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SCHATTEN CLASS HANKEL OPERATORS ON THE BERGMAN SPACES OF STRONGLY PSEUDOCONVEX DOMAINS

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ABSTRACT. In this paper, we characterize holomorphic functions f such that the Hankel operators $H_{\bar{f}}$ are in the Schatten classes on bounded strongly pseudoconvex domains. It is proved that for $p > 2n$, $H_{\bar{f}}$ is in the Schatten class S_p if and only if f is in the Besov space B_p; for $p \leq 2n$, $H_{\bar{f}}$ is in the Schatten class S_p if and only if $f = \text{constant}$.

1. INTRODUCTION

Let D be a bounded strongly pseudoconvex domain with smooth boundary in $\mathbb{C}^n$, $n \geq 2$. Let $H^2(D)$ be the Bergman space consisting of holomorphic L^2 functions. The Bergman projection P is the orthogonal projection from $L^2(D)$ onto $H^2(D)$ defined by $Pf(z) = \int K(z, w) f(w)\, dv(w)$. Here $K(z, w)$ is the Bergman kernel of D. For $f \in L^2(D)$, the Hankel operator H_f from $H^2(D)$ into $L^2(D)$ is defined by $H_f(g) = (I - P)(f \cdot g)$. H_f is densely defined. In [12], we have characterized the functions $f \in H^2(D)$ such that $H_{\bar{f}}$ are bounded and compact by functions in the Bloch space and the little Bloch space, respectively. Recently, Arazy, Fisher, Janson, and Peetre [2], Wallsten [16], and Zhu [17] characterized the functions $f \in H^2(D)$ such that $H_{\bar{f}} \in S_p$ on the unit ball in $\mathbb{C}^n$. In their theorems, there is an interesting cutoff property, i.e., if $p > 2n$, then $H_{\bar{f}} \in S_p$ if and only if f is in the holomorphic Besov space B_p; if $p \leq 2n$, then $H_{\bar{f}} \in S_p$ if and only if f is a constant. In this paper, we extend those results to bounded strongly pseudoconvex domains with smooth boundaries in $\mathbb{C}^n$, $n > 1$. Since there is no nontrivial holomorphic automorphism for general strongly pseudoconvex domains in $\mathbb{C}^n$, the methods used here are new and different from those used in [2, 16, 17]. The main tools used here are the integral representations of solutions to the $\bar{\partial}$-equation and the integral criterion for extending functions on the boundary ∂D holomorphically into D. To state our main results, we need some definitions and notations.

Let $k_z(w) = K(w, z)/K(z, z)^{1/2}$. Define the Berezin transform of $f \in L^2$

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as [4]

$$\tilde{f}(z) = \int f(w)|k_z(w)|^2\,dv(w).$$

If $f \in L^2(D)$, we write $MO(f, z) = (|f|^2)^{\sim}(z) - |\tilde{f}(z)|^2$. Then $MO(f, z)$ is a continuous function on D. It is easy to check that for holomorphic functions f, $\tilde{f}(z) = f(z)$ and

$$MO(f, z) = \int_D |f(z) - f(w)|^2|k_z(w)|^2\,dw.$$

For a (p, q)-form $F(z) = \sum F_{I,J}(z)\,dz_I \wedge d\overline{z}_J$, where $F_{I,J}$ are complex-valued functions on D, $dz_I = dz_{i_1} \wedge dz_{i_2} \wedge \cdots \wedge dz_{i_p}$, and $d\overline{z}_J = d\overline{z}_{j_1} \wedge d\overline{z}_{j_2} \wedge \cdots \wedge d\overline{z}_{j_q}$, let

$$\overline{\partial} F = \sum_1^n \sum \partial F_{I,J}/\partial \overline{z}_i\, d\overline{z}_i \wedge dz_I \wedge d\overline{z}_J,$$

where $\partial/\partial z_i = 1/2\cdot(\partial/\partial x_i - \sqrt{-1}\cdot\partial/\partial y_i)$, $\partial/\partial\overline{z}_i = 1/2\cdot(\partial/\partial x_i + \sqrt{-1}\cdot\partial/\partial y_i)$. We denote

$$|F(z)| = \sum |F_{I,J}(z)|.$$

Let $\rho(z) \in C^\infty(\overline{D})$ be a strictly plurisubharmonic defining function of D such that $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ and $\nabla\rho(z) \neq 0$ for $z \in \partial D$, where $\nabla\rho$ is the gradient of ρ.

The complex tangential space at a boundary point $p \in \partial D$ is the set

$$T_p^{\mathbb{C}}(D) = \left\{\xi \in \mathbb{C}^n : \sum_1^n \partial\rho(p)/\partial z_i \cdot \xi_i = 0\right\}.$$

For $2n < p < \infty$, we define the Besov space B_p of holomorphic functions as

$$B_p = \{f \in H^2(D) : |\overline{\partial} \bar{f}| \cdot |\rho|^{1-(n+1)/p} \in L^p(D)\}.$$

By the work of Grellier [10], it follows that if $f \in B_p$, then $|\overline{\partial}\bar{f} \wedge \overline{\partial}\rho| \cdot |\rho|^{1/2-(n+1)/p} \in L^p(D)$. We will use this result without further comment.

