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THE 2-CHARACTER TABLE DOES NOT DETERMINE A GROUP

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ABSTRACT. Frobenius had defined the group determinant of a group G which is a polynomial in $n = |G|$ variables. Formanek and Sibley have shown that the group determinant determines the group. Hoehnke and Johnson show that the 3-characters (a part of the group determinant) determine the group. In this paper it is shown that the 2-characters do not determine the group. If we start with a group G of a certain type then a group H with the same 2-character table must form a Brauer pair with G. A complete description of such an H is available in Comm. Algebra **9** (1981), 627–640.

1. INTRODUCTION

The group determinant was first introduced in 1896 by Frobenius [3]. It was the problem of how this determinant factorizes which led him to define characters for an arbitrary finite group G. It is a natural question to consider whether G is determined by its group determinant, but it appears that this question was not raised until 1986 (Johnson [7]). In fact the group determinant contains sufficient information to determine a group. This was shown in 1990 by Formanek and Sibley [4], and recently an elementary proof has appeared by Mansfield [9]. In [3] there were also introduced functions $\chi^{(k)}\colon G^k \to \mathbb{C}$ which correspond to a character χ of G, $k = 1, 2, \ldots$. These were named k-characters in [8]. It follows from [3] that if $\{\chi_i\}$, $1 \le i \le m$, is the set of distinct irreducible characters of G then $\{\chi_i^{(k)}, 1 \le k \le \deg(\chi_i), 1 \le i \le m\}$ determines the group determinant of G, and hence G.

Recently it has been announced by Hoehnke and Johnson [5] that the 3-character of the regular representation, or equivalently the knowledge of the 1-, 2-, and 3-characters corresponding to all the irreducible characters of G, is sufficient to determine G. If χ is a character of G, the 1-, 2-, and 3-characters

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corresponding to χ are defined as follows:

$$(1.1)\quad \begin{aligned} &\chi^{(1)}(g) = \chi(g), && g \in G, \\ &\chi^{(2)}(g, h) = \chi(g)\chi(h) - \chi(gh), && g, h \in G, \\ &\chi^{(3)}(g, h, k) = \chi(g)\chi(h)\chi(k) - \chi(g)\chi(hk) - \chi(h)\chi(gk) && \\ &\qquad\qquad - \chi(k)\chi(gh) + \chi(ghk) + \chi(gkh), && g, h, k \in G. \end{aligned}$$

Thus an answer is provided to the question of Brauer in [1] as to which information in addition to the (ordinary) character table of a group is sufficient to determine a group.

There remains the question of whether a group can be determined by its 2-characters. In [8] a 2-character table of a finite group G is defined. If the set of irreducible characters of G is $\{\chi_1, \dots, \chi_m\}$ then two types of "degenerate" 2-characters are defined:

$$(1.2)\quad \begin{aligned} &\text{(i) } \chi_i \circ \chi_j(g, h) = \chi_i(g)\chi_j(h) + \chi_i(h)\chi_j(g), && 1 \le i < j \le m, \\ &\text{(ii) } \chi_i^{(2,+)}(g, h) = \chi_i(g)\chi_i(h) + \chi_i(gh), && 1 \le i \le m. \end{aligned}$$

The 2-character table of G then consists of the $\chi_i^{(2)}$, where $\deg(\chi_i) \ge 2$, and the degenerate 2-characters described in (1.2). It is shown in [8] that orthogonality relations hold among these 2-characters.

A consequence of Theorem 2.1 of this work is that there exist pairs of nonisomorphic groups with the same 2-character table, an explicit example of such a pair being two groups of order $624 \cdot 625$. It follows that the 2-characters are not sufficient to determine a group.

We remark that G and H have the same 2-character tables if and only if there exists a map $\psi: G \to H$ and a correspondence $\chi_i \leftrightarrow \mu_i$ between the irreducible characters of G and H such that for each generalized 2-character ν of G (see above)

$$(1.3)\qquad \nu(g, h) = \tau(\psi(g), \psi(h)),$$

where τ is the 2-character of H which corresponds to ν under the correspondence induced by $\chi_i \leftrightarrow \mu_i$.

Throughout the paper it is assumed that all characters are complex characters.

2. Doubly transitive solvable Frobenius groups

In [2] it is shown that if G is a doubly transitive solvable permutation group and if the group H has the same ordinary character table as G then H must also be a doubly transitive solvable permutation group, and $\{G, H\}$ is a Brauer pair. Moreover, the classification of Huppert [6] may be used to show that, apart from exceptional cases in which the character table of G determines G uniquely, G and H must be subgroups of $FS(p^n)$, the group of semilinear maps of the finite field $F = GF(p^n)$. We prove the following theorem, which depends on results in [2].

