r_k} \right\}.$$

Clearly $\mathscr{D}$ is a dense subset of $\mathscr{H}$. For any f in $\mathscr{D}(T_i)$, $f = \sum \mathbf{a}_k^T \Phi_k$, we have from the definition that

$$T_i f = \sum_{k=0}^{\infty} \mathbf{a}_k^T [A_{k,i}\Phi_{k+1} + B_{k,i}\Phi_k + A_{k-1,i}^T \Phi_{k-1}].$$

Therefore, $\mathscr{D} \subset \mathscr{D}(T_i)$, and it follows that T_i is densely defined on $\mathscr{H}$.

Lemma 1. *If condition* $(*)$ *is satisfied then the operator* T_i *defined in equation* (2) *is selfadjoint.*

Proof. Let $f, g \in \mathscr{D}(T_i)$, $f = \sum \mathbf{a}_k^T \Phi_k$ and $g = \sum \mathbf{b}_k^T \Phi_k$. First we prove that T_i is symmetric, i.e., $\langle T_i f, g\rangle = \langle T_i g, f\rangle$. The orthogonality of $\mathbb{P}_n$ implies that the $B_{n,i}$ are symmetric matrices, $B_{n,i} = \mathscr{L}(x_i \mathbb{P}_n \mathbb{P}_n^T)$. From the definition of T_i we have

$$\langle T_i f, g\rangle = \lim_{n\to\infty} S_n(\langle T_i f, g\rangle),$$

where

$$\begin{aligned} S_n(\langle T_i f, g\rangle) &= \sum_{k=0}^{n} [\mathbf{a}_{k-1}^T A_{k-1,i} + \mathbf{a}_k^T B_{k,i} + \mathbf{a}_{k+1}^T A_{k,i}^T]\mathbf{b}_k \\ &= \sum_{k=0}^{n} \mathbf{b}_k^T [A_{k-1,i}^T \mathbf{a}_{k-1} + B_{k,i}\mathbf{a}_k + A_{k,i}\mathbf{a}_{k+1}] \\ &= \sum_{k=0}^{n-1} \mathbf{b}_{k+1}^T A_{k,i}^T \mathbf{a}_k + \sum_{k=0}^{n} \mathbf{b}_k^T B_{k,i}\mathbf{a}_k + \sum_{k=1}^{n+1} \mathbf{b}_{k-1}^T A_{k-1,i}\mathbf{a}_k. \end{aligned}$$

Therefore, it follows readily that

$$\begin{aligned} &|S_n(\langle T_i f, g\rangle) - S_n(\langle T_i g, f\rangle)| \\ &\quad = |\mathbf{b}_n^T A_{n,i}\mathbf{a}_{n+1} - \mathbf{a}_n^T A_{n,i}\mathbf{b}_{n+1}| \\ &\quad \le \|A_{n,i}\|_2 (\|\mathbf{b}_n\|_2 \|\mathbf{a}_{n+1}\|_2 + \|\mathbf{a}_n\|_2 \|\mathbf{b}_{n+1}\|_2) \\ &\quad \le \|A_{n,i}\|_2 (\|\mathbf{b}_n\|_2^2 + \|\mathbf{a}_{n+1}\|_2^2 + \|\mathbf{a}_n\|_2^2 + \|\mathbf{b}_{n+1}\|_2^2)/2. \end{aligned}$$

If $\langle T_i f, g\rangle - \langle T_i g, f\rangle = \delta$, then for a sufficiently large N we have

$$\frac{\delta}{2} \sum_{n\ge N} \frac{1}{\|A_{n,i}\|_2} \le \sum \|\mathbf{b}_n\|_2^2 + \sum \|\mathbf{a}_n\|_2^2 \le \|f\|^2 + \|g\|^2.$$