Throughout this paper, constants are denoted by the letter C, and they may change from line to line.

Theorem A. *Let D be a connected and bounded strongly pseudoconvex domain with smooth boundary in $\mathbb{C}^n$, $n \geq 2$. Let $f \in H^2(D)$.*

(1) *For $p > 2n$, the following statements are equivalent:*

(a) $H_{\bar{f}} \in S_p$;

(b) $f \in B_p$;

(c) $MO(f, z)^{1/2} \cdot K(z, z)^{1/p} \in L^p(D)$.

(2) *For $p \leq 2n$, $H_{\bar{f}} \in S_p$ if and only if f is a constant.*

In §2, we construct a special weighted integral operator T which solves the $\overline{\partial}$-equation, and prove that for $p > 2n$, if $\psi \in C^2(D) \cap L^2(D)$ satisfies $|\overline{\partial}\psi \wedge \overline{\partial}\rho| \cdot |\rho|^{1/2-(n+1)/p} + |\overline{\partial}\psi| \cdot |\rho|^{1-(n+1)/p} \in L^p(D)$, then the operator $T_\psi(h) = T(h \cdot \overline{\partial}\psi)$ is in the Schatten class S_p as an operator from $H^2(D)$ into $L^2(D)$, and consequently, $H_\psi = (I - P)T_\psi \in S_p$. In §3, we prove that for $f \in H^2(D)$, if $H_{\bar{f}} \in S_p$ with $p > 2n$, then $f \in B_p$; if $H_{\bar{f}} \in S_p$ with $p \leq 2n$, then f

has nontangential limit $f_b \in L^2(\partial D)$ and $\bar{f}_b$ satisfies the weakly tangential Cauchy-Riemann equation [6] on ∂D which implies that f is a constant.

After this paper was written, Marco M. Peloso informed me that he obtained similar results independently.

2. Sufficiency

In this section, we prove (c) $\Rightarrow$ (b) $\Rightarrow$ (a) in Theorem A.

From now on, we will fix a bounded strongly pseudoconvex domain D with smooth boundary and let $\rho(z) \in C^\infty(\overline{D})$ be a strictly plurisubharmonic defining function of D. To simplify notation, we shall write $\rho_i(z) = \partial\rho(z)/\partial z_i$, $\rho_{ij}(z) = \partial^2\rho(z)/\partial z_i \partial z_j$, where $1 \le i,\ j \le n$. Let $F_1(z, w)$ denote the Levi polynomial

$$F_1(z, w) = \sum_{i=1}^{n} \rho_i(w)(w_i - z_i) - \frac{1}{2} \cdot \sum_{i,j=1}^{n} \rho_{ij}(w)(w_i - z_i)(w_j - z_j).$$

It is well known [13] that there exist constants δ and C_1 such that for $z, w \in \overline{D}$ with $|z - w| \le \delta$, $\operatorname{Re}(F_1(z, w) - \rho(w)) \ge C_1 \cdot (-\rho(z) - \rho(w) + |z - w|^2)$.

Before going on, we collect some facts which will be needed later.

Lemma 2.1 [3, 13]. *Let ρ and δ be the same as above. There exist functions $h_i(z, w)$, $1 \le i \le n$, and $\Psi(z, w)$ in $C^\infty(\overline{D} \times \overline{D})$ such that*

(1) *For each fixed $w \in \overline{D}$, $h_i(z, w)$ and $\Psi(z, w)$ are holomorphic in $z \in \overline{D}$.*

(2) *$\Psi(w, w) = -\rho(w)$ and there is a nonvanishing smooth function $g(z, w)$ in $\overline{D} \times \overline{D}$ such that if $|z - w| \le \delta/2$, then $\Psi(z, w) = g(z, w) \cdot (F_1(z, w) - \rho(w))$; if $|z - w| \ge \delta/2$, then $|\Psi(z, w)| \ge 1/C$.*

(3) *$\Psi(z, w) = \sum_{i=1}^{n} h_i(z, w)(w_i - z_i) - \rho(w)$.*

(4) *$h_i(w, w) = \rho_i(w) - \rho(w) \cdot g_i(w)$, where $g_i(w) = \partial g(z, w)/\partial w_i|_{z=w}$.*

Write

$$G(z, w) = \sum \rho_i(z)(z_i - w_i) - \frac{1}{2} \cdot \sum \rho_{ij}(z)(z_i - w_i)(z_j - w_j).$$

Lemma 2.2 [13]. *Let ρ and G be the same as above. There exist constants δ and c such that for any $z \in D$ with $|\rho(z)| \le \delta$, in the ball $B(z, \delta)$ we can perform a smooth change of variables $\tau = \tau(w)$ with the properties*

(1) *$\tau_1(w) = G(z, w)$;*

(2) *$|z - w|/c \le |\tau(w)| \le c \cdot |z - w|$ for $w \in B(z, \delta)$;*

(3) *$1/c \le |\partial\tau/\partial w| \le c$ for $w \in B(z, \delta)$, where $\partial\tau/\partial w$ denotes the Jacobian of τ.*

For any $w \in D$ with $|\rho(w)| \le \delta$, in the ball $B(w, \delta)$ we can perform a smooth change of variables $\lambda = \lambda(z)$ with $\lambda_1(z) = G(z, w)$ such that (2) and (3) hold for $\lambda(z)$.

