ON APPROXIMATE ANTIGRADIENTS

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Abstract. For $n \in \mathbb{N}$ and $I = [0, 1]$, let $I^n$ be the unit cube and $\lambda^n$ the Lebesgue measure in $\mathbb{R}^n$. It is proved that if $f : I^n \to \mathbb{R}^n$ and $F_0 : I^n \to \mathbb{R}$ are continuous and $\varepsilon > 0$, then there exist a continuous $F : I^n \to \mathbb{R}$ and an open set $W \subset (I^n)^o$ with $\lambda^n(W) = 1$ such that

(i) $\nabla F$ exists and is continuous on $W$,
(ii) $\|\nabla F(x) - f(x)\| < \varepsilon \quad \forall x \in W$, and
(iii) $|F(x) - F_0(x)| < \varepsilon \quad \forall x \in I^n$, where $\|y\| = (\sum_{j=1}^n y_j^2)^{1/2}$ $\forall y \in \mathbb{R}^n$.

0. Introduction and definitions

If $f \in C([0, 1])$ and $F(x) = \int_0^x f$ for $0 \leq x \leq 1$, then $F' = f$. A famous theorem of Lusin states that if $\phi : I \to \mathbb{R}$ is a measurable function, then there exists $F \in C([0, 1])$ such that $F' = \phi$ a.e. on $[0, 1]$ (see [1, p. 217]). However, if $f(x, y) = (0, x)$ for $(x, y) \in I^2$, then there is no $F : I^2 \to \mathbb{R}$ for which $\nabla F = f$, because $D_1 F = 0$ implies that $F$ depends only on $y$. Thus, if $n > 1$ and $f \in C(I^n, \mathbb{R}^n)$, then $f$ need not have an antigradient $F : \nabla F = f$.

Can we find some kind of approximate antigradient for $f$? More generally, if $f : I^n \to \mathbb{R}^n$ is only measurable, what can be said? We are going to give some answers to these questions.

Throughout this paper, we use some definitions listed below.

(0.1) Definitions. Let $m, n \in \mathbb{N}$ and a nonvoid set $\Omega$ be given.

(1) If $F : \Omega \to \mathbb{R}^m$ is a mapping, then for $1 \leq i \leq m$, define the $i$th coordinate function $F_i : \Omega \to \mathbb{R}$ of $F$ by letting $F_i(t)$ be the $i$th coordinate of $F(t)$:

$$F(t) = (F_1(t), F_2(t), \ldots, F_m(t)), \quad t \in \Omega.$$  

(2) If $\Omega$ is a topological space, then $C(\Omega, \mathbb{R}^m)$ denotes the family of all continuous mappings from $\Omega$ to $\mathbb{R}^m$. If $m = 1$, write $C(\Omega)$ for $C(\Omega, \mathbb{R})$. Similarly, if $\Omega \subset \mathbb{R}^n$, we denote by $\mathcal{M}(\Omega, \mathbb{R}^m)$ the set of all Lebesgue measurable mappings from $\Omega$ to $\mathbb{R}^m$ and write $\mathcal{M}(\Omega)$ for $\mathcal{M}(\Omega, \mathbb{R})$.

(3) Let $\Omega \subset \mathbb{R}^n$ and let $f \in C(\Omega)$. If $\frac{\partial f}{\partial x_i}(x)$ exists at some $x \in \Omega^o$ (the
interior of $\Omega$) $\forall 1 \leq j \leq n$, we define
\[
\nabla f(x) = \left( \frac{\partial}{\partial x_1} f(x), \frac{\partial}{\partial x_2} f(x), \ldots, \frac{\partial}{\partial x_n} f(x) \right),
\]
and call $\nabla f$ the gradient of the function $f$. We also write $D_j$ for $\frac{\partial}{\partial x_j}$.

Conversely, let $f \in C(\Omega, \mathbb{R}^n)$. If there is $F \in C(\Omega)$ such that $\nabla F = f$, we call $F$ an antigradient of $f$.

