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ON APPROXIMATE ANTIGRADIENTS

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ABSTRACT. For $n \in \mathbb{N}$ and $I = [0, 1]$, let I^n be the unit cube and λ^n the Lebesgue measure in $\mathbb{R}^n$. It is proved that if $f: I^n \to \mathbb{R}^n$ and $F_0: I^n \to \mathbb{R}$ are continuous and $\varepsilon > 0$, then there exist a continuous $F: I^n \to \mathbb{R}$ and an open set $W \subset (I^n)^\circ$ with $\lambda^n(W) = 1$ such that

(i) ∇F exists and is continuous on W,

(ii) $\|\nabla F(x) - f(x)\| < \varepsilon \quad \forall x \in W$, and

(iii) $|F(x) - F_0(x)| < \varepsilon \quad \forall x \in I^n$, where $\|y\| = (\sum_{j=1}^n y_j^2)^{1/2} \quad \forall y \in \mathbb{R}^n$.

0. INTRODUCTION AND DEFINITIONS

If $f \in C([0, 1])$ and $F(x) = \int_0^x f$ for $0 \le x \le 1$, then $F' = f$. A famous theorem of Lusin states that if $\phi: I \to \mathbb{R}$ is a measurable function, then there exists $F \in C([0, 1])$ such that $F' = \phi$ a.e. on $[0, 1]$ (see [1, p. 217]). However, if $f(x, y) = (0, x)$ for $(x, y) \in I^2$, then there is no $F: I^2 \to \mathbb{R}$ for which $\nabla F = f$, because $D_1 F = 0$ implies that F depends only on y. Thus, if $n > 1$ and $f \in C(I^n, \mathbb{R}^n)$, then f need not have an antigradient $F: \nabla F = f$. Can we find some kind of approximate antigradient for f? More generally, if $f: I^n \to \mathbb{R}^n$ is only measurable, what can be said? We are going to give some answers to these questions.

Throughout this paper, we use some definitions listed below.

(0.1) **Definitions.** Let $m, n \in \mathbb{N}$ and a nonvoid set Ω be given.

(1) If $F: \Omega \to \mathbb{R}^m$ is a mapping, then for $1 \le i \le m$, define the ith coordinate function $F_i: \Omega \to \mathbb{R}$ of F by letting $F_i(t)$ be the ith coordinate of $F(t)$:

$$F(t) = (F_1(t), F_2(t), \dots, F_m(t)), \qquad t \in \Omega.$$

(2) If Ω is a topological space, then $C(\Omega, \mathbb{R}^m)$ denotes the family of all continuous mappings from Ω to $\mathbb{R}^m$. If $m = 1$, write $C(\Omega)$ for $C(\Omega, \mathbb{R})$. Similarly, if $\Omega \subset \mathbb{R}^n$, we denote by $\mathfrak{M}(\Omega, \mathbb{R}^m)$ the set of all Lebesgue measurable mappings from Ω to $\mathbb{R}^m$ and write $\mathfrak{M}(\Omega)$ for $\mathfrak{M}(\Omega, \mathbb{R})$.

(3) Let $\Omega \subset \mathbb{R}^n$ and let $f \in C(\Omega)$. If $\frac{\partial f}{\partial x_j}(x)$ exists at some $x \in \Omega^\circ$ (the

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interior of Ω) $\forall 1 \leq j \leq n$, we define

$$\nabla f(x) = \left(\frac{\partial}{\partial x_1} f(x), \frac{\partial}{\partial x_2} f(x), \ldots, \frac{\partial}{\partial x_n} f(x)\right),$$

and call ∇f the *gradient* of the function f. We also write D_j for $\frac{\partial}{\partial x_j}$.

Conversely, let $f \in \mathfrak{M}(\Omega, \mathbb{R}^n)$. If there is $F \in C(\Omega)$ such that $\nabla F = f$, we call F an *antigradient* of f.