Remark. In standard texts [1, 13], the coordinates $\tau = \tau(w)$ with $\operatorname{Re}\tau_1 = \rho(w) - \rho(z)$ (or $\operatorname{Re}\tau_1 = \rho(w)$) and $\operatorname{Im}\tau_1 = \operatorname{Im} G(z, w)$ are used. Since we are going to estimate some special integrals, we need the coordinate system in the lemma.

Choose a smooth function χ on $\mathbb{C}^n \times \mathbb{C}^n$ such that $0 \le \chi \le 1$ and

$$\chi(z, w) = \begin{cases} 1 & \text{if } |z - w| \le \delta/2, \\ 0 & \text{if } |z - w| \ge \delta. \end{cases}$$

Write

$$G_1(z, w) = \chi \cdot G(z, w) + (1-\chi)|z-w|^2,$$
$$\|z-w\|_A^2 = |G_1(z, w)|^2 + |\rho(z)| \cdot |z-w|^2,$$
$$s_i(z, w) = \overline{G_1(z, w)} \cdot \left[\chi \cdot \left(\rho_i(z) - \frac{1}{2} \cdot \sum_{j=1}^{n} \rho_{ij}(z)(z_j - w_j)\right) + (1-\chi) \cdot (\overline{z_i - w_i})\right] + |\rho(z)| \cdot \overline{(z_i - w_i)}.$$

It is obvious that $\|z-w\|_A^2 = \sum s_i(z, w) \cdot (z_i - w_i)$, $|s_i| \leq C \cdot |z-w|$ uniformly for $z, w \in \overline{D}$, and for z in any compact subset $\Omega \subset D$, $\|z-w\|_A^2 \geq C' \cdot |z-w|^2$ uniformly for $w \in \overline{D}$, where C' may depend on the compact subset Ω. Thus, the s_i satisfy condition (1) in [5].

Lemma 2.3. (1) *If* $|z-w| < \delta/2$, *then* $\|z-w\|_A^2 \geq 1/C \cdot (|G(z, w)|^2 + |\rho(w)| \cdot |z-w|^2)$.

(2) *If* $|z-w| \geq \delta/2$, *then* $\|z-w\|_A^2 \geq 1/C \cdot |\rho(w)|$.

Proof. If $|\rho(z)| \geq 1/2 \cdot |\rho(w)|$ or $|z-w| > \delta$, it is obvious that the results hold. If $|\rho(z)| < 1/2 \cdot |\rho(w)|$ and $|z-w| \leq \delta$, note that $\rho(w)$ is a C^∞ strictly plurisubharmonic function in a neighborhood of $\overline{D}$, by the Taylor expansion of $\rho(w)$ at z, then

$$(*) \quad \begin{aligned} \operatorname{Re} G_1(z, w) &\geq \chi/2 \cdot (-\rho(w) + \rho(z) + C_1 \cdot |z-w|^2) + (1-\chi) \cdot |z-w|^2 \\ &\geq \chi/2 \cdot (-1/2 \cdot \rho(w) + C_1 \cdot |z-w|^2) + (1-\chi) \cdot |z-w|^2. \end{aligned}$$

Thus, for $|z-w| < \delta/2$,

$$\begin{aligned} |G_1|^2 = |G|^2 &\geq 1/2 \cdot |G|^2 + C_2 \cdot (|\rho(w)|^2 + |z-w|^4) \\ &\geq 1/2 \cdot |G|^2 + 2 \cdot C_2 \cdot |\rho(w)| \, |z-w|^2. \end{aligned}$$

This finishes the proof of assertion (1).

For $\delta/2 \leq |z-w| \leq \delta$, by $(*)$ we have $|G_1|^2 \geq [\min\{C_1/2, 1\}]^2 \cdot |z-w|^4 \geq C_2 \cdot \delta^4$. Note that $\rho(w) \in C^\infty(\overline{D})$; then $1/|\rho(w)| > C_3$. Therefore, $|G_1|^2 \geq C_3 C_2 \delta^4 |\rho(w)|$. Q.E.D.

Following Berndtsson and Andersson [5], we define

$$s(z, w) = \sum_1^n s_i(z, w) \, dw_i, \qquad h(z, w) = \sum_1^n h_i(z, w) \, dw_i,$$
$$\mu(z, w) = \overline{\partial}_w h(z, w)/\rho(w) - \overline{\partial} \rho(w) \wedge h(z, w)/\rho(w)^2,$$
$$L(z, w) = C_n \cdot \sum_0^{n-1} \gamma_k \cdot [-\rho(w)/\Psi(z, w)]^{k+n+1} \cdot s \wedge \mu^k \wedge (\overline{\partial}_w s)^{n-k-1} / \|z-w\|_A^{2(n-k)},$$

where γ_k and C_n are some constants [5].