Therefore, our condition $(*)$ implies that $\delta = 0$. Thus, T_i is symmetric. We now prove that T_i is selfadjoint, i.e., $T_i|_{\mathscr{D}(T_i^*)} = T_i^*$. Assume $g \in \mathscr{D}(T_i^*)$ and $T^* g = f$. Let again $f = \sum \mathbf{a}_k^T \Phi_k$ and $g = \sum \mathbf{b}_k^T \Phi_k$. Since

$$\langle T_i \Phi_k, g\rangle = A_{k,i}\mathbf{b}_{k+1} + B_{k,i}\mathbf{b}_k + A_{k-1,i}^T \mathbf{b}_{k-1}$$

it follows from $\langle T_i \Phi_k, g\rangle = \langle \Phi_k, T_i^* g\rangle$ that $\mathbf{a}_k = A_{k,i}\mathbf{b}_{k+1} + B_{k,i}\mathbf{b}_k + A_{k-1,i}^T \mathbf{b}_{k-1}$. Therefore,

$$\begin{aligned} f = \sum \mathbf{a}_k^T \Phi_k &= \sum [A_{k,i}\mathbf{b}_{k+1} + B_{k,i}\mathbf{b}_k + A_{k-1,i}^T \mathbf{b}_{k-1}]^T \Phi_k \\ &= \sum \mathbf{b}_k^T [A_{k,i}\Phi_{k+1} + B_{k,i}\Phi_k + A_{k-1,i}^T \Phi_{k-1}] = T_i g. \end{aligned}$$

That is, $T_i^* g = T_i g$ for all $g \in \mathscr{D}^*(T_i)$. Thus, T_i is selfadjoint. □

For the proof of Theorem 2 we only need T_i to be essentially selfadjoint and their closures to commute. The essential selfadjointness of T_i can also be derived from [6]. For bounded operators, the fact that T_i and T_j have commuting spectral measures is equivalent to $T_iT_j = T_jT_i$. But for unbounded operators this is no longer true (cf. [7]). We need the following result due to Nelson [7] (see also [9]).

Lemma 2. *Let T and S be symmetric operators in a Hilbert space $\mathscr{H}$ and let D be a dense linear manifold in $\mathscr{H}$ such that D is contained in the domain of T^2, S^2, TS, and ST, and such that $TSf = STf$ for all f in $\mathscr{H}$. If the restriction of $S^2 + T^2$ to D is essentially selfadjoint then T and S are essentially selfadjoint and $\overline{S}$ and $\overline{T}$ commute, where $\overline{T}$ stands for the closure of T.*

Lemma 3. *If condition $(*)$ is satisfied then the operators $T_1, \dots, T_d$ defined in equation (2) mutually commute.*

Proof. Since our T_i is selfadjoint by the previous lemma, $\overline{T}_i = T_i^{**} = T_i$. The orthogonality of $\mathbb{P}_n$ in Theorem 1 implies that the coefficient matrices satisfy [14]

$$\begin{gathered}
A_{k,i}A_{k+1,j} = A_{k,j}A_{k+1,i}, \\
A_{k,i}B_{k+1,j} + B_{k,i}A_{k,j} = B_{k,j}A_{k,i} + A_{k,j}B_{k+1,i}, \\
A^T_{k-1,i}A_{k-1,j} + B_{k,i}B_{k,j} + A_{k,i}A^T_{k,j} = A^T_{k-1,j}A_{k-1,i} + B_{k,i}B_{k,j} + A_{k,j}A^T_{k,i},
\end{gathered}$$

for $i \neq j$, $1 \le i, j \le d$, and $k \ge 0$, where $A_{-1,i} = 0$. Using these equations it is readily seen that $T_iT_jf = T_jT_if$ on $\mathscr{D}(T_iT_j) \cap \mathscr{D}(T_jT_i)$. In particular, $T_iT_jf = T_jT_if$ for all f in $\mathscr{D}$, where $\mathscr{D}$ contains all finite linear combinations of $\mathbf{\Phi}_k$ as defined before. Since $\mathscr{D}$ is a dense linear manifold in $\mathscr{H}$ and clearly $\mathscr{D}$ is contained in $\mathscr{D}(T_i^2)$, $\mathscr{D}(T_j^2)$, $\mathscr{D}(T_iT_j)$, and $\mathscr{D}(T_jT_i)$, we can apply Lemma 2. For $i \neq j$ let T_{ij} be the restriction of $T_i^2 + T_j^2$ on $\mathscr{D}$. If condition $(*)$ holds, then the selfadjointness of T_i and T_j by Lemma 1 implies that