(4) We always denote by I^n the unit cube in $\mathbb{R}^n$. That is

$$I^n = \mathop{\times}_{k=1}^{n} [0, 1] = \{x \in \mathbb{R}^n : 0 \leq x_j \leq 1, \ j = 1, 2, \ldots, n\}.$$

Also, we always denote by λ^n the Lebesgue measure on $\mathbb{R}^n$.

(5) Let $\Omega \subset \mathbb{R}^n$ and $\phi = (\phi_1, \phi_2, \ldots, \phi_m) \in \mathfrak{M}(\Omega, \mathbb{R}^m)$. If $\varepsilon > 0$, then we say $\|(\phi_1, \phi_2, \ldots, \phi_m)\| < \varepsilon$ a.e. on Ω to mean that

$$\lambda^n(\{x \in \Omega : \|(\phi_1(x), \phi_2(x), \ldots, \phi_m(x))\| \geq \varepsilon\}) = 0,$$

where $\|y\| = (\sum_{i=1}^m y_i^2)^{1/2}$ if $y \in \mathbb{R}^m$.

(6) For any $F : \Omega \to \mathbb{R}$, we define the *uniform norm* of F by

$$\|F\|_u = \sup\{|F(x)| : x \in \Omega\}.$$

(7) We reserve the letter ψ to denote Lebesgue's singular function (see [2, p. 130] or [4, p. 113]). It is a continuous, nondecreasing function $\mathbb{R} \to [0, 1]$ with $\psi(0) = 0$ and $\psi(1) = 1$ which is constant on each component interval of $\mathbb{R} \setminus C$, where C is Cantor's ternary set.

1. Approximation to antigradients

(1.1) **Theorem.** *Let $n \in \mathbb{N}$, let $P \in C(I^n, \mathbb{R}^n)$, and let $\varepsilon > 0$. Then there exist a continuous $Q : I^n \to \mathbb{R}$ and an open $V \subset (I^n)^\circ$ with $\lambda^n(V) = 1$ such that*

(i) *∇Q exists and is continuous on V, and*

(ii) *$\|\nabla Q(x) - P(x)\| < \varepsilon \ \forall x \in V$,*

where $\|y\| = (\sum_{j=1}^n y_j^2)^{1/2} \ \forall y \in \mathbb{R}^n$.

Proof. If $n = 1$, define

$$Q(x) = \int_0^x P(t)\, dt \qquad (0 \leq x \leq 1)$$

and take $V =]0, 1[= I^\circ$. Then $\nabla Q = Q' = P$ on V so the theorem is true if $n = 1$.

Suppose, as an induction hypothesis, that $n > 1$ and that the theorem is true if n is replaced with $n - 1$. The uniform continuity of P provides $\delta > 0$ such that

$$u, v \in I^n, \ \|u - v\| < \delta \Rightarrow \|P(u) - P(v)\| < \varepsilon/6n. \tag{1}$$

Choose $N \in \mathbb{N}$ with $N\delta > 1$.

In this paragraph, let $k \in \{1, 2, \ldots, n\}$ be given and fixed. We have $P(x) = (P_1(x), P_2(x), \ldots, P_n(x))$ for $x \in I^n$, where $P_j \in C(I^n)$ $(1 \leq j \leq n)$. For $r \in \{0, 1, \ldots, N\}$, we identify the *slice*

$$S_r = \{x \in I^n : x_k = r/N\}$$

with I^{n-1} in the obvious way. Thus we write

$$S_r^\circ = \{x \in S_r : 0 < x_j < 1 \ \forall j \neq k\}.$$

For each such r, we apply our induction hypothesis to the restriction of P to S_r to obtain a continuous function $G_r \colon S_r \to \mathbb{R}$ and a set $V_r \subset S_r^\circ$ which is open in S_r such that

$$(2)\qquad \begin{aligned} &\lambda^{n-1}(V_r) = 1, \\ &\frac{\partial G_r}{\partial x_j} \text{ exists and is continuous on } V_r \ \forall j \neq k \ (1 \le j \le n), \text{ and} \\ &\left(\sum_{j=1,\, j\neq k}^{n} \left[\frac{\partial G_r}{\partial x_j}(x) - P_j(x)\right]^2\right)^{1/2} < \frac{\varepsilon}{6n} \quad \forall x \in V_r. \end{aligned}$$

