

## LIFTING GOTTLIEB SETS AND DUALITY

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**ABSTRACT.** Let  $p : E_f \rightarrow X$  be a fibration induced by a map  $f : X \rightarrow Y$  from the path space fibration  $\varepsilon : PY \rightarrow Y$ . Let  $g : A \rightarrow X$  be cyclic. When does  $g$  lift to a map  $A \rightarrow E_f$  which is cyclic? We give an answer of this question for arbitrary  $A$  and  $Y$ . Also, we give an answer in the dual situation.

### 1. INTRODUCTION

A based map  $f : A \rightarrow X$  is cyclic [8] if there exists a map  $\phi : X \times A \rightarrow X$  such that  $\phi j$  is homotopic to  $\nabla(1 \vee f)$ , where  $j : X \vee A \rightarrow X \times A$  is the inclusion and  $\nabla : X \vee X \rightarrow X$  is the folding map. The Gottlieb set denoted  $G(A, X)$  is the set of all homotopy classes of cyclic maps from  $A$  to  $X$ . Dually, a based map  $f : X \rightarrow A$  is cocyclic [8] if there exists a map  $\theta : X \rightarrow X \vee A$  such that  $j\theta$  is homotopic to  $(1 \times f)\Delta$ , where  $j : X \vee A \rightarrow X \times A$  is the inclusion and  $\Delta : X \rightarrow X \times X$  is the diagonal map. The dual Gottlieb set  $DG(X, A)$  is the set of all homotopy classes of cocyclic maps from  $X$  to  $A$ . In this paper we consider the following problem and its dual. Let  $f : X \rightarrow Y$  be a map and  $PY$  the space of paths in  $Y$  which begin at  $*$ . Let  $\varepsilon : PY \rightarrow Y$  be the fibration given by  $\varepsilon(\eta) = \eta(1)$ . Let  $p : E_f \rightarrow X$  be the fibration induced by  $f$  from  $\varepsilon$ , that is,  $E_f = \{(x, \eta) \in X \times PY \mid f(x) = \varepsilon(\eta)\}$  is the pullback of  $f : X \rightarrow Y$  and  $\varepsilon : PY \rightarrow Y$ . Let  $g : A \rightarrow X$  be cyclic, that is, there is a map  $\phi : X \times A \rightarrow X$  such that  $\phi j \sim \nabla(1 \vee g)$ , where  $j : X \vee A \rightarrow X \times A$  is the inclusion. When does  $g$  lift to a map  $A \rightarrow E_f$  which is cyclic? This can be achieved, if there is a map  $\hat{\phi} : E_f \times A \rightarrow E_f$  such that  $\hat{\phi}|_{E_f} \sim 1_{E_f}$  and  $\hat{\phi}(p \times 1) = p\hat{\phi}$ . In case  $A = S^n$  and  $Y$  is an Eilenberg-Mac Lane space, Gottlieb [1, Theorem 6.3], has given a necessary and sufficient condition for the existence of this  $\hat{\phi}$ . In case  $A$  is arbitrary and  $Y$  is a product of Eilenberg-Mac Lane spaces, Halbhavi and Varadarajan [2] have given a necessary and sufficient condition for the existence of such a  $\hat{\phi}$ . In case  $A$  is arbitrary and  $Y$  is an  $H$ -space, Hoo [4] has also given a necessary and sufficient condition for the existence of such a  $\hat{\phi}$ . We can obtain a necessary and sufficient condition for the existence of such a  $\hat{\phi}$  with arbitrary  $A$  and  $Y$ , and we show that Hoo's necessary and sufficient condition