Theorem 2.1. *Let G be a doubly transitive solvable Frobenius group. Then the following are equivalent:*

(a) *The groups G and H form a Brauer pair.*
(b) *The groups G and H have the same 2-character tables.*

Proof. We first prove that (a) $\Rightarrow$ (b). Suppose that G is a doubly transitive Frobenius group and that there is a group H not isomorphic to G such that $\{G, H\}$ is a Brauer pair. In [2] it is shown that G and H must be subgroups of $FS(p^n)$ of the following form. Let σ denote the Frobenius automorphism $\binom{x}{x^p}$ of F, $\bar{\omega}$ denote the map $\binom{x}{x\omega}$ of F (for any $w \in F$), and n^* denote the map $\binom{x}{x+n}$ (for any $n \in F$). From now on let us pick ω to be a generator of the multiplicative group of F. Let $N = \langle n^*; n \in F\rangle$ be the subgroup of $FS(p^n)$ isomorphic to the additive group of F. Then

$$G \simeq G_0 \ltimes N, \quad \text{where } G_0 = \langle \bar{\omega}^k, \sigma^v \bar{\omega}^i \rangle,$$
$$H \simeq H_0 \ltimes N, \quad \text{where } H_0 = \langle \bar{\omega}^k, \sigma^v \bar{\omega}^j \rangle$$

with $k/(p^n - 1)$, $vk = n$, $(k, i) = (k, j) = 1$, and such that p^v has order k modulo $k(p^v - 1)$.

Lemma 2.2. *Suppose $(k, i) = 1$ and $(k, j) = 1$. Then a prime q can be chosen such that $q \nmid p^n - 1$ and $j \equiv iq$ modulo k.*

Proof. Let

$$x \equiv ji^{-1} \bmod k, \tag{2.1}$$

where i^{-1} is the inverse of i in $\mathbb{Z}_k$. Then $x = a + kt$, $a = j \cdot i^{-1}$. Thus any element of the arithmetic progression $\{a + kt; t = 1, 2, \dots\}$ is a solution of (2.1) and by Dirichlet's theorem a prime solution q may be chosen such that $q \nmid p^n - 1$. Then $j \equiv iq \bmod k$. □

Lemma 2.3. *Define the map $\theta\colon G_0 \to H_0$ by*

$$\theta(\bar{\omega}^k) = \bar{\omega}^{kq}, \qquad \theta(\sigma^v \bar{\omega}^i) = \sigma^v \bar{\omega}^{iq},$$

where q is a prime satisfying the conditions of Lemma 2.2. Then θ extends to an isomorphism from G_0 to H_0, such that $\theta(\sigma^{vt}\bar{\omega}^r) = \sigma^{vt}\bar{\omega}^{rq}$ whenever $\sigma^{vt}\bar{\omega}^r$ lies in G_0.

Proof. It is clear that $\theta(\bar{\omega}^k)$ lies in H_0. Now

$$\theta(\sigma^v \bar{\omega}^i) = \sigma^v \bar{\omega}^{iq} = \sigma^v \bar{\omega}^{j+k\lambda}, \qquad \lambda \in \mathbb{Z},$$

since $q \equiv j \bmod k$ and thus

$$\theta(\sigma^v \bar{\omega}^i) = (\sigma^v \bar{\omega}^j)(\bar{\omega}^{k\lambda}) \quad \text{lies in } H_0.$$

The elements of G_0 may be described uniquely as those of the form

$$(\sigma^v \omega^i)^t \bar{\omega}^{k\lambda}, \qquad 0 \le t \le \frac{n}{v},\ 1 \le \lambda \le \frac{p^n - 1}{k}. \tag{2.2}$$

For suppose $(\sigma^v \bar{\omega}^i)^t \bar{\omega}^{k\lambda} = (\sigma^v \omega^i)^{t'} \bar{\omega}^{k\lambda'}$ with $0 \le t' \le t \le n/v$ and $1 \le \lambda, \lambda' \le (p^n - 1)/k$. It follows that $(\sigma^v \bar{\omega}^i)^{t-t'} = \bar{\omega}^{k(\lambda' - \lambda)}$, i.e.,

$$\sigma^{v(t-t')} \bar{\omega}^s = \bar{\omega}^{k(\lambda' - \lambda)} \quad \text{for some } s$$

(using Lemma 1.2(vi) in [2]). Therefore $t - t' = 0$, and hence $\lambda - \lambda' = 0$. Thus the elements in (2.2) are all distinct, and by counting must form all the elements of G_0.