$$\langle (T_i^2 + T_j^2)f, g\rangle = \langle f, (T_i^2 + T_j^2)g\rangle$$

for all f, g in $\mathscr{D}$. Thus, T_{ij} is symmetric. Assume that $g \in \mathscr{D}(T_{ij}^*)$ and $f = T_{ij}^*g$, and write $f = \sum \mathbf{a}^T\mathbf{\Phi}_k$ and $g = \sum \mathbf{b}_k^T\mathbf{\Phi}_k$. First we have

$$T_{i,j}\mathbf{\Phi}_k = C_{k+1}\mathbf{\Phi}_{k+2} + D_{k+1}\mathbf{\Phi}_{k+1} + E_k\mathbf{\Phi}_k + D_k^T\mathbf{\Phi}_{k-1} + C_{k-1}^T\mathbf{\Phi}_{k-2},$$

where

$$\begin{aligned}
C_{k+1} &= A_{k,i}A_{k+1,i} + A_{k,j}A_{k+1,j}, \\
D_{k+1} &= A_{k,i}B_{k+1,i} + B_{k,i}A_{k,i} + A_{k,j}B_{k+1,j} + B_{k,j}A_{k+1,j}, \\
E_k &= A_{k,i}A^T_{k,i} + B_{k,i}B_{k,i} + A^T_{k-1,i}A_{k-1,i} \\
&\quad + A_{k,j}A^T_{k,j} + B_{k,j}B_{k,j} + A^T_{k-1,j}A_{k-1,j}.
\end{aligned}$$

Then as in the proof of Lemma 1, we can compute $\langle T_{ij}\mathbf{\Phi}_k, g\rangle$ and use $\langle \mathbf{\Phi}_k, T_{ij}^*g\rangle = \langle T_{ij}\mathbf{\Phi}_k, g\rangle$ to write $\mathbf{a}_k$ in terms of $\{\mathbf{b}_k\}$. Thus we obtain

$$\mathbf{a}_k = C_{k+1}\mathbf{b}_{k+2} + D_{k+1}\mathbf{b}_{k+1} + E_k\mathbf{b}_k + D_k^T\mathbf{b}_{k-1} + C_{k-1}^T\mathbf{b}_{k-2}.$$

Using this relation in $f = \sum \mathbf{a}_k^T \Phi_k$ and changing the summation variables, we obtain

$$f = \sum \mathbf{a}_k^T \Phi_k = \sum \mathbf{b}_k^T T_{i,j} \Phi_k = T_{ij} g.$$

Thus, T_{ij} is selfadjoint. From Lemma 2 it follows that T_i and T_j commute. □

The next lemma is proved in [15] for bounded operators, but its proof does not depend on the boundedness of T_i.

Lemma 4. *The vector* $\Phi_0 \in \mathscr{H}$ *is a cyclic vector with respect to* $T_1, \ldots, T_d$ *and*

$$\Phi_n = \mathbb{P}_n(T_1, \ldots, T_d)\Phi_0,$$

where $\mathbb{P}_n(x_1, \ldots, x_d)$ *is of the form used in equation* (1).

Proof of Theorem 2. The proof follows the same line as in the bounded case [15]; we shall be brief. Let $\{\mathbb{P}_n\}$ satisfy the three-term relation in the theorem. If condition $(*)$ is satisfied, then it follows from previous lemmas and Theorem 3 that $T_1, \ldots, T_d$ are unitarily equivalent to the multiplication operators $X_1, \ldots, X_d$ in $L^2(\mathbb{R}^d, \mu)$, where the measure $\mu \in \mathscr{M}$ is defined by $\mu(B) = \langle E(B)\Phi_0, \Phi_0 \rangle$. The polynomials $\{\mathbb{P}_n\}$ in Lemma 4 are the polynomials that satisfy the three-term relation, they are orthonormal with respect to μ as

$$\int \mathbb{P}_n(\mathbf{x})\mathbb{P}_m^T(\mathbf{x})\, d\mu(\mathbf{x}) = \langle \mathbb{P}_n \Phi_0, \mathbb{P}_m^T \Phi_0 \rangle = \langle \Phi_n, \Phi_m^T \rangle.$$

Therefore, the existence of the measure of orthogonality is proved. Since the unitary equivalence implies that each of the multiplication operators X_j is selfadjoint, the determinacy of μ follows from [5, Theorem 7]. □

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