(4) We always denote by $I^n$ the unit cube in $\mathbb{R}^n$. That is
\[
I^n = \prod_{k=1}^n [0, 1] = \{ x \in \mathbb{R}^n : 0 \leq x_k \leq 1, \ j = 1, 2, \ldots, n \}.
\]
Also, we always denote by $\lambda^n$ the Lebesgue measure on $\mathbb{R}^n$.

(5) Let $\Omega \subset \mathbb{R}^n$ and $\phi = (\phi_1, \phi_2, \ldots, \phi_m) \in C(\Omega, \mathbb{R}^m)$. If $\varepsilon > 0$, then we say $\| (\phi_1, \phi_2, \ldots, \phi_m) \| < \varepsilon$ a.e. on $\Omega$ to mean that
\[
\lambda^n(\{ x \in \Omega : \| (\phi_1(x), \phi_2(x), \ldots, \phi_m(x)) \| \geq \varepsilon \}) = 0,
\]
where $\| y \| = (\sum_{j=1}^m y_j^2)^{1/2}$ if $y \in \mathbb{R}^m$.

(6) For any $F : \Omega \to \mathbb{R}$, we define the uniform norm of $F$ by
\[
\| F \|_u = \sup \{ |F(x)| : x \in \Omega \}.
\]

(7) We reserve the letter $\psi$ to denote Lebesgue’s singular function (see [2, p. 130] or [4, p. 113]). It is a continuous, nondecreasing function $\mathbb{R} \to [0, 1]$ with $\psi(0) = 0$ and $\psi(1) = 1$ which is constant on each component interval of $\mathbb{R} \setminus C$, where $C$ is Cantor’s ternary set.

1. Approximation to antigradients

(1.1) Theorem. Let $n \in \mathbb{N}$, let $P \in C(I^n, \mathbb{R}^n)$, and let $\varepsilon > 0$. Then there exist a continuous $Q : I^n \to \mathbb{R}$ and an open $V \subset (I^n)^\circ$ with $\lambda^n(V) = 1$ such that

(i) $\nabla Q$ exists and is continuous on $V$, and

(ii) $\| \nabla Q(x) - P(x) \| < \varepsilon$ $\forall x \in V$,

where $\| y \| = (\sum_{j=1}^n y_j^2)^{1/2}$ $\forall y \in \mathbb{R}^n$.

Proof. If $n = 1$, define
\[
Q(x) = \int_0^x P(t) \, dt \quad (0 \leq x \leq 1)
\]
and take $V = ]0, 1[ = I^\circ$. Then $\nabla Q = Q' = P$ on $V$ so the theorem is true if $n = 1$.

Suppose, as an induction hypothesis, that $n > 1$ and that the theorem is true if $n$ is replaced with $n - 1$. The uniform continuity of $P$ provides $\delta > 0$ such that
\[
(1) \quad u, v \in I^n, \quad \| u - v \| < \delta \Rightarrow \| P(u) - P(v) \| < \varepsilon/6n.
\]
Choose $N \in \mathbb{N}$ with $N\delta > 1$.

In this paragraph, let $k \in \{1, 2, \ldots, n\}$ be given and fixed. We have $P(x) = (P_1(x), P_2(x), \ldots, P_n(x))$ for $x \in I^n$, where $P_j \in C(I^n)$ ($1 \leq j \leq n$). For $r \in \{0, 1, \ldots, N\}$, we identify the slice
\[
S_r = \{ x \in I^n : x_k = r/N \} \]
with $I^{n-1}$ in the obvious way. Thus we write

$$S^n_r = \{ x \in S_r : 0 < x_j < 1 \forall j \neq k \}.$$ 

For each such $r$, we apply our induction hypothesis to the restriction of $P$ to $S_r$ to obtain a continuous function $G_r : S_r \to \mathbb{R}$ and a set $V_r \subset S^n_r$ which is open in $S_r$ such that

$$\lambda^{n-1}(V_r) = 1,$$

$$\frac{\partial G_r}{\partial x_j}$$

exists and is continuous on $V_r \forall j \neq k (1 \leq j \leq n)$, and

$$\left( \sum_{j=1, j \neq k}^{n} \left[ \frac{\partial G_r}{\partial x_j}(x) - P_j(x) \right]^2 \right)^{1/2} < \frac{\varepsilon}{6n} \forall x \in V_r.$$ 