Now define

$$\theta((\sigma^v\bar{\omega}^i)^t\bar{\omega}^{k\lambda}) = (\theta(\sigma^v\bar{\omega}^i))^t(\theta(\bar{\omega}^k))^\lambda = (\sigma^v\bar{\omega}^{iq})^t\bar{\omega}^{kq\lambda}$$
$$= \sigma^{vt}\bar{\omega}^{iq((p^{vt}-1)/(p-1))+kq\lambda}$$

again using Lemma 1.2(vi) in [2]. Note that since

$$(\sigma^v\bar{\omega}^i)^t\bar{\omega}^{k\lambda} = \sigma^{vt}\bar{\omega}^{i((p^{vt}-1)/(p-1))+k\lambda}$$

it follows that $\theta(\sigma^{vt}\bar{\omega}^r) = \sigma^{vt}\bar{\omega}^{rq}$ whenever $\sigma^{vt}\bar{\omega}^r$ lies in G_0.

Therefore for elements $\sigma^{vt}\bar{\omega}^r$ and $\sigma^{vt'}\bar{\omega}^{r'}$ of G_0 we obtain

$$\begin{aligned}\theta[(\sigma^{vt}\bar{\omega}^r)(\sigma^{vt'}\bar{\omega}^{r'})] &= \theta[\sigma^{v(t+t')}\bar{\omega}^{rp^{vt'}+r'}] \quad \text{(using Lemma 1.2(v) in [2])}\\ &= \sigma^{v(t+t')}\bar{\omega}^{q(rp^{vt'}+r')} = \sigma^{vt}\bar{\omega}^{qr}\sigma^{vt'}\bar{\omega}^{qr'}\\ &= \theta(\sigma^{vt}\bar{\omega}^r)\theta(\sigma^{vt'}\bar{\omega}^{r'}).\end{aligned}$$

Hence θ is a homomorphism of G_0 into H_0.

Suppose $\theta(\sigma^{vt}\bar{\omega}^r) = e$. Then $\sigma^{vt}\bar{\omega}^{rq} = e$, and thus $\sigma^{vt} = e$ and $\bar{\omega}^{rq} = e$ since $\langle\sigma\rangle \cap \langle\bar{\omega}\rangle = \{e\}$. Hence $\ker\theta = \{e\}$ and Lemma 2.3 is proved. □

Now define the map $\psi: G \to H$ by

$$\psi(g_0n^*) = \theta(g_0)(n^q)^*, \qquad n^* \in N,\ g_0 \in G_0.$$

We will show that ψ induces an identification of the 2-character tables of G and H.

In [2] the character table of G is determined (and is the same as that of H). It consists of characters $\chi_1, \dots, \chi_l$ which are obtained from the irreducible characters $\bar{\chi}_1, \dots, \bar{\chi}_l$ of G by composing the corresponding representations with the homomorphism $G \to G_0$ given by $g_0n^* \to g_0$, together with a single extra character χ_{l+1} which is $\rho - 1$, where ρ is the permutation character corresponding to the representation of G as a permutation group on F. Thus

$$\begin{aligned}\chi_i(g_0n^*) &= \bar{\chi}_i(g_0), & & i = 1, \dots, l,\ g_0 \neq e,\ g_0 \in G_0, n^* \in N,\\ \chi_i(n^*) &= \chi_i(e), & & i = 1, \dots, l,\ n^* \in N,\\ \chi_{l+1}(e) &= p^n - 1, & &\\ \chi_{l+1}(g_0n^*) &= 0 & & \text{if } g_0 \neq e,\\ \chi_{l+1}(n^*) &= -1 & & \text{if } n^* \neq e \in N.\end{aligned}$$

We claim that the conjugacy classes of G are of the following form. Let $Cl_G(g)$ denote the conjugacy class of the element g in G. Then if $e \neq g_0 \in G_0$, it follows that $Cl_G(g_0) = \{xn^*;\ x \in Cl_{G_0}(g),\ n^* \in N\}$. The remaining two classes are $\{e\}$ and $N - \{e\}$. Suppose $g_0 \neq e$ lies in G_0. An arbitrary element $y \in G$ may be written $y = y_0n^*$, $y_0 \in G_0$, $n^* \in N$. Then

$$\begin{aligned}g_0^y &= (g_0^{y_0})^{n^*} = (n^*)^{-1}g_0'n^* \quad (g_0' = g_0^{y_0} \in G_0)\\ &= g_0'(n^*)^{-1}g_0'n^* = g_0'n'^* \quad \text{(for some } n' \in N).\end{aligned}$$

