

## FOR RIGHT-ANGLED COXETER GROUPS $z^{|g|}$ IS A COEFFICIENT OF A UNIFORMLY BOUNDED REPRESENTATION

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**ABSTRACT.** A Coxeter group  $\Gamma$  is right angled if any exponent in the Coxeter diagram is either 2 or  $\infty$ . Using the action of  $\Gamma$  on its Davis complex, we construct a family of cocycles that we use to perturb the left regular representation of  $\Gamma$ . In this way, we obtain a family  $(\pi_z)_{|z|<1}$  of uniformly bounded representations of  $\Gamma$ , of which the function  $g \rightarrow |g|$  is a coefficient (where  $|g|$  denotes the word length of  $g \in \Gamma$ ).

### INTRODUCTION

Recall [1] that a Coxeter system  $(\Gamma, S)$  is a group  $\Gamma$  with a distinguished set of generators  $w_i \in S$  and relations  $w_i^2 = 1 = (w_i w_j)^{m_{ij}}$ , where  $m_{ij}$  is  $\infty$  (and then there is no relation between  $w_i$  and  $w_j$ ) or an integer  $\geq 2$ . If all  $m_{ij}$  are  $\infty$  or 2, we call the Coxeter system *right angled*.

The Coxeter system gives rise to a function  $|g|$  on  $\Gamma$ , defined as the minimal length of the word in the  $w_i$ 's representing  $g$ . We call it the length function. Abusing language we will refer to Coxeter systems as Coxeter groups.

The purpose of this paper is to prove the following:

**Theorem.** *Let  $(\Gamma, S)$  be a finitely generated right-angled Coxeter group, and let  $|g|$  be the (word) length function on  $\Gamma$ . Then for any complex number  $z$  such that  $|z| < 1$ , the function  $z^{|g|}$  is a coefficient of a uniformly bounded representation  $\pi_z$ .*

The theorem is proved by studying cocycles following [4, 6]. The representations  $\pi_z$  constructed in the proof of the theorem are unitary for real  $z$ ; the family  $\pi_z$  depends holomorphically on  $z$  (compare [6]).

### COCYCLES

Let  $G$  be a group acting on the space  $X$  and  $\pi : G \rightarrow \text{GL}(V)$  be its representation on some vector space  $V$ .

**Definition.** A cocycle on  $(X, G)$  twisted by  $\pi$  is a map  $c : X \times X \rightarrow \text{GL}(V)$  such that:

$$(1) \quad c(x, x) = \text{id},$$

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- (2)  $c(x, y)c(y, z) = c(x, z),$
- (3)  $c(gx, gy) = \pi(g)c(x, y)\pi(g)^{-1}.$

Property (2) is called the chain rule and property (3) is called equivariance. One should think of this definition as follows. The action of  $G$  on  $X$  and  $V$  gives rise to a “bundle” over  $G\backslash X$ :

$$V \longrightarrow X \times_G V \longrightarrow G\backslash X.$$

Here  $X \times_G V = G\backslash(X \times V)$  and the  $G$ -action on the product is a diagonal one. A cocycle  $c$  is a parallel translation in this bundle lifted to the “covering”  $X \longrightarrow G\backslash X$ . The convention is that  $c(x, y)$  maps the fiber over  $y$  to the fiber over  $x$ .

As usual in such a situation we have the associated (monodromy) representation  $\pi_c(g) = c(x_0, gx_0)\pi(g)$ . It is straightforward to check using properties (1)–(3) that it is indeed a representation of  $G$  on  $V$ .

### THE COMPLEX

For any Coxeter group  $(\Gamma, S)$  there is a cell complex  $C(\Gamma)$  on which  $\Gamma$  acts properly. It is defined as follows: cells are indexed by right cosets  $\Gamma/\Gamma_P$ , where  $\Gamma_P$  is a finite group generated by a subset  $P$  of  $S$ . A cell  $[\gamma]$  is a face of  $[\eta]$  if  $[\gamma] \subset [\eta]$  as cosets.

The obvious action of  $\Gamma$  on  $C(\Gamma)$  coming from the left action of  $\Gamma$  on itself is a reflection group action. Any reflection (i.e., an element conjugated to a generator in  $S$ ) has its mirror of fix-points. Any mirror is two-sided, i.e., its complement has two components. Closures of connected components of the set of all mirrors are called fundamental domains. We say that a fundamental domain  $F$  is adjacent to the mirror of  $s$  if  $sF \cap F \neq \emptyset$ .