Next we construct a function $Q_k$ on $I^n$ which extends each $G_r$. For $x = (x_1, x_2, \ldots, x_n) \in I^n$, define $x' \in S_k$ by $(x'_j) = x_j$ if $j \neq k$ and $(x'_k) = r/N$. Let $\psi$ be Lebesgue's singular function in terms of Cantor's ternary set $C$ as in (0.1) (7), and define $f_r$ on the slab

$$\{ x \in I^n : (r-1)/N \leq x_k \leq r/N \}$$

for $r = 1, 2, \ldots, N$ by the rule

$$f_r(x) = G_{r-1}(x'_{r-1}) + \int_{x_{r-1}}^{x_r} G_r(x') - G_{r-1}(x'_{r-1}) \cdot \psi(Nx_k - r + 1).$$

Since $\psi(0) = 0$ and $\psi(1) = 1$, we have

$$x \in S_{r-1} \Rightarrow x_k = (r - 1)/N \Rightarrow x'_{r-1} = x \Rightarrow f_r(x) = G_{r-1}(x),$$

and

$$x \in S_{r} \Rightarrow x_k = r/N \Rightarrow x' = x \Rightarrow f_r(x) = G_r(x).$$

Thus the formula

$$Q_k(x) = f_r(x) \quad \text{if } (r-1)/N \leq x_k \leq r/N$$

unambiguously defines the function $Q_k : I^n \to \mathbb{R}$. The continuity of the functions $x \to x'$, each $G_r$, and $\psi$ shows that $Q_k$ is continuous on $I^n$. Define

$$W_k = (I^n)^{\circ} \cap \left( \bigcap_{r=0}^{N} \{ x \in I^n : x_k \in V_r \} \right) \cap \left( \bigcup_{r=1}^{N} \{ x \in I^n : r - 1 \leq Nx_k \leq r, \ (Nx_k - r + 1) \notin C \} \right),$$

where $C$ is Cantor's ternary set. Then $W_k$ is an open subset of $(I^n)^{\circ}$ and $\lambda^n(W_k) = 1$. By (3), (4), and the properties of the functions $G_r$, we see that for $j \neq k$ the partial derivative $\frac{\partial Q_k}{\partial x_j}$ exists and is continuous on $W_k$. Since $\psi' = 0$ on $[0, 1] \setminus C$, we have

$$\frac{\partial Q_k}{\partial x_k}(x) = 0 \forall x \in W_k.$$

Thus $\nabla Q_k$ is continuous on $W_k$. Now let $x \in W_k$ with $r - 1 < Nx_k < r$. Then we have

$$\|x - x'\| < \|x'_{r-1} - x'\| = 1/N < \delta.$$
so we use (1)-(4) and the fact that $0 \leq \psi \leq 1$ to see that $j \neq k$ $(1 \leq j \leq n) \Rightarrow$

$$\left| \frac{\partial Q_k}{\partial x_j}(x) - \frac{\partial G_{r-1}}{\partial x_j}(x^{r-1}) \right| = \left| \frac{\partial f_r}{\partial x_j}(x) - \frac{\partial G_{r-1}}{\partial x_j}(x^{r-1}) \right|$$

$$= \left| \left[ \frac{\partial G_r}{\partial x_j}(x^r) - \frac{\partial G_{r-1}}{\partial x_j}(x^{r-1}) \right] \cdot \psi(Nx_k - r + 1) \right|$$

$$\leq \left| \frac{\partial G_r}{\partial x_j}(x^r) - P_j(x^r) \right| + \left| P_j(x^r) - P_j(x^{r-1}) \right|$$