Next we construct a function Q_k on I^n which extends each G_r. For $x = (x_1, x_2, \dots, x_n) \in I^n$, define $x^r \in S_r$ by $(x^r)_j = x_j$ if $j \neq k$ and $(x^r)_k = r/N$. Let ψ be Lebesgue's singular function in terms of Cantor's ternary set C as in (0.1) (7), and define f_r on the *slab*

$$\{x \in I^n : (r-1)/N \le x_k \le r/N\}$$

for $r = 1, 2, \dots, N$ by the rule

$$(3)\qquad f_r(x) = G_{r-1}(x^{r-1}) + [G_r(x^r) - G_{r-1}(x^{r-1})] \cdot \psi(Nx_k - r + 1).$$

Since $\psi(0) = 0$ and $\psi(1) = 1$, we have

$$x \in S_{r-1} \Rightarrow x_k = (r-1)/N \Rightarrow x^{r-1} = x \Rightarrow f_r(x) = G_{r-1}(x),$$

and

$$x \in S_r \Rightarrow x_k = r/N \Rightarrow x^r = x \Rightarrow f_r(x) = G_r(x).$$

Thus the formula

$$(4)\qquad Q_k(x) = f_r(x) \quad \text{if } (r-1)/N \le x_k \le r/N$$

unambiguously defines the function $Q_k \colon I^n \to \mathbb{R}$. The continuity of the functions $x \to x^r$, each G_r, and ψ shows that Q_k is continuous on I^n. Define

$$\begin{aligned} W_k = (I^n)^\circ &\cap \left(\bigcap_{r=0}^{N} \{x \in I^n : x^r \in V_r\}\right) \\ &\cap \left(\bigcup_{r=1}^{N} \{x \in I^n : r-1 \le Nx_k \le r,\ (Nx_k - r + 1) \notin C\}\right), \end{aligned}$$

where C is Cantor's ternary set. Then W_k is an open subset of $(I^n)^\circ$ and $\lambda^n(W_k) = 1$. By (3), (4), and the properties of the functions G_r, we see that for $j \neq k$ the partial derivative $\frac{\partial Q_k}{\partial x_j}$ exists and is continuous on W_k. Since $\psi' = 0$ on $[0, 1] \setminus C$, we have

$$(5)\qquad \frac{\partial Q_k}{\partial x_k}(x) = 0 \quad \forall x \in W_k.$$

Thus ∇Q_k is continuous on W_k. Now let $x \in W_k$ with $r - 1 < Nx_k < r$. Then we have

$$\|x - x^r\| < \|x^{r-1} - x^r\| = 1/N < \delta$$

so we use (1)–(4) and the fact that $0 \le \psi \le 1$ to see that $j \ne k$ $(1 \le j \le n) \Rightarrow$

$$\begin{aligned}\left|\frac{\partial Q_k}{\partial x_j}(x) - \frac{\partial G_{r-1}}{\partial x_j}(x^{r-1})\right| &= \left|\frac{\partial f_r}{\partial x_j}(x) - \frac{\partial G_{r-1}}{\partial x_j}(x^{r-1})\right| \\ &= \left|\left[\frac{\partial G_r}{\partial x_j}(x^r) - \frac{\partial G_{r-1}}{\partial x_j}(x^{r-1})\right] \cdot \psi(Nx_k - r + 1)\right| \\ &\le \left|\frac{\partial G_r}{\partial x_j}(x^r) - P_j(x^r)\right| + |P_j(x^r) - P_j(x^{r-1})| \\ &\quad + \left|P_j(x^{r-1}) - \frac{\partial G_{r-1}}{\partial x_j}(x^{r-1})\right| \\ &< \frac{\varepsilon}{6n} + \frac{\varepsilon}{6n} + \frac{\varepsilon}{6n} = \frac{\varepsilon}{2n},\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\partial G_{r-1}}{\partial x_j}(x^{r-1}) - P_j(x)\right| &\le \left|\frac{\partial G_{r-1}}{\partial x_j}(x^{r-1}) - P_j(x^{r-1})\right| + |P_j(x^{r-1}) - P_j(x)| \\ &< \frac{\varepsilon}{6n} + \frac{\varepsilon}{6n} < \frac{\varepsilon}{2n}.\end{aligned}$$

