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LIFTING GOTTLIEB SETS AND DUALITY

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ABSTRACT. Let $p : E_f \to X$ be a fibration induced by a map $f : X \to Y$ from the path space fibration $\varepsilon : PY \to Y$. Let $g : A \to X$ be cyclic. When does g lift to a map $A \to E_f$ which is cyclic? We give an answer of this question for arbitrary A and Y. Also, we give an answer in the dual situation.

1. INTRODUCTION

A based map $f : A \to X$ is cyclic [8] if there exists a map $\phi : X \times A \to X$ such that ϕj is homotopic to $\nabla(1 \vee f)$, where $j : X \vee A \to X \times A$ is the inclusion and $\nabla : X \vee X \to X$ is the folding map. The Gottlieb set denoted $G(A, X)$ is the set of all homotopy classes of cyclic maps from A to X. Dually, a based map $f : X \to A$ is cocyclic [8] if there exists a map $\theta : X \to X \vee A$ such that $j\theta$ is homotopic to $(1 \times f)\Delta$, where $j : X \vee A \to X \times A$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. The dual Gottlieb set $DG(X, A)$ is the set of all homotopy classes of cocyclic maps from X to A. In this paper we consider the following problem and its dual. Let $f : X \to Y$ be a map and PY the space of paths in Y which begin at *. Let $\varepsilon : PY \to Y$ be the fibration given by $\varepsilon(\eta) = \eta(1)$. Let $p : E_f \to X$ be the fibration induced by f from ε, that is, $E_f = \{(x, \eta) \in X \times PY \mid f(x) = \varepsilon(\eta)\}$ is the pullback of $f : X \to Y$ and $\varepsilon : PY \to Y$. Let $g : A \to X$ be cyclic, that is, there is a map $\phi : X \times A \to X$ such that $\phi j \sim \nabla(1 \vee g)$, where $j : X \vee A \to X \times A$ is the inclusion. When does g lift to a map $A \to E_f$ which is cyclic? This can be achieved, if there is a map $\hat{\phi} : E_f \times A \to E_f$ such that $\hat{\phi}_{|E_f} \sim 1_{E_f}$ and $\phi(p \times 1) = p\hat{\phi}$. In case $A = S^n$ and Y is an Eilenberg-Mac Lane space, Gottlieb [1, Theorem 6.3], has given a necessary and sufficient condition for the existence of this $\hat{\phi}$. In case A is arbitrary and Y is a product of Eilenberg-Mac Lane spaces, Halbhavi and Varadarajan [2] have given a necessary and sufficient condition for the existence of such a $\hat{\phi}$. In case A is arbitrary and Y is an H-space, Hoo [4] has also given a necessary and sufficient condition for the existence of such a $\hat{\phi}$. We can obtain a necessary and sufficient condition for the existence of such a $\hat{\phi}$ with arbitrary A and Y, and we show that Hoo's necessary and sufficient condition