$$+ \left| P_j(x^{r-1}) - \frac{\partial G_{r-1}}{\partial x_j}(x^{r-1}) \right|$$

$$< \frac{\varepsilon}{6n} + \frac{\varepsilon}{6n} + \frac{\varepsilon}{2n} = \varepsilon,$$

and

$$\left| \frac{\partial G_{r-1}}{\partial x_j}(x^{r-1}) - P_j(x) \right| \leq \left| \frac{\partial G_{r-1}}{\partial x_j}(x^{r-1}) - P_j(x^{r-1}) \right| + \left| P_j(x^{r-1}) - P_j(x) \right|$$

$$< \frac{\varepsilon}{6n} + \frac{\varepsilon}{6n} < \frac{\varepsilon}{2n}.$$

This shows that

$$(6) \left| \frac{\partial Q_k}{\partial x_j}(x) - P_j(x) \right| < \frac{\varepsilon}{n} \text{ if } x \in W_k \text{ and } j \neq k \ (1 \leq j \leq n).$$

Define $P^k : I^n \to \mathbb{R}^n$ by

$$P^k(x) = (P^k_1(x), P^k_2(x), \ldots, P^k_n(x)),$$

where $P^k_j = P_j$ if $j \neq k$ and $P^k_k = 0$. Then (5) and (6) yield that $x \in W_k \Rightarrow$

$$\|\nabla Q_k(x) - P^k(x)\| = \left( \sum_{j=1, j \neq k}^n \left| \frac{\partial Q_k}{\partial x_j}(x) - P_j(x) \right|^2 \right)^{1/2}$$

$$\leq \left( (n-1) \cdot \left( \frac{\varepsilon}{n} \right)^2 \right)^{1/2} \leq \frac{n-1}{n} \varepsilon .$$

Thus we have constructed $Q_k$ and $W_k$ for each $k \in \{1, 2, \ldots, n\}$. Finally, notice that $P = \frac{1}{n-1} \sum_{k=1}^n P^k$ and define

$$V = \bigcap_{k=1}^n W_k \text{ and } Q = \frac{1}{n-1} \sum_{k=1}^n Q_k .$$

Then $V$ is an open subset of $(I^n)^\circ$, $\lambda^n(V) = 1$, $Q \in C(I^n)$, and

(i) $\nabla Q$ is defined and continuous on $V$.

Also, (7) yields $x \in V \Rightarrow$

$$\|\nabla Q(x) - P(x)\| \leq \frac{1}{n-1} \sum_{k=1}^n \|\nabla Q_k(x) - P^k(x)\| < \frac{1}{n-1} \sum_{k=1}^n \frac{n-1}{n} \varepsilon = \varepsilon .$$

This is (ii). We have completed the proof. $\square$
The next theorem tells us what can be done if the mapping \( P \) in the above theorem is only measurable.

**Theorem.** Let \( n \in \mathbb{N} \) and let \( \phi \in \mathcal{M}(I^n, \mathbb{R}^n) \). Then there exists a sequence \( \{Q_k\}_{k=1}^\infty \subset C(I^n) \) such that

(i) \( \nabla Q_k \) exists and is continuous on some open set \( V_k \subset (I^n)^\circ \) with \( \lambda^n(V_k) = 1 \) \( \forall k \in \mathbb{N} \), and

(ii) \( \lim_{k \to \infty} \nabla Q_k(x) = \phi(x) \) for almost every \( x \in \bigcap_{k=1}^\infty V_k \).