This shows that

$$\left|\frac{\partial Q_k}{\partial x_j}(x) - P_j(x)\right| < \frac{\varepsilon}{n} \quad \text{if } x \in W_k \text{ and } j \ne k \ (1 \le j \le n). \tag{6}$$

Define $P^k \colon I^n \to \mathbb{R}^n$ by

$$P^k(x) = (P_1^k(x), P_2^k(x), \ldots, P_n^k(x)),$$

where $P_j^k = P_j$ if $j \ne k$ and $P_k^k = 0$. Then (5) and (6) yield that $x \in W_k \Rightarrow$

$$\begin{aligned}\|\nabla Q_k(x) - P^k(x)\| &= \left(\sum_{j=1,\, j\ne k}^{n} \left[\frac{\partial Q_k}{\partial x_j}(x) - P_j(x)\right]^2\right)^{1/2} \\ &\le \left((n-1)\cdot\left(\frac{\varepsilon}{n}\right)^2\right)^{1/2} \le \frac{n-1}{n}\varepsilon.\end{aligned} \tag{7}$$

Thus we have constructed Q_k and W_k for each $k \in \{1, 2, \ldots, n\}$.

Finally, notice that $P = \frac{1}{n-1}\sum_{k=1}^{n} P^k$ and define

$$V = \bigcap_{k=1}^{n} W_k \quad \text{and} \quad Q = \frac{1}{n-1}\sum_{k=1}^{n} Q_k.$$

Then V is an open subset of $(I^n)^\circ$, $\lambda^n(V) = 1$, $Q \in C(I^n)$, and

(i) ∇Q is defined and continuous on V.

Also, (7) yields $x \in V \Rightarrow$

$$\|\nabla Q(x) - P(x)\| \le \frac{1}{n-1}\sum_{k=1}^{n}\|\nabla Q_k(x) - P^k(x)\| < \frac{1}{n-1}\sum_{k=1}^{n}\frac{n-1}{n}\varepsilon = \varepsilon.$$

This is (ii). We have completed the proof. □

The next theorem tells us what can be done if the mapping P in the above theorem is only measurable.

(1.2) **Theorem.** *Let $n \in \mathbb{N}$ and let $\phi \in \mathfrak{M}(I^n, \mathbb{R}^n)$. Then there exists a sequence $\{Q_k\}_{k=1}^{\infty} \subset C(I^n)$ such that*

(i) *∇Q_k exists and is continuous on some open set $V_k \subset (I^n)^\circ$ with $\lambda^n(V_k) = 1$ $\forall k \in \mathbb{N}$, and*

(ii) *$\lim_{k\to\infty} \nabla Q_k(x) = \phi(x)$ for almost every $x \in \bigcap_{k=1}^{\infty} V_k$.*

Proof. Let $\phi = (\phi_1, \phi_2, \dots, \phi_n)$ as we mentioned in (0.1).