Now suppose $g_0^x = g_0$ for $x = x_0n^* \in G$. Then $g_0 = n^{*-1}g_0^{x_0}n^*$. But since G is Frobenius, $(n^*)^{-1}g_0^{x_0}n^*$ fixes 0 if and only if $n^* = e$. Hence $C_G(g_0) = C_{G_0}(g_0)$. Thus $[G : C_G(g_0)] = [G : G_0][G_0 : C_{G_0}(g_0)]$, i.e.,

$|Cl_G(g_0)| = |N|\,|Cl_{G_0}(g_0)|$ and hence $Cl_G(g_0) = \{xn^* ;\, n^* \in N ,\ x \in Cl_{G_0}(g_0)\}$. Since $C_G(n^*) = N - \{e\}$ for $n^* \in N - \{e\}$, the above claim is justified.

We now set up the correspondence between the characters of G_0 and H_0, $\bar{\chi}_i \leftrightarrow \bar{\mu}_i$, by means of the isomorphism θ:

$$\bar{\chi}_i(g_0) = \bar{\mu}_i(\theta(g_0)), \qquad g_0 \in G_0.$$

The characters $\mu_1, \ldots, \mu_l$ of H are defined by

$$\mu_i(h_0 n^*) = \bar{\mu}_i(h_0), \qquad i = 1, \ldots, l,$$

and again $\mu_{l+1} = \rho - 1$. Thus

$$\chi_i(g_0 n^*) = \mu_i(\theta(g_0)(n^q)^*), \qquad i = 1, \ldots, l+1,$$

and

$$\chi_i(g) = \mu_i(\psi(g)), \qquad i = 1, \ldots, l+1, \ g \in G.$$

We now verify that (1.3) is satisfied by all possible choices for ν.

(i) $$\nu = \chi_i \circ \chi_j, \quad \tau = \mu_i \circ \mu_j, \qquad 1 \le i < j \le l+1,$$
$$\begin{aligned} \nu(g, g') &= \chi_i(g)\chi_j(g') + \chi_i(g')\chi_j(g) \\ &= \mu_i(\psi(g))\mu_j(\psi(g')) + \mu_i(\psi(g'))\mu_j(\psi(g)) \\ &= \tau(\psi(g), \psi(g')). \end{aligned}$$

(ii) $$\nu = \chi_j^{(2)} \quad \text{or} \quad \chi_j^{(2,+)}, \qquad 1 \le j \le l.$$

If $\nu = \chi_j^{(2)}$, then $\nu(g, g') = \chi_j(g)\chi_j(g') - \chi_j(gg')$. Let $g = g_0 n^*$ and $g' = g_0' n'^*$. Then $gh = g_0 g_0' n''^*$ for some $n'' \in N$. Hence

$$\begin{aligned} \nu(g, g') &= \chi_j(\psi(g))\chi_j(\psi(g')) - \bar{\chi}_j(g_0 g_0') \\ &= \mu_j(\psi(g))\mu_j(\psi(g')) - \bar{\mu}_j(\theta(g_0 g_0')) \\ &= \mu_j(\psi(g))\mu_j(\psi(g')) - \mu_j(\psi(g_0)\psi(g_0')) \\ &= \tau(\psi(g), \psi(g')). \end{aligned}$$

The second case is similar (note that $\chi_j^{(2)}$ occurs as a 2-character only if $\deg(\chi_j) \ge 2$).

(iii) $$\nu = \chi_{l+1}^{(2)} \text{ or } \chi_{l+1}^{(2,+)};$$

let $\nu = \chi_{l+1}^{(2)}$. For convenience we omit the suffix $l+1$, thus

$$\nu(g, g') = \chi(g)\chi(g') - \chi(gg').$$

Let $g = g_0 n^*$ and $g' = g_0' n'^*$.