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follows from ours. Our method is a modification of Haslam's which was used to study Postnikov systems and G-spaces. We now consider the dual situation. Let $f : X \to Y$ be a map and cX the reduced cone of X. There is a cofibration $X \xrightarrow{\imath} cX \to \Sigma X$, where $\imath(x) = [x, 1]$. Let $i : Y \to C_f$ be the cofibration induced by f from $\imath$, that is, $C_f = cX \amalg Y/\imath(x) \sim f(x)$ is the pushout of $f : X \to Y$ and $\imath : X \to cX$. Let $g : Y \to A$ be cocyclic, that is, there is a map $\theta : Y \to Y \vee A$ such that $j\theta \sim (1 \times g)\Delta$, where $j : Y \vee A \to Y \times A$ is the inclusion. When does g extend to a map $C_f \to A$ which is cocyclic? When there is a map $\hat{\theta} : C_f \to C_f \vee A$ such that $p_1 j' \hat{\theta} \sim 1_{C_f}$ and $(i \vee 1)\theta = \hat{\theta} i$, this can be achieved, where $j' : C_f \vee A \to C_f \times A$ is the inclusion and $p_1 : C_f \times A \to C_f$ is the projection. In case A is arbitrary and X is a wedge product of Moore spaces, Halbhavi and Varadarajan [2] have given a necessary and sufficient condition for the existence of this $\hat{\theta}$. In case A is arbitrary and X is a co-H-space, Hoo [4] has given a necessary and sufficient condition for the existence of such a $\hat{\theta}$. We can obtain a necessary and sufficient condition for the existence of such a $\hat{\theta}$ with arbitrary A and X, and we show that Hoo's necessary and sufficient condition follows from ours. All our spaces will be homotopy type of connected locally finite CW complexes. We assume also that spaces have nondegenerate base points. All homotopies and maps are to respect base points. The base point as well as the constant map will be denoted by $*$. Also, we denote by $[X, Y]$ the set of homotopy classes of pointed maps $X \to Y$. The identity map of space will be denoted by 1 when it is clear from the context. ΣX and ΩX denote the reduced suspension and the loop space of X respectively. The adjoint functor from the group $[\Sigma X, Y]$ to the group $[X, \Omega Y]$ will be denoted by τ.

2. Lifting Gottlieb sets

Let $f : X \to Y$ be a map and PY the space of paths in Y which begin at $*$. Let $\varepsilon : PY \to Y$ be the fibration given by evaluating a path at its end point. Let $p : E_f \to X$ be the fibration induced by f from ε. That is, E_f is the pullback of $f : X \to Y$ and $\varepsilon : PY \to Y$;

$$\begin{array}{ccc} E_f & \longrightarrow & PY \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ X & \xrightarrow{f} & Y \end{array}$$

where $E_f = \{(x, \eta) \in X \times PY \mid f(x) = \varepsilon(\eta)\}$, $p(x, \eta) = x$. In fact, the fibration $p : E_f \to X$ is principal.

The following two lemmas are standard.

Lemma 2.1. *A map $g : A \to X$ can be lifted to a map $A \to E_f$ if and only if $fg \sim *$.*

Lemma 2.2 [3]. *Given maps $g_k : A_k \to E_f$, $k = 1, 2$, and $g : A_1 \times A_2 \to E_f$ satisfying $pg_{|A_k} \sim pg_k$, $k = 1, 2$, then there is a map $h : A_1 \times A_2 \to E_f$ such that $ph = pg$ and $h_{|A_k} \sim g_k$, $k = 1, 2$.*

Theorem 2.3. *Let $g : A \to X$ be cyclic, that is, there is a map $\phi : X \times A \to X$ such that $\phi j \sim \nabla(1 \vee g)$, where $j : X \vee A \to X \times A$ is the inclusion. Then there* *exists a map* $\hat{\phi}: E_f \times A \to E_f$ *such that* $\hat{\phi}_{|E_f} \sim 1_{E_f}$ *and the diagram*

$$\begin{array}{ccc} E_f \times A & \xrightarrow{\hat{\phi}} & E_f \\ {\scriptstyle p\times 1}\downarrow & & {\scriptstyle p}\downarrow \\ X \times A & \xrightarrow{\phi} & X \end{array}$$

commutes if and only if $f\phi(p \times 1) \sim *$.

Proof. If such a $\hat{\phi}$ exists, we have, from Lemma 2.1, that $f\phi(p \times 1) \sim *$. Conversely, suppose $f\phi(p\times 1) \sim *$. By Lemma 2.1, there is a map $\phi' : E_f \times A \to E_f$ such that $p\phi' = \phi(p \times 1)$. Then $p\phi'_{|E_f} = \phi(p \times 1)_{|E_f} \sim p1_{E_f}$. Thus we have, from Lemma 2.2, that there is a map $\hat{\phi} : E_f \times A \to E_f$ such that $p\hat{\phi} = p\phi' = \phi(p \times 1)$, $\hat{\phi}_{|E_f} \sim 1_{E_f}$, and $\hat{\phi}_{|A} \sim \phi'_{|A}$. This proves the theorem.