It is easy to check that for $|z - w| < \delta/2$,

$$s(z, w) = \overline{G(z, w)} \cdot \left[\sum_1^n \rho_i(z)\, dw_i + \beta_1(z, w)\right] + |\rho(z)| \cdot \sum_1^n (\overline{z}_i - \overline{w}_i)\, dw_i,$$

$$\begin{aligned}\overline{\partial}_w s(z, w) = & \left[-\sum_1^n \overline{\rho_i(z)}\, d\overline{w}_i + \overline{\beta_2(z, w)}\right] \\ & \wedge \left[\sum_1^n \rho_i(z)\, dw_i + \beta_1(z, w)\right] + \rho(z) \sum_1^n d\overline{w}_i \wedge dw_i,\end{aligned}$$

where β_k, $k = 1, 2$, are $(1, 0)$-forms with $|\beta_i(z, w)| \le C \cdot |z - w|$.

Write $\tilde{s}(z, w) = \sum_1^n \rho_i(z)\, dw_i + \beta_1(z, w)$. By direct computation we have

$$\mu^k = [\rho(w)(\overline{\partial}_w h)^k - (k-1)\overline{\partial}\rho(w) \wedge h \wedge (\overline{\partial}_w h)^{k-1}]/\rho(w)^{k+1}, \tag{2.1}$$

and for $|z - w| < \delta/2$,

$$\begin{aligned}(\overline{\partial}_w s)^m = & \left[\rho(z) \sum d\overline{w}_i \wedge dw_i + (m-1)\overline{\partial}_w \overline{G(z, w)} \wedge \tilde{s}(z, w)\right] \\ & \wedge (\rho(z) \sum d\overline{w}_i \wedge dw_i)^{m-1}.\end{aligned} \tag{2.2}$$

Lemma 2.4 [5, pp. 103–104]. *If u is a $\overline{\partial}$-closed $(0, 1)$-form with coefficients in $C^1(\overline{D})$, then*

$$v(z) = T(u)(z) = \int_D u(w) \wedge L(z, w)$$

is a solution to the equation $\overline{\partial} v = u$.

Remark 1. In [5], the theorem was proved for strictly convex domains by letting

$$s_i = \left[\sum_{k=1}^k \overline{\rho_k(z)}(\overline{z}_k - \overline{w}_k)\right] \cdot \rho_i(z) + |\rho(z)| \cdot (\overline{z}_i - \overline{w}_i),$$

and $h_i = \rho_i(w)$, where $1 \le i \le n$. As indicated in [5, p. 104], an application of the same arguments yields the results here.

Remark 2. By a standard argument (see [13, p. 297]), it follows that Lemma 2.4 holds for the $\overline{\partial}$-closed $(0, 1)$-forms u with coefficients in $C^1(D) \cap L^1(D)$.

The next lemma is crucial to our analysis. It seems to me that the standard integral representations and estimates [1, 13] do not work in our case; the following estimates should have their own interest. For each $a > 0$, we shall write $D_a = \{z \in \overline{D}\colon |\rho(z)| < a\}$.

Lemma 2.5. *If $f \in B_p$ with $p > 2n$, then for $1/q + 1/p = 1$,*

$$\int_D \left(\int_D |\overline{\partial} \bar{f} \wedge L(z, w)|^q\, dw\right)^{p/q} dz < \infty, \tag{2.3}$$

$$\int_D \left(\int_D |\overline{\partial} \bar{f} \wedge L(z, w)|^q\, dz\right)^{p/q} dw < \infty. \tag{2.4}$$

Proof. By Lemmas 2.1 and 2.3, it is easy to check that for $|z - w| \ge \delta/2$,

$$|\overline{\partial} \bar{f} \wedge L| \le C \cdot |\rho \cdot \overline{\partial f}|.$$

Note that for $|z-w| < \delta/2$, $\partial_w G_1(z, w) = \partial_w G(z, w) = -\partial\rho(w) + e_1(z, w)$, $\tilde{s}(z, w) = \sum \rho_i(z)\, dw_i + \beta_1(z, w) = \partial\rho(w) + e_2(z, w)$, and $h(z, w) = \partial\rho(w) + e_3(z, w)$, where e_i $(i = 1, 2, 3)$ are $(1, 0)$-forms with $|e_k(z, w)| \le C \cdot |z-w|$ for $k = 1, 2$, and $|e_3| \le C \cdot (|z-w| + |\rho(w)|)$. It follows that

$$|\overline{\partial}\bar{f}(w) \wedge \overline{\partial}_w \overline{G_1(z, w)}| \le |\overline{\partial}\bar{f}(w) \wedge \overline{\partial}\rho(w)| + C \cdot |z-w|\, |\overline{\partial f}|,$$
$$|\tilde{s}(z, w) \wedge h(z, w)| \le C \cdot [|z-w| + |\rho(w)|],$$
$$|\overline{\partial f}(w) \wedge \overline{\partial}_w \overline{G}_1 \wedge \tilde{s}(z, w) \wedge \overline{\partial}\rho(w) \wedge h(z, w)|$$
$$\le C \cdot |\overline{\partial}\bar{f}(w) \wedge \overline{\partial}\rho(w)|\, |z-w| \cdot (|z-w| + |\rho(w)|).$$