This construction is described in full in [3].

### COCYCLES FOR RIGHT-ANGLED GROUPS

In this section we present a natural perturbation of the left regular representation of a right-angled Coxeter group with  $z^{|g|}$  as a coefficient. This is a generalization of a construction in [4, 6]. Recall that  $\Gamma$  acts on  $C(\Gamma)$  on the left and that fundamental domains are indexed by group elements. Recall that the left regular representation acts on the Hilbert space  $l^2(\Gamma)$ , which has a basis indexed by group elements  $\delta_\gamma$ .

Suppose that a reflection  $s$  is given. The set of all fundamental domains splits into the disjoint union  $A_s \cup B_s = \Gamma$ , where  $A_s = \{F : sF \cap F \neq \emptyset\}$  are the fundamental chambers adjacent to the mirror of  $s$ , and  $B_s$  are the nonadjacent chambers. The cocycle is defined first for pairs  $(F, sF)$  and then extended to all pairs by the chain rule. Suppose that  $a$  and  $z$  are two complex numbers such that  $a^2 + z^2 = 1$ . Then the formula is as follows:

$$c_z(F, sF)(\delta_x) = \begin{cases} \delta_x & \text{if } x \in B_s; \\ a\delta_x + z\delta_{sx} & \text{if } x \in A_s \text{ and} \\ & x, F \text{ are on the same side of } s; \\ -z\delta_x + a\delta_{sx} & \text{otherwise.} \end{cases}$$

**Lemma 1.** *The map  $c_z$  is a cocycle.*

*Proof.* Since the equivariance is clear and the chain rule is built into the definition, all we have to check is that the definition is correct, that is, that the two ways of extending the cocycle give the same result. Thus we have to prove that

$$(1) \quad c(F, sF) \circ c(sF, tsF) = c(F, tF) \circ c(tF, stF)$$

whenever  $st = ts$  and, moreover, we have to do it for generators in  $S$  only. This property is clear away from the set of  $F$ 's where both maps are not identity, that is, away from the  $c_z$ -invariant subspace spanned by  $\{\delta_F, \delta_{sF}, \delta_{tF}, \delta_{stF}\}$  for any  $F$  adjacent to both  $s$  and  $t$  mirrors. Now we can check (1) by a direct computation, or use abstract nonsense: if  $G_1$  acts on  $X_1$ ,  $G_2$  on  $X_2$ , and  $c_1$  and  $c_2$  are cocycles for these actions twisted by  $\pi_1$  and  $\pi_2$  respectively, then the product cocycle is given by the formula

$$c\{(x_1, y_1), (x_2, y_2)\}(v_1 \otimes v_2) = c(x_1, x_2)(v_1) \otimes c(y_1, y_2)(v_2).$$

We check that in the case  $G = Z_2$  the formula for  $c_z$  gives a cocycle twisted by the regular representation and take the product cocycle. It is equal to  $c_z$  on the space spanned by  $\{\delta_F, \delta_{sF}, \delta_{tF}, \delta_{stF}\}$ ; hence,  $c_z$  is indeed a cocycle.

In fact, the above construction supplies a much larger family of cocycles. Suppose again that  $(\Gamma, S)$  is a right-angled group and assign to each generator  $w_i \in S$  a complex number  $z_i$ ; let  $a_i$  be a number such that  $a_i^2 + z_i^2 = 1$ . Again define the cocycle first for pairs of the form  $(F, sF)$ , and then extend it to all pairs by the chain rule. Suppose the reflection  $s$  is conjugated to a (unique) generator  $w_i$ ; then  $a_i(s), z_i(s)$  denote complex numbers corresponding to  $w_i$ . The cocycle is defined by the formula

$$c_{z_1, \dots, z_n}(F, sF)(\delta_x) = \begin{cases} \delta_x, & \text{if } x \in B_s; \\ a_i(s)\delta_x + z_i(s)\delta_{sx} & \text{if } x \in A_s \text{ and} \\ & F \text{ are on the same side of } s; \\ -z_i(s)\delta_x + a_i(s)\delta_{sx} & \text{otherwise.} \end{cases}$$