Consider the following diagram where each square homotopy commutes and each column is the Puppe sequences of the cofibration:

$$\begin{array}{ccccccc} E_f \times A & \xrightarrow{p\times 1} & X \times A & & & & \\ \downarrow & & \downarrow & & & & \\ E_f \times cA & \xrightarrow{p\times 1} & X \times cA & & & & \\ \downarrow & & \downarrow & & & & \\ E_f \times cA/E_f \times A & \xrightarrow{\hat{p}} & X \times cA/X \times A & & & & \\ {\scriptstyle q}\downarrow & & {\scriptstyle q}\downarrow & & & & \\ \Sigma(E_f \times A) & \xrightarrow{\Sigma(p\times 1)} & \Sigma(X \times A) & \xrightarrow{\Sigma\phi} & \Sigma X & \xrightarrow{\Sigma f} & \Sigma Y \end{array}$$

where $\hat{p}$ is induced by $p \times 1$.

Corollary 2.4 [4, Theorem 1]. *Let* Y *be an* H*-space and* $g : A \to X$ *cyclic, that is, there is a map* $\phi : X \times A \to X$ *such that* $\phi j \sim \nabla(1 \vee g)$, *where* $j : X \vee A \to X \times A$ *is the inclusion. Then there exists a map* $\hat{\phi} : E_f \times A \to E_f$ *such that* $\hat{\phi}_{|E_f} \sim 1_{E_f}$ *and* $p\hat{\phi} = \phi(p \times 1)$ *if and only if* $\Sigma(f\phi)q\hat{p} \sim *$.

Proof. From Theorem 2.3, it is sufficient to show that $\Sigma(f\phi)q\hat{p} \sim *$ if and only if $f\phi(p \times 1) \sim *$. If $f\phi(p \times 1) \sim *$, then $\Sigma(f\phi)q\hat{p} \sim \Sigma(f\phi(p \times 1))q \sim *$. Suppose $\Sigma(f\phi)q\hat{p} \sim *$. Since $E_f \times A \xrightarrow{(1\times\imath)} E_f \times cA \to E_f \times cA/E_f \times A$ is a cofibration, there is an exact sequence $\to [\Sigma(E_f \times cA), \Sigma Y] \xrightarrow{(\Sigma(1\times\imath))^*} [\Sigma(E_f \times A), \Sigma Y] \xrightarrow{q^*} [E_f \times A/E_f \times A, \Sigma Y] \to$. Since $q^*(\Sigma(f\phi(p \times 1))) \sim \Sigma(f\phi)q\hat{p} \sim *$ and $\operatorname{Ker} q^* = \operatorname{Im}(\Sigma(1 \times \imath))^*$, there is a map $k : \Sigma(E_f \times cA) \to \Sigma Y$ such that $k_{|\Sigma(E_f\times A)} = \Sigma(f\phi(p \times 1))$. Taking adjoints, we get a map $E_f \times cA \to \Omega\Sigma Y$ extending $e'f\phi(p \times 1)$, where $e' = \tau(1_{\Sigma Y})$. Since Y is an H-space, there is a map $r : \Omega\Sigma Y \to Y$ such that $re' \sim 1_Y$. Hence we have a map $H : E_f \times A \times I \to Y$ satisfying $H(\ ,0) = f\phi(p\times 1)$ and $H(\ ,\ ,1) = fpp_1$, where $p_1 : E_f \times A \to E_f$ is the projection. From Lemma 2.1, $H(\ ,\ ,1) = fpp_1 \sim *$. Thus $f\phi(p \times 1) \sim *$.

We can also obtain the following proposition . The proof of Proposition 2.5 is similar to that of Theorem 2.3. So we will omit the proof.