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follows from ours. Our method is a modification of Haslam's which was used to study Postnikov systems and  $G$ -spaces. We now consider the dual situation. Let  $f : X \rightarrow Y$  be a map and  $cX$  the reduced cone of  $X$ . There is a cofibration  $X \xrightarrow{i} cX \rightarrow \Sigma X$ , where  $i(x) = [x, 1]$ . Let  $i : Y \rightarrow C_f$  be the cofibration induced by  $f$  from  $i$ , that is,  $C_f = cX \amalg Y/i(x) \sim f(x)$  is the pushout of  $f : X \rightarrow Y$  and  $i : X \rightarrow cX$ . Let  $g : Y \rightarrow A$  be cocyclic, that is, there is a map  $\theta : Y \rightarrow Y \vee A$  such that  $j\theta \sim (1 \times g)\Delta$ , where  $j : Y \vee A \rightarrow Y \times A$  is the inclusion. When does  $g$  extend to a map  $C_f \rightarrow A$  which is cocyclic? When there is a map  $\hat{\theta} : C_f \rightarrow C_f \vee A$  such that  $p_1 j' \hat{\theta} \sim 1_{C_f}$  and  $(i \vee 1)\theta = \hat{\theta} i$ , this can be achieved, where  $j' : C_f \vee A \rightarrow C_f \times A$  is the inclusion and  $p_1 : C_f \times A \rightarrow C_f$  is the projection. In case  $A$  is arbitrary and  $X$  is a wedge product of Moore spaces, Halbhavi and Varadarajan [2] have given a necessary and sufficient condition for the existence of this  $\hat{\theta}$ . In case  $A$  is arbitrary and  $X$  is a co- $H$ -space, Hoo [4] has given a necessary and sufficient condition for the existence of such a  $\hat{\theta}$ . We can obtain a necessary and sufficient condition for the existence of such a  $\hat{\theta}$  with arbitrary  $A$  and  $X$ , and we show that Hoo's necessary and sufficient condition follows from ours. All our spaces will be homotopy type of connected locally finite CW complexes. We assume also that spaces have nondegenerate base points. All homotopies and maps are to respect base points. The base point as well as the constant map will be denoted by  $*$ . Also, we denote by  $[X, Y]$  the set of homotopy classes of pointed maps  $X \rightarrow Y$ . The identity map of space will be denoted by  $1$  when it is clear from the context.  $\Sigma X$  and  $\Omega X$  denote the reduced suspension and the loop space of  $X$  respectively. The adjoint functor from the group  $[\Sigma X, Y]$  to the group  $[X, \Omega Y]$  will be denoted by  $\tau$ .

## 2. LIFTING GOTTLIEB SETS

Let  $f : X \rightarrow Y$  be a map and  $PY$  the space of paths in  $Y$  which begin at  $*$ . Let  $\varepsilon : PY \rightarrow Y$  be the fibration given by evaluating a path at its end point. Let  $p : E_f \rightarrow X$  be the fibration induced by  $f$  from  $\varepsilon$ . That is,  $E_f$  is the pullback of  $f : X \rightarrow Y$  and  $\varepsilon : PY \rightarrow Y$ ;

$$\begin{array}{ccc}
 E_f & \longrightarrow & PY \\
 p \downarrow & & \varepsilon \downarrow \\
 X & \xrightarrow{f} & Y
 \end{array}$$

where  $E_f = \{(x, \eta) \in X \times PY \mid f(x) = \varepsilon(\eta)\}$ ,  $p(x, \eta) = x$ . In fact, the fibration  $p : E_f \rightarrow X$  is principal.

The following two lemmas are standard.

**Lemma 2.1.** *A map  $g : A \rightarrow X$  can be lifted to a map  $A \rightarrow E_f$  if and only if  $fg \sim *$ .*

**Lemma 2.2** [3]. *Given maps  $g_k : A_k \rightarrow E_f$ ,  $k = 1, 2$ , and  $g : A_1 \times A_2 \rightarrow E_f$  satisfying  $pg|_{A_k} \sim pg_k$ ,  $k = 1, 2$ , then there is a map  $h : A_1 \times A_2 \rightarrow E_f$  such that  $ph = pg$  and  $h|_{A_k} \sim g_k$ ,  $k = 1, 2$ .*

**Theorem 2.3.** *Let  $g : A \rightarrow X$  be cyclic, that is, there is a map  $\phi : X \times A \rightarrow X$  such that  $\phi j \sim \nabla(1 \vee g)$ , where  $j : X \vee A \rightarrow X \times A$  is the inclusion. Then there*

exists a map  $\hat{\phi} : E_f \times A \rightarrow E_f$  such that  $\hat{\phi}|_{E_f} \sim 1_{E_f}$  and the diagram

$$\begin{array}{ccc} E_f \times A & \xrightarrow{\hat{\phi}} & E_f \\ p \times 1 \downarrow & & p \downarrow \\ X \times A & \xrightarrow{\phi} & X \end{array}$$

commutes if and only if  $f\phi(p \times 1) \sim *$ .

*Proof.* If such a  $\hat{\phi}$  exists, we have, from Lemma 2.1, that  $f\phi(p \times 1) \sim *$ . Conversely, suppose  $f\phi(p \times 1) \sim *$ . By Lemma 2.1, there is a map  $\phi' : E_f \times A \rightarrow E_f$  such that  $p\phi' = \phi(p \times 1)$ . Then  $p\phi'|_{E_f} = \phi(p \times 1)|_{E_f} \sim p1_{E_f}$ . Thus we have, from Lemma 2.2, that there is a map  $\hat{\phi} : E_f \times A \rightarrow E_f$  such that  $p\hat{\phi} = p\phi' = \phi(p \times 1)$ ,  $\hat{\phi}|_{E_f} \sim 1_{E_f}$ , and  $\hat{\phi}|_A \sim \phi'|_A$ . This proves the theorem.