Recall that for $|z-w| < \delta/2$,

$$|G(z, w)|^2 + |\rho(w)| \cdot |z-w|^2 \le C \cdot \|z-w\|_A^2,$$
$$|G_1(z, w)| = |G(z, w)| \le C \cdot |\Psi(z, w)|,$$
$$|\rho(z)| + |\rho(w)| + |z-w|^2 + |\operatorname{Im}\Psi(z, w)| \le C \cdot |\Psi(z, w)|.$$

By the equations given before Lemma 2.4, a straightforward computation yields that

$$(**) \qquad |\overline{\partial f} \wedge L| \le C \cdot \left(E_0 + F_0 + \sum_1^{n-1} E_k + 1\right) \cdot Q.$$

Here

$$E_0 = |\rho(z)|^{n-1} |\rho(w)|^{n+1/2+(n+1)/p} |z-w| / (\|z-w\|_A^{2n} \cdot |\Psi(z, w)|^{n+1}),$$
$$F_0 = \frac{|\rho(z)|^{n-1} \cdot |\rho(w)|^n}{\|z-w\|_A^{2n} \cdot |\Psi(z, w)|^{n+1}} \cdot |\rho(w)|^{(n+1)/p}$$
$$\cdot (|G(z, w)| + |\rho(z)|\, |z-w| + |z-w|^2),$$
$$E_k = |\rho(z)|^{n-k-1} \cdot |\rho(w)|^{n-1/2+(n+1)/p} / (\|z-w\|_A^{2(n-k)} |\Psi(z, w)|^{n+k-1/2}),$$

where $1 \le k < n-1$,

$$E_{n-1} = |\rho(w)|^{(n+1)/p} / (|z-w| \cdot |\Psi(z, w)|^{n+1/2}),$$

and

$$Q(w) = (|\overline{\partial}\rho(w) \wedge \overline{\partial}\, \overline{f}| \cdot |\rho(w)|^{1/2} + |\overline{\partial}\bar{f}(w)| \cdot |\rho(w)|) / |\rho(w)|^{(n+1)/p}.$$

To prove (2.3) and (2.4), we show that for $q - 1 < \varepsilon < 1$ and $0 \le i \le n-1$,

$$(2.5) \qquad \int_D E_i(z, w)^q \cdot |\rho(w)|^{-\varepsilon}\, dw \le C \cdot |\rho(z)|^{-\varepsilon},$$

$$(2.6) \qquad \int_D E_i(z, w)^q \cdot |\rho(z)|^{-\varepsilon}\, dz \le C \cdot |\rho(w)|^{-\varepsilon},$$

$$(2.7) \qquad \int_D E_i(z, w)^q\, dz \le C,$$

where C is a constant independent of z, $w \in D$; the same estimates hold for F_0.

By Lemmas 2.1 and 2.3, if $|z-w| \ge \delta/2$, then E_i and F_0 are bounded by a constant C. If $|\rho(z)| \ge \delta$ and $|z-w| < \delta/2$, then $E_i^q \cdot |\rho(w)|^{-\varepsilon}$ and

$F_0^q \cdot |\rho(w)|^{-\varepsilon}$ are bounded above by $C/|z-w|^{(2n-1)q}$, and similar results hold when $|\rho(w)| \geq \delta$. Note that $(2n-1)q < 2n$ when $p > 2n$; it is easy to prove (2.5)–(2.7) for those cases. Thus, it suffices to prove (2.5)–(2.7) for z, $w \in D_\delta$ and integrals over $|z-w| < \delta/2$. Since the proofs of those estimates for E_0 and F_0 are similar, we shall only prove (2.5)–(2.7) for $E_0(z, w)$ and E_k, $1 \leq k \leq n-1$, respectively.

For $z \in D_\delta$ and $w \in B = B(z, \delta/2)$, note that

$$|\Psi(z, w)| \geq C \cdot (|\rho(z)| + |\rho(w)| + |z-w|^2);$$

then

$$\begin{aligned} I_0 &= \int_B E_0(z, w)^q |\rho(w)|^{-\varepsilon}\, dw \\ &\leq C/|\rho(z)|^q \cdot \int_B (|z-w|^2 + |G|^2/|\rho(z)|)^{-(n-1/2)q} \\ &\qquad \cdot (|\rho(z)| + |z-w|^2)^{-\varepsilon - q/2 + (n+1)q/p}\, dw. \end{aligned}$$