Proposition 2.5. *Let* $g : A \to E_f$ *be a map such that* $pg : A \to X$ *is cyclic, that is, there is a map* $\phi : X \times A \to X$ *such that* $\phi j \sim \nabla(1 \vee pg)$, *where* $j : X \vee A \to X \times A$ *is the inclusion. Then there exists a map* $\hat{\phi} : E_f \times A \to E_f$ *such that* $\hat{\phi}_{|E_f} \sim 1_{E_f}$, $\hat{\phi}_{|A} \sim g$, *and* $p\hat{\phi} = \phi(p \times 1)$ *if and only if* $f\phi(p \times 1) \sim *$.

Corollary 2.6. *Let* $[A, X] = 0$. *Then, for any map* $f : X \to Y$, $[A, E_f] = G(A, E_f)$.

Proof. Consider the map $\phi : X \times A \to X$ is given by $\phi(x, a) = x$. Then $\phi(p \times 1) = pp_1$, where $p_1 : E_f \times A \to E_f$ is the projection. That is, $\phi(p \times 1) : E_f \times A \to X$ can be lifted to the map $p_1 : E_f \times A \to E_f$. Thus we know, from Lemma 2.1, that $f\phi(p \times 1) \sim *$. Let $[g] \in [A, E_f]$. Since $p_*([g]) \in [A, X] = 0$, $pg \sim *$. Thus $\phi j \sim \nabla(1 \vee *) \sim \nabla(1 \vee pg)$. From the fact $f\phi(p \times 1) \sim *$ and Proposition 2.5, $[g] \in G(A, E_f)$.

Remark 2.7. Consider the map $\psi : E_f \times \Omega Y \to E_f$ given by $\psi((x, \eta), \omega) = (x, \eta + \omega)$, where $+$ denotes the usual product of two paths. Then the inclusion $i : \Omega Y \to E_f$ is cyclic. Thus the induced map i_* maps $[A, \Omega Y]$ into $G(A, E_f)$ [8]. Therefore, Corollary 2.6 can also be obtained from the long exact sequence of homotopy sets for the fibration $\Omega Y \to E_f \to X$.

3. Extending dual Gottlieb sets

We now consider the dual situation. There is a well-known cofibration $X \xrightarrow{\imath} cX \to \Sigma X$, where $\imath(x) = [x, 1]$, cX is the reduced cone, and ΣX is the reduced suspension. Given a map $f : X \to Y$, consider the cofibration $Y \xrightarrow{i} C_f \to \Sigma X$ induced by $f : X \to Y$ from $\imath$. That is, C_f is the pushout of $f : X \to Y$ and $\imath : X \to cX$:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \imath}\downarrow & & \downarrow{\scriptstyle i} \\ cX & \longrightarrow & C_f \end{array}$$

where C_f is the mapping cone of f, the space obtained from $cX \amalg Y$ by identifying $[x, 1] \in cX$ with $f(x)$, $i(y) = y$. In fact, the cofibration $Y \xrightarrow{i} C_f \to \Sigma X$ is a principal cofibration [7]. Thus there is a map $\phi : C_f \to \Sigma X \vee C_f$ such that the following diagram is commutative:

$$\begin{array}{ccc} Y & \xrightarrow{i_2} & \Sigma X \vee Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle 1\vee i} \\ C_f & \xrightarrow{\phi} & \Sigma X \vee C_f \end{array}$$

where i_2 is the inclusion and $\phi : C_f \to \Sigma X \vee C_f$ is given by $\phi(y) = (*, y)$, $\phi([x, t]) = (\langle x, 2t \rangle, *)$, $0 \le t \le 1/2$, $\phi([x, t]) = (*, [x, 2t - 1])$, $1/2 \le t \le 1$.

The following two lemmas are standard.