By Luzin's theorem, for each $k \in \mathbb{N}$ and $i \in \{1, 2, \dots, n\}$ there exist an open set $E_{k,i} \subset I^n$ with $\lambda^n(E_{k,i}) < 2^{-k}/n$ and a function $f_{k,i} \in C(I^n)$ such that

$$f_{k,i}(x) = \phi_i(x) \quad \forall x \in I^n \setminus E_{k,i}.$$

Let $E_k = \bigcup_{i=1}^{n} E_{k,i}$. Then $\lambda^n(E_k) < 2^{-k}$ and

$$f_{k,i}(x) = \phi_i(x) \quad \forall x \in I^n \setminus E_k, \qquad 1 \le i \le n.$$

Take $f_k = (f_{k,1}, f_{k,2}, \dots, f_{k,n})$. Then $f_k \in C(I^n, \mathbb{R}^n)$ and

$$f_k(x) = \phi(x) \quad \forall x \in I^n \setminus E_k \text{ for each } k \in \mathbb{N}. \tag{1}$$

Theorem (1.1) says that, for each $k \in \mathbb{N}$, there exist a $Q_k \in C(I^n)$ and an open set $V_k \subset (I^n)^\circ$ with $\lambda^n(V_k) = 1$ such that ∇Q_k exists and is continuous on V_k and

$$\|\nabla Q_k(x) - f_k(x)\| < 2^{-k} \quad \forall x \in V_k. \tag{2}$$

Define $E = \bigcap_{r=1}^{\infty} \bigcup_{k=r}^{\infty} E_k$ and $V = \bigcap_{k=1}^{\infty} V_k$. For each $r \in \mathbb{N}$, we have

$$\lambda^n(E) \le \lambda^n\left(\bigcup_{k=r}^{\infty} E_k\right) \le \sum_{k=r}^{\infty} \lambda^n(E_k) \le 2^{-r+1},$$

so $\lambda^n(E) = 0$. Obviously $\lambda^n(V) = 1$.

Now let $x \in V \setminus E$ be given. Then

$$x \in \bigcap_{k=r_x}^{\infty} (V \setminus E_k)$$

for some $r_x \in \mathbb{N}$. Thus

$$f_{k,i}(x) = \phi_i(x) \quad \forall k \ge r_x, \quad 1 \le i \le n,$$

and by (2) we have

$$\|\nabla Q_k(x) - f_k(x)\| < 2^{-k} \quad \forall k \ge r_x.$$

Thus (1) yields

$$\|\nabla Q_k(x) - \phi(x)\| < 2^{-k} \quad \forall k \ge r_x.$$

Hence we have

$$\lim_{k\to\infty} \nabla Q_k(x) = \phi(x).$$

Since $x \in V \setminus E$ was arbitrary and $\lambda^n(V \setminus E) = 1$, we are done. □

As a consequence of (1.2), we obtain an interesting dense subspace of the F-space $\mathfrak{M}(I^n, \mathbb{R}^n)$ with its metric of coordinatewise convergence in measure.

(1.3) **Corollary.** *For* $f = (f_1, f_2, \dots, f_n)$ *and* $g = (g_1, g_2, \dots, g_n)$ *in* $\mathfrak{M}(I^n, \mathbb{R}^n)$ *put*

$$\rho(f, g) = \max_{1\le j\le n} \int_{I^n} \frac{|f_j - g_j|}{1 + |f_j - g_j|}\, d\lambda^n .$$

Then ρ *is a complete invariant metric for* $\mathfrak{M}(I^n, \mathbb{R}^n)$ *and the linear subspace* $\{\nabla F : F \in C(I^n)$ *and* ∇F *exists and is continuous on an open set* $V \subset (I^n)^\circ$ *with* $\lambda^n(V) = 1\}$ *is dense in this space.*

Proof. The completeness of ρ is immediate from the well-known special case $n = 1$ in which ρ-convergence is equivalent to convergence in measure (see [3, p. 93, Theorem E] for completeness). Since a.e. convergence on a finite measure space implies convergence in measure (see [4, (11.31)]), we need only apply (1.2). □

2. Density in $C(I^n)$ of approximate antigradients

In this section we present our main theorem as stated in the abstract above. It improves Theorem (1.1) in that it shows that, for a given P and ε, the set of Q's that satisfy the conclusion of (1.1) is dense in $C(I^n)$. We need the results of this section for our work on *universal primitives* that is in preparation.