Using the coordinate system in Lemma 2.2 and writing $\tau' = (\tau_2, \dots, \tau_n) \in \mathbb{C}^{n-1}$, one has

$$I_0 \leq C/|\rho(z)|^q \cdot \int_{|\tau| \leq 1} (|\tau'|^2 + |\tau_1|^2/|\rho(z)|)^{-(n-1/2)q} (|\rho(z)| + |\tau'|^2)^{-\varepsilon - q/2 + (n+1)q/p}\, d\tau.$$

Integrate with respect to τ_1 over the unit disc $|\tau_1| \leq 1$ in $\mathbb{C}$, and then let $\tau' = \sqrt{|\rho(z)|} \cdot \eta'$; we have

$$I_0 \leq C/|\rho(z)|^\varepsilon \cdot \int_{\mathbb{C}^{n-1}} |\eta'|^{-(2n-1)q+2} (1 + |\eta'|^2)^{-\varepsilon - q/2 + (n+1)q/p}\, d\eta'. \tag{2.8}$$

Note that for $p > 2n$ and $\varepsilon > q-1$, $(2n-1)q - 2 < 2n-2$ and $(2n-1)q - 2 + 2\varepsilon + q - 2(n+1)q/p > 2n-2$. It follows that the integral on the right side of (2.8) is finite. Thus, $I_0 < \infty$.

Next we prove (2.6) for $E_0(z, w)$. By Lemmas 2.2 and 2.3, we have

$$\begin{aligned} I_1 &= \int_{B(w, \delta/2)} E_0(z, w)^q |\rho(z)|^{-\varepsilon}\, dz \\ &\leq C \cdot |\rho(w)|^{(n+1)q/p} \int_B (|z-w|^2 + |G|^2/|\rho(w)|)^{-(n-1/2)q} \\ &\qquad \cdot (|\rho(w)| + |z-w|^2)^{-3q/2-\varepsilon}\, dz \\ &\leq C \cdot |\rho(w)|^{(n+1)q/p} \int_{|\tau| \leq 1} (|\tau'|^2 + |\tau_1|^2/|\rho(w)|)^{-(n-1/2)q} \\ &\qquad \cdot (|\rho(w)| + |\tau'|^2)^{-3q/2-\varepsilon}\, d\tau. \end{aligned}$$

By the same arguments as those used in the proof of $I_0 < \infty$, one can prove that $I_1 < \infty$. This completes the proof of (2.6) for $E_0(z, w)$. Similarly, we can prove (2.7) for E_0.

Now we prove (2.5)–(2.7) for E_k, $1 \le k < n-1$. By Lemmas 2.1–2.3, it follows that

$$\begin{aligned} I_2 &= \int_{B(z,\delta/2)} E_k(z,w)^q |\rho(w)|^{-\varepsilon}\, dw \\ &\le C \cdot \int_{|\tau|\le 1} |\rho(z)|^{(n-k-1)q} (|\rho(z)|\,|\tau'|^2 + |\tau_1|^2)^{-nq+kq} \\ &\qquad \cdot (|\rho(z)| + |\tau'|^2)^{-kq+(n+1)q/p-\varepsilon}\, d\tau . \end{aligned}$$

Let $\tau_1 = |\rho(z)| \cdot \eta_1$ and $\tau' = \sqrt{|\rho(z)|} \cdot \eta'$. Then

$$I_2 \le C/|\rho(z)|^{\varepsilon} \cdot \int_{\mathbb{C}^n} (|\eta|^2)^{-nq+kq} \cdot (1+|\eta'|^2)^{-kq+(n+1)q/p-\varepsilon}\, d\eta . \tag{2.9}$$

Note that $-1 > -nq+kq > -n$ and $-nq+kq-kq+(n+1)q/p-\varepsilon < -n$. It is obvious that the integral on the right side of (2.9) is finite. Thus, we get (2.5) for E_k.

Again, by Lemma 2.3, $\|z-w\|_A^2 \ge 1/C \cdot (|G(z,w)|^2 + |\rho(w)| \cdot |z-w|^2)$. Then

$$\begin{aligned} E_k &\le C|\rho(z)|^{\varepsilon/q} |\rho(w)|^{n-1/2+(n+1)/p} \\ &\quad \cdot (|G|^2 + |\rho(w)|\,|z-w|^2)^{k-n} |\Psi(z,w)|^{-2k-1/2-\varepsilon/q} . \end{aligned}$$

By repeating the procedure above, one can prove (2.6) and (2.7) for E_k, $1 \le k < n-1$.

It remains to prove (2.5)–(2.7) for E_{n-1}. In this case, the results can be proved by using the coordinate systems and methods given in [13, pp. 299–300] except for obvious modifications. We omit the details here.

Note that $f \in B_p$ implies that $Q \in L^p(D)$. An application of the arguments used in the proof of Lemma 5 in [18] to (**), (2.5), and (2.6) yields (2.3); by (**) and (2.7) it follows that (2.4) holds. This finishes the proof of our lemma. Q.E.D.