Lemma 3.1 [6, Proposition 2.34, p. 25]. *A map $g : Y \to A$ can be extended to C_f (there is a map $h : C_f \to A$ such that $hi = g$) if and only if $gf \sim *$.*

Lemma 3.2 [7]. *Let $g_1, g_2 : C_f \to A$ be maps. Then $g_1 i \sim g_2 i$ if and only if there is a map $\gamma : \Sigma X \to A$ such that $g_1 \sim \nabla(\gamma \vee g_2)\phi$, where $\phi : C_f \to \Sigma X \vee C_f$ is given by $\phi(y) = (*, y)$, $\phi([x, t]) = (\langle x, 2t\rangle, *)$, $0 \le t \le 1/2$, $\phi([x, t]) = (*, [x, 2t-1])$, $1/2 \le t \le 1$.*

Lemma 3.3. *Let $g_k : C_f \to A_k$ be maps, $k = 1, 2$, and $g : C_f \to A_1 \vee A_2$ a map such that $p_k j g i \sim g_k i$, $k = 1, 2$, where $j : A_1 \vee A_2 \to A_1 \times A_2$ is the inclusion and $p_k : A_1 \times A_2 \to A_k$, $k = 1, 2$, are projections. Then there is a map $h : C_f \to A_1 \vee A_2$ such that $gi = hi$ and $p_k j h \sim g_k$, $k = 1, 2$.*

Proof. By Lemma 3.2, there are maps $\gamma_k : \Sigma X \to A_k$ such that $g_k \sim \nabla(\gamma_k \vee p_k j g)\phi$, $k = 1, 2$. Let $\gamma = (\gamma_1 \vee \gamma_2)\mu : \Sigma X \to A_1 \vee A_2$, where $\mu : \Sigma X \to \Sigma X \vee \Sigma X$ is the co-H-structure. Consider the map $h = \nabla(\gamma \vee g)\phi : C_f \to A_1 \vee A_2$. Then $hi = gi$ follows from the fact $\phi i = (1 \vee i)i_2$. Moreover,

$$\begin{aligned} p_k j h &= p_k j \nabla(\gamma \vee g)\phi = \nabla(p_k \vee p_k)(j \vee j)(\gamma \vee g)\phi \\ &= \nabla(p_k j \gamma \vee p_k j g)\phi \sim \nabla(\gamma_k \vee p_k j g)\phi \sim g_k. \end{aligned}$$

This proves the lemma.

Now we have the following theorem which is the dual of Theorem 2.3.

Theorem 3.4. *Let $g : Y \to A$ be cocyclic, that is, there is a map $\theta : Y \to Y \vee A$ such that $j\theta \sim (1 \times g)\Delta$, where $j : Y \vee A \to Y \times A$ is the inclusion and $\Delta : Y \to Y \times Y$ is the diagonal map. Then there exists a map $\hat{\theta} : C_f \to C_f \vee A$ such that $p_1 j' \hat{\theta} \sim 1_{C_f}$ and the diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\theta} & Y \vee A \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i \vee 1} \\ C_f & \xrightarrow{\hat{\theta}} & C_f \vee A \end{array}$$

commutes if and only if $(i \vee 1)\theta f \sim *$, where $j' : C_f \vee A \to C_f \times A$ is the inclusion and $p_1 : C_f \times A \to C_f$ is the projection.

Proof. If such a $\hat{\theta}$ exists, we have, from Lemma 3.1, that $(i \vee 1)\theta f \sim *$. Conversely, suppose $(i \vee 1)\theta f \sim *$. By Lemma 3.1, there is a map $\theta' : C_f \to C_f \vee A$ such that $\theta' i = (i \vee 1)\theta$. Then $p_1 j' \theta' i = p_1 j'(i \vee 1)\theta = p_1(i \times 1)j\theta \sim p_1(i \times g)\Delta = 1_{C_f} i$. Thus we have, from Lemma 3.3, that there is a map $\hat{\theta} : C_f \to C_f \vee A$ such that $\hat{\theta} i = \theta' i = (i \vee 1)\theta$, $p_1 j' \hat{\theta} \sim 1_{C_f}$, and $p_2 j' \hat{\theta} \sim p_2 j' \theta'$. This proves the theorem.