**Proof.** Let \( \phi = (\phi_1, \phi_2, \ldots, \phi_n) \) as we mentioned in (0.1).

By Luzin's theorem, for each \( k \in \mathbb{N} \) and \( i \in \{1, 2, \ldots, n\} \) there exist an open set \( E_{k,i} \subset I^n \) with \( \lambda^n(E_{k,i}) < 2^{-k}/n \) and a function \( f_{k,i} \in C(I^n) \) such that

\[
f_{k,i}(x) = \phi_i(x) \quad \forall x \in I^n \setminus E_{k,i}. \]

Let \( E_k = \bigcup_{i=1}^n E_{k,i} \). Then \( \lambda^n(E_k) < 2^{-k} \) and

\[
f_{k,i}(x) = \phi_i(x) \quad \forall x \in I^n \setminus E_k, \quad 1 \leq i \leq n. \]

Take \( f_k = (f_{k,1}, f_{k,2}, \ldots, f_{k,n}) \). Then \( f_k \in C(I^n, \mathbb{R}^n) \) and

(1) \( f_k(x) = \phi(x) \quad \forall x \in I^n \setminus E_k \) for each \( k \in \mathbb{N} \).

Theorem (1.1) says that, for each \( k \in \mathbb{N} \), there exist a \( Q_k \in C(I^n) \) and an open set \( V_k \subset (I^n)^\circ \) with \( \lambda^n(V_k) = 1 \) such that \( \nabla Q_k \) exists and is continuous on \( V_k \) and

(2) \( \|\nabla Q_k(x) - f_k(x)\| < 2^{-k} \quad \forall x \in V_k. \)

Define \( E = \bigcap_{r=1}^\infty \bigcup_{k=r}^\infty E_k \) and \( V = \bigcap_{k=1}^\infty V_k \). For each \( r \in \mathbb{N} \), we have

\[
\lambda^n(E) \leq \lambda^n \left( \bigcup_{k=r}^\infty E_k \right) \leq \sum_{k=r}^\infty \lambda^n(E_k) \leq 2^{-r+1},
\]

so \( \lambda^n(E) = 0 \). Obviously \( \lambda^n(V) = 1 \).

Now let \( x \in V \setminus E \) be given. Then

\[
x \in \bigcap_{k=r_x}^\infty (V \setminus E_k)
\]

for some \( r_x \in \mathbb{N} \). Thus

\[
f_{k,i}(x) = \phi_i(x) \quad \forall k \geq r_x, \quad 1 \leq i \leq n,
\]

and by (2) we have

\[
\|\nabla Q_k(x) - f_k(x)\| < 2^{-k} \quad \forall k \geq r_x.
\]

Thus (1) yields

\[
\|\nabla Q_k(x) - \phi(x)\| < 2^{-k} \quad \forall k \geq r_x.
\]

Hence we have

\[
\lim_{k \to \infty} \nabla Q_k(x) = \phi(x).
\]

Since \( x \in V \setminus E \) was arbitrary and \( \lambda^n(V \setminus E) = 1 \), we are done. \( \square \)

As a consequence of (1.2), we obtain an interesting dense subspace of the \( F \)-space \( \mathcal{M}(I^n, \mathbb{R}^n) \) with its metric of coordinatewise convergence in measure.
1206 X.-X. Gan and K. R. Stromberg

(1.3) **Corollary.** For \( f = (f_1, f_2, \ldots, f_n) \) and \( g = (g_1, g_2, \ldots, g_n) \) in \( \mathcal{M}(I^n, \mathbb{R}^n) \) put

\[
\rho(f, g) = \max_{1 \leq i \leq n} \frac{1}{\int_{I^n} 1 + |f_j - g_j|} \ d\lambda^n.
\]

Then \( \rho \) is a complete invariant metric for \( \mathcal{M}(I^n, \mathbb{R}^n) \) and the linear subspace \( \{VF : F \in C(I^n) \text{ and } \nabla F \text{ exists and is continuous on an open set } V \subset (I^n)^{o} \} \) is dense in this space.

**Proof.** The completeness of \( \rho \) is immediate from the well-known special case \( n = 1 \) in which \( \rho \)-convergence is equivalent to convergence in measure (see [3, p. 93, Theorem E] for completeness). Since a.e. convergence on a finite measure space implies convergence in measure (see [4, (11.31)]), we need only apply (1.2). \( \square \)

2. **Density in \( C(I^n) \) of approximate antigradients**

In this section we present our main theorem as stated in the abstract above. It improves Theorem (1.1) in that it shows that, for a given \( P \) and \( \varepsilon \), the set of \( Q \)'s that satisfy the conclusion of (1.1) is dense in \( C(I^n) \). We need the results of this section for our work on universal primitives that is in preparation.