We begin with three lemmas.

(2.1) **Lemma.** *Let* $n \in \mathbb{N}$ *be given and let* $\{0, 1\}^n = \{u \in \mathbb{R}^n : u_j \in \{0, 1\}, j = 1, 2, \dots, n\}$. *Suppose that* $y_u \in \mathbb{R}$ *is given* $\forall u \in \{0, 1\}^n$, *and let* m *be the minimum and* M *the maximum of* $\{y_u : u \in \{0, 1\}^n\}$. *Then there exist a function* $f \in C(I^n)$ *and an open set* $V \subset (I^n)^\circ$ *with* $\lambda^n(V) = 1$ *such that*

(i) $\nabla f(x) = 0 \ \ \forall x \in V$,
(ii) $f(u) = y_u \ \ \forall u \in \{0, 1\}^n$, *and*
(iii) $m \le f(x) \le M \ \ \forall x \in I^n$.

Proof. If $n = 1$, let f be defined by

$$f(x) = y_0 + (y_1 - y_0) \cdot \psi(x) \quad \forall x \in I = [0, 1],$$

and take $V = I \backslash C$, where ψ is Lebesgue's singular function and C is Cantor's ternary set. Then (i)–(iii) are obvious.

Suppose, as an induction hypothesis, that $n > 1$ and the theorem is true if n is replaced with $n - 1$.

For $l \in \{0, 1\}$, we identify the *slice*

$$S_l = \{x \in I^n : x_n = l\}$$

with I^{n-1} in the obvious way and write

$$S_l^\circ = \{x \in S_l : 0 < x_j < 1,\ \ 1 \le j \le n-1\},$$
$$A_l = \{u \in \{0, 1\}^n : u_n = l\}.$$

By the induction hypothesis, we obtain an $f_l \in C(S_l)$ and an open set $V_l \subset S_l^\circ$ with $\lambda^{n-1}(V_l) = 1 \ \ (l = 0, 1)$ such that

$$\frac{\partial f_l}{\partial x_j}(x) = 0 \quad \forall x \in V_l,\ \ 1 \le j < n,$$
$$f_l(u) = y_u \quad \forall u \in A_l \quad \text{and} \quad m \le f_l(x) \le M \quad \forall x \in S_l.$$

For any $x = (x_1, x_2, \ldots, x_n) \in I^n$, define $x^l \in S_l$ by $(x^l)_j = x_j$ if $j < n$ and $(x^l)_n = l$ for $l = 0, 1$. Define $f\colon I^n \to \mathbb{R}$ by

$$\begin{aligned} f(x) &= f_0(x^0) + [f_1(x^1) - f_0(x^0)] \cdot \psi(x_n) \\ &= [1 - \psi(x_n)] \cdot f_0(x^0) + \psi(x_n) \cdot f_1(x^1). \end{aligned}$$

Also define

$$V = (I^n)^\circ \cap \left(\bigcap_{l=0}^{1} \{x \in I^n : x^l \in V_l\} \right) \cap (\{x \in I^n : x_n \notin C\}).$$

It is obvious that V is open and $\lambda^n(V) = 1$. The continuity of the functions $x \to x^l$ $(l = 0, 1)$, each f_l, and ψ ensures that f is continuous on I^n. Plainly, $\nabla f(x) = 0 \ \forall x \in V$ by the definition of f and the definition of V. This proves (i).

If $u \in A_0$, then $u_n = 0$, $u = u^0$, and $\psi(u_n) = 0$, hence $f(u) = f_0(u) = y_u$. If $u \in A_1$, then $u_n = 1$, $u = u^1$, and $f(u) = f_1(u) = y_u$. Thus (ii) holds.