The arguments used in the proof of Lemma 1 also yield

**Lemma 2.** *The map  $c_{z_1, \dots, z_n}(x, y)$  is a cocycle.*

The monodromy representation obtained from  $c_{z_1, \dots, z_n}$  will be denoted by  $\pi_{z_1, \dots, z_n}$  or if  $z_1 = \dots = z_n = z$  simply by  $\pi_z$ . Notice that

$$\langle \pi_z(g)(\delta_e), \delta_e \rangle = z^{|g|}$$

and

$$\langle \pi_{z_1, \dots, z_n}(g)(\delta_e), \delta_e \rangle = \prod_i z_i^{|g|_i}.$$

Here  $|g|_i$  is the length function which counts how many times the generator  $w_i$  occurs in the shortest word expressing  $g$  in terms of generators from  $S$ , or, equivalently, how many times a path from  $e$  to  $g$  must cross a mirror of a reflection conjugated to  $w_i$ . It is fairly clear that  $|g| = \sum_i^n |g|_i$ . With the second definition the proof of both formulas is a straightforward computation (compare [6]).

### TREES

The cocycles  $c_{z_1, \dots, z_n}$  are closely linked to cocycles for groups acting on trees. Namely, we can perform the following construction.

Fix a generator  $w \in S$ . Define a graph  $C_w(\Gamma)$  as follows. Its vertices are connected components of  $C(\Gamma) - \bigcup_s \text{Mirror}(s)$  where  $s$  are reflections conjugated to  $w$ . Two vertices are joined by an edge if the components are adjacent in  $C(\Gamma)$ ; i.e., if they intersect after taking closures.

**Lemma 3.**  $C_w(\Gamma)$  is a tree.

*Proof.* Any loop  $\lambda$  in  $C_w(\Gamma)$  lifts to a path in  $C(\Gamma)$  which can be closed up to a loop  $\Lambda$  without crossing the  $w$ -mirror. Projection of  $\Lambda$  to the tree is again  $\lambda$ . Since  $C(\Gamma)$  is contractible (cf. [3]),  $\Lambda$  and, hence,  $\lambda$  are homologous to zero and thus  $C_w(\Gamma)$  is a tree.

*Proof of the theorem.* The map which sends a chamber  $x$  in  $C(\Gamma)$  to the vertex  $[x]_w$  of  $C_w(\Gamma)$  that it is contained in is clearly equivariant and yields a (cellular) map  $\mu_w : C(\Gamma) \rightarrow C_w(\Gamma)$ . We take the diagonal of the family  $\mu : C(\Gamma) \rightarrow \prod_w C_w(\Gamma)$  to get an equivariant embedding. This map also induces the isometric embedding of the Hilbert spaces  $\mu : l^2(C(\Gamma)) \rightarrow \bigotimes_w l^2(C_w(\Gamma))$  by  $\mu(\delta_x) = \bigotimes_w \delta_{[x]_w}$ .

On each tree we have the cocycle  $pv_z$ , provided by the construction of Pimsner and Valette. We will not discuss it here in detail but rather refer to [6]. It is a simple but important observation which we can check by direct calculation that the map  $\mu$  is cocycle preserving, that is,

$$\bigotimes pv_{z_i}(\mu(x), \mu(y)) = \mu \circ c_{z_1, \dots, z_n}(x, y).$$

Hence we can estimate norms

$$\|\pi_{z_1, \dots, z_n}\| = \|c_{z_1, \dots, z_n}\| \leq \|\bigotimes pv_{z_i}\| \leq \prod_i \frac{2|1 - z_i^2|}{1 - |z_i|}.$$

The first inequality comes from the fact that  $\mu$  is an isometric embedding, and the second from an estimate of the norms of representations coming from trees, adapted by Valette from [5]. This proves the theorem.

*Remark.* One can prove the bare statement of the theorem with the tensor product representation that we used in the proof, without mentioning  $\pi_z$ . However, it seems important to us that the coefficient comes from a rather natural perturbation of the regular representation.

The unsatisfactory aspect of our story is that one would like to prove the theorem for an arbitrary Coxeter group. It is known that  $r^{|g|}$  is a coefficient of a unitary representation [2]. Bożejko has remarked that the proof given there shows that, in fact,  $\prod_{i \in I} r_i^{|g|_i}$  is a coefficient of a unitary representation. Here  $|g|_i$  is defined as follows. Let  $I$  denote the set of conjugacy classes of reflections and  $i \in I$ ;  $|g|_i$  is the number of occurrences of generators  $w \in W$  from  $i$  in a shortest presentation of  $g$  in terms of  $w$ 's. For right-angled groups this definition agrees with the definition of  $|g|_i$  given previously.

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