We begin with three lemmas.

(2.1) **Lemma.** Let \( n \in \mathbb{N} \) be given and let \( \{0, 1\}^n = \{ u \in \mathbb{R}^n : u_j \in \{0, 1\}, \ j = 1, 2, \ldots, n \} \). Suppose that \( y_u \in \mathbb{R} \) is given \( \forall u \in \{0, 1\}^n \), and let \( m \) be the minimum and \( M \) the maximum of \( \{y_u : u \in \{0, 1\}^n\} \). Then there exist a function \( f \in C(I^n) \) and an open set \( V \subset (I^n)^{o} \) with \( \lambda^n(V) = 1 \) such that

(i) \( \nabla f(x) = 0 \ \forall x \in V \),
(ii) \( f(u) = y_u \ \forall u \in \{0, 1\}^n \), and
(iii) \( m \leq f(x) \leq M \ \forall x \in I^n \).

**Proof.** If \( n = 1 \), let \( f \) be defined by

\[
f(x) = y_0 + (y_1 - y_0) \cdot \psi(x) \ \forall x \in I = [0, 1],
\]

and take \( V = I \setminus C \), where \( \psi \) is Lebesgue's singular function and \( C \) is Cantor's ternary set. Then (i)–(iii) are obvious.

Suppose, as an induction hypothesis, that \( n > 1 \) and the theorem is true if \( n \) is replaced with \( n - 1 \).

For \( l \in \{0, 1\} \), we identify the slice

\[
S_l = \{ x \in I^n : x_n = l \}
\]

with \( I^{n-1} \) in the obvious way and write

\[
S_l^o = \{ x \in S_l : 0 < x_j < 1, \ 1 \leq j \leq n - 1 \},
\]

\[
A_l = \{ u \in \{0, 1\}^n : u_n = l \}.
\]

By the induction hypothesis, we obtain an \( f_l \in C(S_l) \) and an open set \( V_l \subset S_l^o \) with \( \lambda^{n-1}(V_l) = 1 \) \( (l = 0, 1) \) such that

\[
\frac{\partial f_l}{\partial x_j}(x) = 0 \ \forall x \in V_l, \ 1 \leq j < n,
\]

\[
f_l(u) = y_u \ \forall u \in A_l \quad \text{and} \quad m \leq f_l(x) \leq M \ \forall x \in S_l.
\]
For any $x = (x_1, x_2, \ldots, x_n) \in I^n$, define $x^l \in S_l$ by $(x^l)_j = x_j$ if $j < n$ and $(x^l)_n = l$ for $l = 0, 1$. Define $f: I^n \to \mathbb{R}$ by

$$f(x) = f_0(x^0) + [f_1(x^1) - f_0(x^0)] \cdot \psi(x_n) = [1 - \psi(x_n)] \cdot f_0(x^0) + \psi(x_n) \cdot f_1(x^1).$$

Also define

$$V = (I^n)^o \cap \left( \bigcap_{j=0}^{1} \{x \in I^n : x^l \in V_j \} \right) \cap \{x \in I^n : x_n \notin C \}.$$}

It is obvious that $V$ is open and $\lambda^n(V) = 1$. The continuity of the functions $x \to x^l$ ($l = 0, 1$), each $f_j$, and $\psi$ ensures that $f$ is continuous on $I^n$. Plainly, $\nabla f(x) = 0 \forall x \in V$ by the definition of $f$ and the definition of $V$. This proves (i).

If $u \in A_0$, then $u_n = 0$, $u = u^0$, and $\psi(u_n) = 0$, hence $f(u) = f_0(u) = y_u$. If $u \in A_1$, then $u_n = 1$, $u = u^1$, and $f(u) = f_1(u) = y_u$. Thus (ii) holds.