Since $0 \le \psi \le 1$, $f(x)$ is between $f_0(x^0)$ and $f_1(x^1)$ so (iii) holds too. □

(2.2) *Remark.* It is not difficult to replace I^n with any closed interval

$$\Omega = [a, b] = \{x \in \mathbb{R}^n : a_j \le x_j \le b_j, \ 1 \le j \le n\},$$

where $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$.

(2.3) **Lemma.** *Let $n \in \mathbb{N}$, let $G \in C(I^n)$, and let $\varepsilon > 0$. Then there exist a function $H \in C(I^n)$ and an open set $V \subset (I^n)^\circ$ with $\lambda^n(V) = 1$ such that*

(i) $\nabla H(x) = 0 \ \forall x \in V$,

(ii) $\|G - H\|_u < \varepsilon$,

where $\|F\|_u = \sup\{|F(x)| : x \in I^n\}$ for any $F \in C(I^n)$.

Proof. Suppose $n = 1$. Find $N \in \mathbb{N}$ such that $|G(u) - G(v)| < \varepsilon/2$ if $u, v \in I$ and $|u - v| \le 1/N$. As before, let C be Cantor's ternary set and let ψ be Lebesgue's singular function. Define

$$H(x) = G\left(\frac{k}{N}\right) + \left[G\left(\frac{k+1}{N}\right) - G\left(\frac{k}{N}\right)\right] \cdot \psi(Nx - k)$$

if $k \in \{0, 1, \ldots, N-1\}$ and $\frac{k}{N} \le x \le \frac{k+1}{N}$, also define

$$V = \bigcup_{k=0}^{N-1} \{x \in I : k \le Nx \le k+1, \ (Nx - k) \notin C\}.$$

Notice that $H(\frac{k}{N}) = G(\frac{k}{N})$ for $k = 0, 1, \ldots, N$, $H \in C(I)$, $\lambda(V) = 1$, and $k \le Nx \le k + 1 \Rightarrow$

$$\begin{aligned} |H(x) - G(x)| &\le \left|H(x) - G\left(\frac{k}{N}\right)\right| + \left|G\left(\frac{k}{N}\right) - G(x)\right| \\ &< \left|G\left(\frac{k+1}{N}\right) - G\left(\frac{k}{N}\right)\right| \cdot \psi(Nx - k) + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Plainly, $x \in V \Rightarrow \nabla H(x) = H'(x) = 0$ so the lemma holds for $n = 1$.

Suppose $n > 1$ and, as an induction hypothesis, that the lemma is true if n is replaced by $n - 1$. The uniform continuity of G provides $\delta > 0$ such that

$$u, v \in I^n, \ \|u - v\| < \delta \Rightarrow \|G(u) - G(v)\| < \varepsilon/8. \tag{1}$$

Choose $N \in \mathbb{N}$ with $N\delta > 1$. For any $r \in \{0, 1, \ldots, N\}$, we identify the *slice*

$$S_r = \{x \in I^n : x_n = r/n\}$$

with I^{n-1} in the obvious way, and we write

$$S_r^\circ = \{x \in S_r : 0 < x_j < 1 \ \forall j < n\}.$$

For each such r, we apply our induction hypothesis to the restriction of G to S_r to obtain a continuous function $G_r \in C(S_r)$ and an open set $V_r \subset S_r^\circ$ which is open in S_r such that

$$\begin{aligned} &\lambda^{n-1}(V_r) = 1, \\ &\frac{\partial G_r}{\partial x_j}(x) = 0 \quad \forall x \in V_r \ (1 \le j < n), \ \text{ and} \\ &|G(x) - G_r(x)| < \frac{\varepsilon}{8} \quad \forall x \in S_r. \end{aligned} \tag{2}$$

Next we construct H on I^n which extends each G_r and satisfies the requirements of the lemma.