Theorem 2.6. *For $f \in B_p$ with $p > 2n$, let $T_0(g) = T(g \cdot \overline{\partial}\bar{f})$ for $g \in H^2(D)$. Then T_0 is in the Schatten class S_p as an operator from $H^2(D)$ into $L^2(D)$.*

Proof. By Lemma 2.5 and Russo's theorem [14], it follows that $T_0(g) = T(\overline{\partial}(\bar{f}g))$ is an operator in S_p. Q.E.D.

Theorem 2.7. *If $f \in B_p$ with $p > 2n$, then $H_{\bar{f}} \in S_p$.*

Proof. For $g \in H^\infty(D)$, the space of bounded holomorphic functions in D, it is easy to check that $f \in B_p$ implies that $|\overline{\partial}(\bar{f} \cdot g)| \in L^1(D)$. By Lemma 2.4 and Theorem 2.6, $u = T_0(g) = T(\overline{\partial}(\bar{f}g))$ is a solution to the equation $\overline{\partial}u = \overline{\partial}(\bar{f}g) = g \cdot \overline{\partial}\bar{f}$ and $u \in L^2(D)$. Obviously, $\bar{f} \cdot g \in L^2(D)$ is a solution to the same equation. By the uniqueness of the solution orthogonal to $H^2(D)$, it follows that $H_{\bar{f}}g = (I-P)(\bar{f}g) = (I-P)(u) = (I-P)(T_0(g))$. Since $H^\infty(D)$ is dense in $H^2(D)$ [13], Theorem 2.6 implies that $H_{\bar{f}} \in S_p$. Q.E.D.

Remark. In the proofs of Lemma 2.5 and Theorem 2.7, we have actually proved that if $\psi \in C^2(D) \cap L^2(D)$ satisfies $(|\overline{\partial}\psi \wedge \overline{\partial}\rho| \cdot |\rho|^{1/2} + |\rho \cdot \overline{\partial}\psi|)/|\rho|^{(n+1)/p} \in L^p(D)$, then the Hankel operator $H_\psi \in S_p$.

Theorem 2.8. *For* $f \in H^2(D)$, *if* $MO(f, z)^{1/2}K(z, z)^{1/p} \in L^p(D)$, *then* $f \in B_p$ *when* $p > 2n$.

Proof. Note that if $f \in H^2(D)$, then $\tilde{f}(z) = f(z)$. By Theorem F in [4], it follows that for any $\xi \in \mathbb{C}^n$, $|f_*(z)\xi| \le C \cdot MO(f, z)^{1/2} \cdot S(z, \xi)$, where $S(z, \xi)$ is the infinitesimal form of the Bergman metric (for the definition of $S(z, \xi)$ see [13]), and $f_*(z)\xi = \sum \partial f(z)/\partial z_i \cdot \xi_i$. By the boundary behaviors of the Bergman metric [9], it follows that for z close to ∂D, $|f_*(z)\xi| \le C \cdot MO(f, z)^{1/2} \cdot [|\xi_T|/|\rho(z)|^{1/2} + |\xi_N|/|\rho(z)|]$, where ξ_T and ξ_N are the components of ξ in the complex tangent directions and complex normal direction at $\pi(z)$, and $\pi(z)$ is the normal projection of z on ∂D. Thus, $|\rho \cdot \partial f| \le C \cdot MO(f, z)^{1/2}$. Note that [8] $K(z, z)/C \le |\rho(z)|^{-(n+1)} \le C \cdot K(z, z)$. Thus, $MO(f, z)^{1/2}K(z, z)^{1/p} \in L^p(D)$ implies that $|\rho \cdot \overline{\partial f}| \cdot |\rho|^{-(n+1)/p} \in L^p(D)$. Therefore, when $p > 2n$, $f \in B_p$. Q.E.D.

3. Necessity

In this section, we prove (a) $\Rightarrow$ (c) and (2) of Theorem A.

Theorem 3.1. *For* $f \in H^2(D)$ *and* $p \ge 2$, *if* $H_{\bar{f}} \in S_p$, *then*

$$MO(f, z)^{1/2}K(z, z)^{1/p} \in L^p(D).$$

Proof. It is easy to check that $H_{\bar{f}}k_z(w) = (\overline{f(w)} - \overline{f(z)}) \cdot k_z(w)$ and $\|H_{\bar{f}}k_z\|^2 = MO(f, z)$. Note that for $z \in D$, $k_z(\cdot)$ are unit vectors in $H^2(D)$. By the same arguments as those given in [2, 17], the result follows. Q.E.D.

Theorem 3.2. *Let* $f \in H^2(D)$ *and* $p \ge 2$. *If* $H_{\bar{f}} \in S_p$, *then* $|\nabla f| \in L^\alpha(D)$ *for some* $\alpha > 1$. *Moreover, if* $p > 2n$, *then* $H_{\bar{f}} \in S_p$ *implies that* $f \in B_p$.