Since $0 \leq \psi \leq 1$, $f(x)$ is between $f_0(x^0)$ and $f_1(x^1)$ so (iii) holds too. □

(2.2) Remark. It is not difficult to replace $I^n$ with any closed interval $I = [a, b] = \{x \in \mathbb{R}^n : a \leq x_j \leq b_j, \ 1 \leq j \leq n\}$, where $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$.

(2.3) Lemma. Let $n \in \mathbb{N}$, let $G \in C(I^n)$, and let $\varepsilon > 0$. Then there exist a function $H \in C(I^n)$ and an open set $V \subset (I^n)^o$ with $\lambda^n(V) = 1$ such that

(i) $\nabla H(x) = 0 \forall x \in V$,

(ii) $\|G - H\|_u < \varepsilon$,

where $\|F\|_u = \sup\{\|F(x)\| : x \in I^n\}$ for any $F \in C(I^n)$.

Proof. Suppose $n = 1$. Find $N \in \mathbb{N}$ such that $|G(u) - G(v)| < \varepsilon/2$ if $u, v \in I$ and $|u - v| < 1/N$. As before, let $C$ be Cantor's ternary set and let $\psi$ be Lebesgue's singular function. Define

$$H(x) = G \left( \frac{k}{N} \right) + \left[ G \left( \frac{k + 1}{N} \right) - G \left( \frac{k}{N} \right) \right] \cdot \psi(Nx - k)$$

if $k \in \{0, 1, \ldots, N - 1\}$ and $\frac{k}{N} \leq x \leq \frac{k + 1}{N}$, also define

$$V = \bigcup_{k=0}^{N-1} \{x \in I : k \leq Nx \leq k + 1, \ (Nx - k) \notin C \}.$$}

Notice that $H(\frac{k}{N}) = G(\frac{k}{N})$ for $k = 0, 1, \ldots, N$, $H \in C(I)$, $\lambda(V) = 1$, and $k \leq Nx \leq k + 1 \Rightarrow$

$$|H(x) - G(x)| \leq \left| H(x) - G \left( \frac{k}{N} \right) \right| + \left| G \left( \frac{k + 1}{N} \right) - G(x) \right|$$

$$< \left| G \left( \frac{k + 1}{N} \right) - G \left( \frac{k}{N} \right) \right| \cdot \psi(Nx - k) + \frac{\varepsilon}{2} < \varepsilon.$$}

Plainly, $x \in V \Rightarrow \nabla H(x) = H'(x) = 0$ so the lemma holds for $n = 1$.

Suppose $n > 1$ and, as an induction hypothesis, that the lemma is true if $n$ is replaced by $n - 1$. The uniform continuity of $G$ provides $\delta > 0$ such that

(1) $u, v \in I^n, \ |u - v| < \delta \Rightarrow \|G(u) - G(v)\| < \varepsilon/8$. 

Choose $N \in \mathbb{N}$ with $N\delta > 1$. For any $r \in \{0, 1, \ldots, N\}$, we identify the slice $S_r = \{x \in I^n : x_n = r/n\}$ with $I^{n-1}$ in the obvious way, and we write $S_r^c = \{x \in S_r : 0 < x_j < 1 \; \forall j < n\}$.

For each such $r$, we apply our induction hypothesis to the restriction of $G$ to $S_r$ to obtain a continuous function $G_r \in C(S_r)$ and an open set $V_r \subset S_r^c$ which is open in $S_r$ such that

$$\lambda^{n-1}(V_r) = 1,$$

$$\frac{\partial G_r}{\partial x_j}(x) = 0 \; \forall x \in V_r \; (1 \leq j \leq n), \text{ and}$$

$$|G(x) - G_r(x)| < \frac{\varepsilon}{8} \; \forall x \in S_r.$$

Next we construct $H$ on $I^n$ which extends each $G_r$ and satisfies the requirements of the lemma.