The construction of H is exactly the same as that of Q_n in the proof of Theorem (1.1). That is,

$$H(x) = G_{r-1}(x^{r-1}) + [G_r(x^r) - G_{r-1}(x^{r-1})] \cdot \psi(Nx_n - r + 1)$$

if $(r-1)/N \le x_n \le r/N$. Also define

$$\begin{aligned} V = (I^n)^\circ &\cap \left(\bigcap_{r=0}^{N} \{x \in I^n : x^r \in V_r\} \right) \\ &\cap \left(\bigcup_{r=1}^{N} \{x \in I^n : r - 1 \le Nx_n \le r, \ (Nx_n - r + 1) \notin C\} \right), \end{aligned}$$

where C is Cantors ternary set. Plainly V is open, $\lambda^n(V) = 1$, $H \in C(I^n)$, and $\nabla H(x) = 0 \ \ \forall x \in V$.

Now let $x \in V$ with $r - 1 < Nx_n < r$. Then we have

$$\|x - x^r\| < \|x^{r-1} - x^r\| = 1/N < \delta \quad \text{and also} \quad \|x - x^{r-1}\| < \delta.$$

From (1) and (2) we have

$$\begin{aligned} |H(x) - G(x)| &= |G_{r-1}(x^{r-1}) + [G_r(x^r) - G_{r-1}(x^{r-1})] \cdot \psi(Nx_n - r + 1) - G(x)| \\ &\le |G_{r-1}(x^{r-1}) - G(x^{r-1})| + |G(x^{r-1}) - G(x)| \\ &\quad + |G_r(x^r) - G_{r-1}(x^{r-1})| \cdot |\psi(Nx_n - r + 1)| \\ &\le \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + |G_r(x^r) - G_{r-1}(x^{r-1})| \\ &\le \frac{\varepsilon}{4} + |G_r(x^r) - G(x^r)| + |G(x^r) - G(x^{r-1})| \\ &\quad + |G(x^{r-1}) - G_{r-1}(x^{r-1})| \\ &\le \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} < \varepsilon. \end{aligned}$$

Thus $\|H - G\|_u < \varepsilon$ on I^n. This completes the proof. □

(2.4) **Lemma.** *Let $n \in \mathbb{N}$ and let $Q \in C(I^n)$ be such that there exists an open set $V \subset (I^n)^\circ$ with $\lambda^n(V) = 1$ on which ∇Q exists and is continuous. Suppose* *that* $F_0 \in C(I^n)$ *and* $\varepsilon > 0$. *Then there exist* $F \in C(I^n)$ *and an open set* $W \subset V$ *with* $\lambda^n(W) = 1$ *such that*

(i) ∇F *exists and is continuous on* W,
(ii) $\nabla F(x) = \nabla Q(x) \ \ \forall x \in W$, *and*
(iii) $\|F - F_0\|_u < \varepsilon$.

Proof. Apply Lemma (2.3) to $G = F_0 - Q$ to obtain $H \in C(I^n)$ and an open set $V_0 \subset (I^n)^\circ$ with $\lambda^n(V_0) = 1$ such that $\|H - G\|_u < \varepsilon$ and $\nabla H(x) = 0$ $\forall x \in V_0$. Take $F = Q + H$ and $W = V \bigcap V_0$ to complete the proof.

(2.5) **Main Theorem.** *Let* $n \in \mathbb{N}$, *let* $f \in C(I^n, \mathbb{R}^n)$, *let* $F_0 \in C(I^n)$, *and let* $\varepsilon > 0$. *Then there exist* $F \in C(I^n)$ *and an open set* $W \subset (I^n)^\circ$ *with* $\lambda^n(W) = 1$ *such that*

(i) ∇F *exists and is continuous on* W,
(ii) $\|\nabla F(x) - f(x)\| < \varepsilon \ \ \forall x \in W$, *and*
(iii) $|F - F_0|_u < \varepsilon$,

where $\|y\| = (\sum_{j=1}^n y_j^2)^{1/2} \ \ \forall y \in \mathbb{R}^n$.

Proof. Take $P = f$ in (1.1) to produce a Q and then use (2.4) to replace Q with F. □

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