Consider the following diagram where each square homotopy commutes and each column is the Puppe sequences of the cofibration:

$$\begin{array}{ccccc} E_f \times A & \xrightarrow{p \times 1} & X \times A & & \\ \downarrow & & \downarrow & & \\ E_f \times cA & \xrightarrow{p \times 1} & X \times cA & & \\ \downarrow & & \downarrow & & \\ E_f \times cA/E_f \times A & \xrightarrow{\hat{p}} & X \times cA/X \times A & & \\ q \downarrow & & q \downarrow & & \\ \Sigma(E_f \times A) & \xrightarrow{\Sigma(p \times 1)} & \Sigma(X \times A) & \xrightarrow{\Sigma\phi} & \Sigma X \xrightarrow{\Sigma f} \Sigma Y \end{array}$$

where  $\hat{p}$  is induced by  $p \times 1$ .

**Corollary 2.4** [4, Theorem 1]. *Let  $Y$  be an  $H$ -space and  $g : A \rightarrow X$  cyclic, that is, there is a map  $\phi : X \times A \rightarrow X$  such that  $\phi j \sim \nabla(1 \vee g)$ , where  $j : X \vee A \rightarrow X \times A$  is the inclusion. Then there exists a map  $\hat{\phi} : E_f \times A \rightarrow E_f$  such that  $\hat{\phi}|_{E_f} \sim 1_{E_f}$  and  $p\hat{\phi} = \phi(p \times 1)$  if and only if  $\Sigma(f\phi)q\hat{p} \sim *$ .*

*Proof.* From Theorem 2.3, it is sufficient to show that  $\Sigma(f\phi)q\hat{p} \sim *$  if and only if  $f\phi(p \times 1) \sim *$ . If  $f\phi(p \times 1) \sim *$ , then  $\Sigma(f\phi)q\hat{p} \sim \Sigma(f\phi(p \times 1))q \sim *$ . Suppose  $\Sigma(f\phi)q\hat{p} \sim *$ . Since  $E_f \times A \xrightarrow{(1 \times i)} E_f \times cA \rightarrow E_f \times cA/E_f \times A$  is a cofibration, there is an exact sequence  $\rightarrow [\Sigma(E_f \times cA), \Sigma Y] \xrightarrow{(\Sigma(1 \times i))^*} [\Sigma(E_f \times A), \Sigma Y] \xrightarrow{q^*} [E_f \times A/E_f \times A, \Sigma Y] \rightarrow$ . Since  $q^*(\Sigma(f\phi(p \times 1))) \sim \Sigma(f\phi)q\hat{p} \sim *$  and  $\text{Ker } q^* = \text{Im}(\Sigma(1 \times i))^*$ , there is a map  $k : \Sigma(E_f \times cA) \rightarrow \Sigma Y$  such that  $k|_{\Sigma(E_f \times A)} = \Sigma(f\phi(p \times 1))$ . Taking adjoints, we get a map  $E_f \times cA \rightarrow \Omega\Sigma Y$  extending  $e'f\phi(p \times 1)$ , where  $e' = \tau(1_{\Sigma Y})$ . Since  $Y$  is an  $H$ -space, there is a map  $r : \Omega\Sigma Y \rightarrow Y$  such that  $re' \sim 1_Y$ . Hence we have a map  $H : E_f \times A \times I \rightarrow Y$  satisfying  $H(\cdot, 0) = f\phi(p \times 1)$  and  $H(\cdot, 1) = fpp_1$ , where  $p_1 : E_f \times A \rightarrow E_f$  is the projection. From Lemma 2.1,  $H(\cdot, 1) = fpp_1 \sim *$ . Thus  $f\phi(p \times 1) \sim *$ .

We can also obtain the following proposition . The proof of Proposition 2.5 is similar to that of Theorem 2.3. So we will omit the proof.