Consider the following diagram where each square homotopy commutes and each column is the Puppe sequence of the fibration:

$$\begin{array}{ccccccc}
C_f \vee A & \xleftarrow{i\vee 1} & Y \vee A & & & & \\
\uparrow{\scriptstyle 1\vee\varepsilon} & & \uparrow{\scriptstyle 1\vee\varepsilon} & & & & \\
C_f \vee PY & \xleftarrow{i\vee 1} & Y \vee PA & & & & \\
\uparrow & & \uparrow & & & & \\
F_2 & \xleftarrow{\hat{i}} & F_1 & & & & \\
\uparrow{\scriptstyle q} & & \uparrow{\scriptstyle q} & & & & \\
\Omega(C_f \vee A) & \xleftarrow{\Omega(i\vee 1)} & \Omega(Y \vee A) & \xleftarrow{\Omega\theta} & \Omega Y & \xleftarrow{\Omega f} & \Omega X
\end{array}$$

where $\hat{i}$ is induced by $i \vee 1$, F_1, F_2 are fibres of maps $1 \vee \varepsilon : Y \vee PA \to Y \vee A$, $1 \vee \varepsilon : C_f \vee PA \to C_f \vee A$ respectively.

Corollary 3.5 [4, Theorem 6]. *Let X be a co-H-space and $g : Y \to A$ cocyclic, that is, there is a map $\theta : Y \to Y \vee A$ such that $j\theta \sim (1 \times g)\Delta$, where $j : Y \vee A \to Y \times A$ is the inclusion and $\Delta : Y \to Y \times Y$ is the diagonal map. Then there exists a map $\hat{\theta} : C_f \to C_f \vee A$ such that $p_1 j' \hat{\theta} \sim 1_{C_f}$ and $\hat{\theta} i = (i \vee 1)\theta$ if and only if $\hat{i} q \Omega(\theta f) \sim *$, where $j' : C_f \vee A \to C_f \times A$ is the inclusion and $p_1 : C_f \times A \to C_f$ is the projection.*

Proof. From Theorem 3.4, it is sufficient to show that $\hat{i} q \Omega(\theta f) \sim *$ if and only if $(i \vee 1)\theta f \sim *$. If $(i \vee 1)\theta f \sim *$, then $\hat{i} q \Omega(\theta f) \sim q\Omega((i \vee 1)\theta f) \sim *$. Now suppose $\hat{i} q \Omega(\theta f) \sim *$. Since $F_2 \to C_f \vee PA \xrightarrow{(1\vee\varepsilon)} C_f \vee A$ is a fibration, there is an exact sequence $\to [\Omega X, \Omega(C_f \vee PA)] \xrightarrow{\Omega(1\vee\varepsilon)_*} [\Omega X, \Omega(C_f \vee A)] \xrightarrow{q_*} [\Omega X, F_2] \to [\Omega X, C_f \vee PA] \to$. Since $q_*(\Omega((i \vee 1)\theta f)) = \hat{i} q \Omega(\theta f) \sim *$ and $\operatorname{Ker} q_* = \operatorname{Im} \Omega(1 \vee \varepsilon)_*$, there is a map $\beta : \Omega X \to \Omega(C_f \vee PA)$ such that $\Omega(1 \vee \varepsilon)\beta \sim \Omega((i \vee 1)\theta f)$. Since X is a co-H-space, there is a map $s_X : X \to \Sigma\Omega X$ such that $e_X s_X \sim 1_X$, where $e_X = \tau^{-1}(1_{\Omega X})$. Let $\nu = \tau^{-1}(\beta)s : X \to C_f \vee PA$. Since $\tau^{-1}(\beta) = e_{(C_f \vee PA)}\Sigma\beta$ and $\Omega e_{(C_f \vee PA)}\Omega\Sigma\beta = \beta\Omega e_X$, $\Omega\nu = \Omega(e_{(C_f \vee PA)}\Sigma\beta s_X) = \beta\Omega(e_X s_X) \sim \beta$. Thus $\Omega((i\vee 1)\theta f) \sim \Omega(1\vee\varepsilon)\beta \sim \Omega((1\vee\varepsilon)\nu) : \Omega X \to \Omega(C_f \vee A)$. Since X is a co-H-space, the function $\Omega : [X, C_f \vee A] \to [\Omega X, \Omega(C_f \vee A)]$, given by $f \mapsto \Omega f$, is injective. Thus $(i \vee 1)\theta f \sim (1 \vee \varepsilon)\nu$. Let $l : C_f \vee PA \to (C_f \vee A)^I$ be given by $l(z, *) =$ constant path at $(z, *)$ and by $l(*, \eta) = i_2\eta$, where $i_2 : A \to C_f \vee A$ is the inclusion. Consider the map $h = l\nu : X \to (C_f \vee A)^I$. Then $h : X \to (C_f \vee A)^I$ gives rise to a map $H : X \times I \to C_f \vee A$ with $H(x, t) = h(x)(t)$. Then $H(\ ,0) = (1 \vee *)\nu$ and $H(\ ,1) = (1 \vee \varepsilon)\nu$, where $* : PA \to A$ is the constant map. Thus $(i \vee 1)\theta f \sim (1 \vee \varepsilon)\nu \sim (1 \vee *)\nu = i_1 p_1 j(1 \vee \varepsilon)\nu \sim i_1 p_1 j(i \vee 1)\theta f \sim i_1 i f$, where $i_1 : C_f \to C_f \vee A$, $j : C_f \vee A \to C_f \times A$ are natural inclusions and $p_1 : C_f \times A \to C_f$ is the projection. Thus we know, from Lemma 3.1, that $(i \vee 1)\theta f \sim *$.