The construction of $H$ is exactly the same as that of $Q_n$ in the proof of Theorem (1.1). That is,

$$H(x) = G_{r-1}(x^{r-1}) + [G_r(x^r) - G_{r-1}(x^{r-1})] \cdot \psi(N x_n - r + 1)$$

if $(r-1)/N \leq x_n \leq r/N$. Also define

$$V = (I^n)^c \cap \left( \bigcap_{r=0}^{N} \{x \in I^n : x^r \in V_r\} \right)$$

$$\cap \left( \bigcup_{r=1}^{N} \{x \in I^n : r - 1 \leq N x_n \leq r, (N x_n - r + 1) \notin C\} \right),$$

where $C$ is Cantor's ternary set. Plainly $V$ is open, $\lambda^n(V) = 1$, $H \in C(I^n)$, and $\nabla H(x) = 0 \; \forall x \in V$.

Now let $x \in V$ with $r - 1 < N x_n < r$. Then we have

$$\|x - x^r\| < \|x^{r-1} - x^r\| = 1/N < \delta \; \text{ and also } \|x - x^{r-1}\| < \delta.$$

From (1) and (2) we have

$$|H(x) - G(x)| = |G_{r-1}(x^{r-1}) + [G_r(x^r) - G_{r-1}(x^{r-1})] \cdot \psi(N x_n - r + 1) - G(x)|$$

$$\leq |G_{r-1}(x^{r-1}) - G(x^{r-1})| + |G(x^{r-1}) - G(x)|$$

$$+ |G_r(x^r) - G_{r-1}(x^{r-1})| \cdot |\psi(N x_n - r + 1)|$$

$$\leq \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + |G_r(x^r) - G_{r-1}(x^{r-1})|$$

$$\leq \frac{\varepsilon}{4} + |G_r(x^r) - G(x^r)| + |G(x^r) - G(x^{r-1})|$$

$$+ |G(x^{r-1}) - G_{r-1}(x^{r-1})|$$

$$\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} < \varepsilon.$$

Thus $\|H - G\|_u < \varepsilon$ on $I^n$. This completes the proof. □

(2.4) **Lemma.** Let $n \in \mathbb{N}$ and let $Q \in C(I^n)$ be such that there exists an open set $V \subset (I^n)^c$ with $\lambda^n(V) = 1$ on which $\nabla Q$ exists and is continuous. Suppose
that $F_0 \in C(I^n)$ and $\varepsilon > 0$. Then there exist $F \in C(I^n)$ and an open set $W \subset V$ with $\lambda^n(W) = 1$ such that

(i) $\nabla F$ exists and is continuous on $W$,

(ii) $\nabla F(x) = \nabla Q(x)$ $\forall x \in W$, and

(iii) $\|F - F_0\|_u < \varepsilon$.

Proof. Apply Lemma (2.3) to $G = F_0 - Q$ to obtain $H \in C(I^n)$ and an open set $V_0 \subset (I^n)^o$ with $\lambda^n(V_0) = 1$ such that $\|H - G\|_u < \varepsilon$ and $\nabla H(x) = 0$ $\forall x \in V_0$. Take $F = Q + H$ and $W = V \cap V_0$ to complete the proof.

(2.5) **Main Theorem.** Let $n \in \mathbb{N}$, let $f \in C(I^n, \mathbb{R}^n)$, let $F_0 \in C(I^n)$, and let $\varepsilon > 0$. Then there exist $F \in C(I^n)$ and an open set $W \subset (I^n)^o$ with $\lambda^n(W) = 1$ such that

(i) $\nabla F$ exists and is continuous on $W$,

(ii) $\|\nabla F(x) - f(x)\| < \varepsilon$ $\forall x \in W$, and

(iii) $\|F - F_0\|_u < \varepsilon$,

where $\|y\| = (\sum_{j=1}^n y_j^2)^{1/2}$ $\forall y \in \mathbb{R}^n$.

Proof. Take $P = f$ in (1.1) to produce a $Q$ and then use (2.4) to replace $Q$ with $F$. □

**REFERENCES**


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