Proof. The second assertion follows from Theorem 2.8 and Lemma 3.1. To prove the first result, by Lemma 3.1 and the proof of Theorem 2.8, one has $|\nabla f| \cdot |\rho|^{1-(n+1)/p} \in L^p(D)$. Note that $|\rho|^{-\beta} \in L^1(D)$ for any $0 < \beta < 1$. By direct calculation using Hölder's inequality we have $|\nabla f| \in L^\alpha(D)$ for some $\alpha > 1$. Q.E.D.

Theorem 3.3. *If* $H_{\bar{f}} \in S_p$, $p \le 2n$, *then* $f =$ constant.

Proof. Since $S_\alpha \subset S_\beta$ if $\alpha < \beta$, it suffices to prove the result for $p = 2n$. If $H_{\bar{f}} \in S_{2n}$, then $H_{\bar{f}}$ is compact. By Theorem E in [12], $f \in B_0(D) = \{g \in H^2(D)\colon |\nabla g \cdot \rho| \to 0 \text{ as } z \to \partial D\}$. Note that [7] $B_0(D) \subset P(L^\infty(D))$ and the Bergman projection is bounded [18] on $L^r(D)$ for $1 < r < \infty$. Thus $f \in H^r(D)$ for any $1 < r < \infty$. By Lemma 3.2, $f \in S_{2n}$ implies that $|\nabla f| \in L^\alpha(D)$ for some $\alpha > 1$. By direct computation using Stokes theorem one can easily check that f is in the Hardy space $h^2(D)$. It is well known [15] that $f \in h^2$ implies that f has nontangential boundary values $f_b \in L^2(\partial D)$. We claim that for any smooth $(n, n-2)$-form γ on $\overline{D}$, $\int_{\partial D} \bar{f}_b \cdot \overline{\partial}\gamma = 0$. In fact, by the definition of f_b and the Stokes Theorem, we have

$$\int_{\partial D} \bar{f}_b \, \overline{\partial}\gamma = \lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} \bar{f} \, \overline{\partial}\gamma = \lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} -\overline{\partial}\bar{f} \wedge \gamma,$$

where $D_\varepsilon = \{z \in D\colon \rho(z) = -\varepsilon\}$. Note that [6, 11] $\overline{\partial}\bar{f} = \overline{\partial}_b\bar{f} + \overline{\partial}_\nu\bar{f}$, where $\overline{\partial}_b\bar{f}$ and $\overline{\partial}_\nu\bar{f}$ are complex tangential and complex normal components of $\overline{\partial}\bar{f}$

on ∂D_ε, respectively. Since γ is a smooth $(n, n-2)$-form, it follows that [6, p. 617] $\int_{\partial D_\varepsilon} \overline{\partial} \bar{f} \wedge \gamma = \int_{\partial D_\varepsilon} \overline{\partial}_b \bar{f} \wedge \gamma$. Thus

$$\int_{\partial D} \bar{f}_b \overline{\partial} \gamma = \lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} -\overline{\partial}_b \bar{f} \wedge \gamma. \tag{3.1}$$

Note that the coefficients of $\overline{\partial}_b \bar{f}(z)$ are of the forms $\sum_1^n \overline{\partial f / \partial z_i} \cdot \bar{\xi}_i$, where ξ are vectors in the complex tangential space at $z \in \partial D_\varepsilon$ and $|\xi| \leq C$. By Lemma 3.1 and the proof of Theorem 2.8, $H_{\bar{f}} \in S_{2n}$ implies that $|\overline{\partial}_b \bar{f}| \, |\rho|^{1/2} \cdot K(z, z)^{1/2n} \in L^{2n}(D)$. Note that $K(z, z) \approx |\rho(z)|^{-(n+1)}$. We have

$$\int_{D_a} |\overline{\partial}_b \bar{f}|^{2n} / |\rho(z)| \, dz = \int_0^a t^{-1} \, dt \int_{\partial D_t} W(z) \cdot |\overline{\partial}_b \bar{f}|^{2n} \, d\sigma < \infty, \tag{3.2}$$

where $D_a = \{z \in D : |\rho(z)| < a\}$, $\partial D_t = \{z \in D : \rho(z) = -t\}$, and $W > 0$ is a smooth function bounded from 0 when a is small.

From (3.2) one can easily obtain that there is a sequence $t_n \to 0$ such that

$$\lim_{n \to \infty} \int_{\partial D_{t_n}} |\overline{\partial}_b \bar{f}|^{2n} \, d\sigma \to 0.$$

Consequently,

$$\lim_{n \to \infty} \int_{\partial D_{t_n}} |\overline{\partial}_b \bar{f}| \, d\sigma = 0. \tag{3.3}$$

An application of (3.3) to (3.1) yields that $\int_{\partial D} \bar{f}_b \wedge \overline{\partial} \gamma = 0$.

By Theorem 10 in [6, p. 617], it follows that $\bar{f}_b$ has a holomorphic extension $f_1 \in h^2(D)$. Note that f is the holomorphic extension of f_b in D. By the reproducing property of the Poisson-Szegö kernel [15], it follows that both f and $\bar{f}$ are holomorphic in D. Consequently, $f \equiv$ constant. Q.E.D.

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