We can also obtain the following proposition. The proof of Proposition 3.6 is similar to that of Theorem 3.4. So we will omit the proof.

Proposition 3.6. *Let* $g : C_f \to A$ *be a map such that* $gi : Y \to A$ *is cocyclic, that is, there is a map* $\theta : Y \to Y \vee A$ *such that* $j\theta \sim (1 \times gi)\Delta$, *where* $j : Y \vee A \to Y \times A$ *is the inclusion. Then there exists a map* $\hat{\theta} : C_f \to C_f \vee A$ *such that* $p_1 j'\hat{\theta} \sim 1_{C_f}$, $p_2 j'\hat{\theta} \sim g$, *and* $\hat{\theta} i = (i \vee 1)\theta$ *if and only if* $(i \vee 1)\theta f \sim *$, *where* $j' : C_f \vee A \to C_f \times A$ *is the inclusion and* $p_1 : C_f \times A \to C_f$, $p_2 : C_f \times A \to A$ *are projections.*

Corollary 3.7. *Let* $[Y, A] = 0$. *Then, for any map* $f : X \to Y$, $[C_f, A] = DG(C_f, A)$.

Proof. Let $\theta : Y \to Y \vee A$ be given by $\theta(y) = (y, *)$. Then $(i \vee 1)\theta = i_1 i$, where $i_1 : C_f \to C_f \vee A$ is the inclusion. That is, $(i \vee 1)\theta : Y \to C_f \vee A$ can be extended to the map $i_1 : C_f \to C_f \vee A$. Thus we know, from Lemma 3.1, that $(i \vee 1)\theta f \sim *$. Let $[g] \in [C_f, A]$. Since $i^*([g]) \in [Y, A] = 0$, $gi \sim *$. Thus $j\theta \sim (1 \times *)\Delta \sim (1 \times gi)\Delta$ and gi is cocyclic. From the fact $(i \vee 1)\theta f \sim *$ and Proposition 3.6, $[g] \in DG(C_f, A)$.

Remark 3.8. Consider the map $\phi : C_f \to \Sigma X \vee C_f$ given by $\phi(y) = (*, y)$, $\phi([x, t]) = (\langle x, 2t\rangle, *)$, $0 \le t \le 1/2$, $\phi([x, t]) = (*, [x, 2t-1])$, $1/2 \le t \le 1$. Then the quotient map $q : C_f \to \Sigma X$ is cocyclic. Thus the induced map q^* maps $[\Sigma X, A]$ into $DG(C_f, A)$ [8]. Therefore, Corollary 3.7 can also be obtained from the Puppe sequence of the cofibration $Y \to C_f \to \Sigma X$.

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