We consider two cases:

Case 1. $g_0 g_0' \ne e$. Then $\chi(gg') = \chi(g_0 g_0' n''^*) = 0$ and

$$\begin{aligned} \psi(g)\psi(g') &= \theta(g_0)(n^q)^*\theta(g_0')(n'^q)^* \\ &= \theta(g_0)\theta(g_0')n'''^* \quad (n'''^* \in N) \\ &= \theta(g_0 g_0')n'''^*. \end{aligned}$$

So $\psi(g)\psi(g')$ is not an element of N. Hence $\mu(\psi(g)\psi(g')) = 0$. Thus $\nu(g, g') = \chi(g)\chi(g') = \mu(\psi(g))\mu(\psi(g')) = \tau(\psi(g), \psi(g'))$.

Case 2. $gg' \in N$. We show that $gg' = e$ if and only if $\psi(g)\psi(g') = e$. It will then follow that

$$\chi(gg') = \mu(\psi(g)\psi(g')) \quad \text{for all } g, g' \text{ such that } gg' \in N,$$

and as in Case 1 that $\nu(g, g') = \lambda(\psi(g), \psi(g'))$. Thus suppose $g_0 g_0' = e$. Then

$$gg' = g_0 g_0' (n^*)^{g_0'} n'^* = (n^*)^{g_0'} n'^*.$$

Now let $g_0' = \sigma^{vt}\bar{\omega}^\lambda$. It follows from Lemma 1.2 in [2] that

$$(n^*)^{g_0'}(n'^*) = (n^{p^{vt}}\omega^\lambda)^*(n')^* = (n^{p^{vt}}\omega^\lambda + n')^*$$

and thus $gg' = e$ if and only if $np^{vt}\omega^\lambda + n' = 0$, i.e., if and only if $-n^{p^v}\omega^\lambda = n'$. Now

$$\begin{aligned}
\psi(g)\psi(g') &= \theta(g_0)(n^q)^*\theta(g_0')(n'^q)^* \\
&= \theta(g_0)\theta(g_0')(n^q)^{*\theta(g_0')}(n'^q)^* \\
&= (n^q)^{*\theta(g_0')}(n'^q)^* = (n^q)^{*\sigma^{vt}\bar{\omega}^{\lambda q}}(n'^q)^* \\
&= (n^{qp^{vt}}\omega^{\lambda q})^*(n'^q)^* = (n^{qp^t}\omega^{\lambda q} + n'^q)^*
\end{aligned}$$

which is e if and only if $-n^{qp^{vt}}\omega^{\lambda q} = n'^q$. Thus $gg' = e$ if and only if $\psi(g)\psi(g') = e$.

The case $v = \chi^{(2,+)}$ is similar.

Hence in all cases we have shown that (1.3) holds; i.e., the 2-character tables of G and H are the same.

Proof that (b) $\Rightarrow$ (a). This is immediate on noting that

(i) If G and H have the same 2-character tables then they necessarily have the same ordinary character tables (see §1).

(ii) As quoted above, in [2] it is shown that if G is any doubly transitive solvable group and H has the same ordinary character table as G then $\{G, H\}$ form a Brauer pair. □

Example. Suppose $p = 5$, $n = 4$, $k = 4$, $v = 1$, $i = 1$, and $j = 3$. Then G and H are nonisomorphic groups of order $624 \cdot 625$. By Theorem 2.1, G and H have the same 2-character tables. An explicit value for q in this case is 7.

3. Some open problems

A consequence of the work in [5] is that if a representation is sufficiently large its 3-character is sufficient to determine the group. There remains the question of how much information the 3-character of an arbitrary faithful representation contains. In [8] the case of an irreducible representation of degree 2 is considered. Here the 2-character alone contains sufficient information to construct an explicit matrix representation. Thus the 2-character of a faithful irreducible representation of degree 2 determines the group.

Problem 1. Let χ be a faithful irreducible representation of G of degree greater than 2. Does $\chi^{(3)}$ determine G?

By the results of §2, the condition that G and H have the same 2-character tables is not sufficient to ensure that G and H are isomorphic. Since Brauer pairs have been the subject of investigation, we pose the following.

Problem 2. If G and H have the same 2-character table must G and H necessarily form a Brauer pair?

Finally we consider representations over fields of finite characteristic. By [4], the group determinant over any field whose characteristic does not divide $|G|$ determines G. In [5] it is shown that if $\operatorname{char}(K) \neq 2$ and $\operatorname{char}(K) \nmid |G|$ the 1-, 2-, and 3-characters of the regular representation over the field K determine G.

Problem 3. Let G be a group of odd order and K be a field of characteristic 2. Which is the smallest value of k for which the 1-, 2-, ..., k-characters of the regular representation over K determine G?

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