**Proposition 2.5.** *Let  $g : A \rightarrow E_f$  be a map such that  $pg : A \rightarrow X$  is cyclic, that is, there is a map  $\phi : X \times A \rightarrow X$  such that  $\phi j \sim \nabla(1 \vee pg)$ , where  $j : X \vee A \rightarrow X \times A$  is the inclusion. Then there exists a map  $\hat{\phi} : E_f \times A \rightarrow E_f$  such that  $\hat{\phi}|_{E_f} \sim 1_{E_f}$ ,  $\hat{\phi}|_A \sim g$ , and  $p\hat{\phi} = \phi(p \times 1)$  if and only if  $f\phi(p \times 1) \sim *$ .*

**Corollary 2.6.** *Let  $[A, X] = 0$ . Then, for any map  $f : X \rightarrow Y$ ,  $[A, E_f] = G(A, E_f)$ .*

*Proof.* Consider the map  $\phi : X \times A \rightarrow X$  is given by  $\phi(x, a) = x$ . Then  $\phi(p \times 1) = pp_1$ , where  $p_1 : E_f \times A \rightarrow E_f$  is the projection. That is,  $\phi(p \times 1) : E_f \times A \rightarrow X$  can be lifted to the map  $p_1 : E_f \times A \rightarrow E_f$ . Thus we know, from Lemma 2.1, that  $f\phi(p \times 1) \sim *$ . Let  $[g] \in [A, E_f]$ . Since  $p_*([g]) \in [A, X] = 0$ ,  $pg \sim *$ . Thus  $\phi j \sim \nabla(1 \vee *) \sim \nabla(1 \vee pg)$ . From the fact  $f\phi(p \times 1) \sim *$  and Proposition 2.5,  $[g] \in G(A, E_f)$ .

*Remark 2.7.* Consider the map  $\psi : E_f \times \Omega Y \rightarrow E_f$  given by  $\psi((x, \eta), \omega) = (x, \eta + \omega)$ , where  $+$  denotes the usual product of two paths. Then the inclusion  $i : \Omega Y \rightarrow E_f$  is cyclic. Thus the induced map  $i_*$  maps  $[A, \Omega Y]$  into  $G(A, E_f)$  [8]. Therefore, Corollary 2.6 can also be obtained from the long exact sequence of homotopy sets for the fibration  $\Omega Y \rightarrow E_f \rightarrow X$ .

### 3. EXTENDING DUAL GOTTLIEB SETS

We now consider the dual situation. There is a well-known cofibration  $X \xrightarrow{i} cX \rightarrow \Sigma X$ , where  $i(x) = [x, 1]$ ,  $cX$  is the reduced cone, and  $\Sigma X$  is the reduced suspension. Given a map  $f : X \rightarrow Y$ , consider the cofibration  $Y \xrightarrow{i} C_f \rightarrow \Sigma X$  induced by  $f : X \rightarrow Y$  from  $i$ . That is,  $C_f$  is the pushout of  $f : X \rightarrow Y$  and  $i : X \rightarrow cX$ :

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i \downarrow & & i \downarrow \\ cX & \longrightarrow & C_f \end{array}$$

where  $C_f$  is the mapping cone of  $f$ , the space obtained from  $cX \amalg Y$  by identifying  $[x, 1] \in cX$  with  $f(x)$ ,  $i(y) = y$ . In fact, the cofibration  $Y \xrightarrow{i} C_f \rightarrow \Sigma X$  is a principal cofibration [7]. Thus there is a map  $\phi : C_f \rightarrow \Sigma X \vee C_f$  such that the following diagram is commutative:

$$\begin{array}{ccc} Y & \xrightarrow{i_2} & \Sigma X \vee Y \\ i \downarrow & & \vee i \downarrow \\ C_f & \xrightarrow{\phi} & \Sigma X \vee C_f \end{array}$$

where  $i_2$  is the inclusion and  $\phi : C_f \rightarrow \Sigma X \vee C_f$  is given by  $\phi(y) = (*, y)$ ,  $\phi([x, t]) = ((x, 2t), *)$ ,  $0 \leq t \leq 1/2$ ,  $\phi([x, t]) = (*, [x, 2t - 1])$ ,  $1/2 \leq t \leq 1$ .

The following two lemmas are standard.

**Lemma 3.1** [6, Proposition 2.34, p. 25]. *A map  $g : Y \rightarrow A$  can be extended to  $C_f$  (there is a map  $h : C_f \rightarrow A$  such that  $hi = g$ ) if and only if  $gf \sim *$ .*

**Lemma 3.2** [7]. *Let  $g_1, g_2 : C_f \rightarrow A$  be maps. Then  $g_1i \sim g_2i$  if and only if there is a map  $\gamma : \Sigma X \rightarrow A$  such that  $g_1 \sim \nabla(\gamma \vee g_2)\phi$ , where  $\phi : C_f \rightarrow \Sigma X \vee C_f$  is given by  $\phi(y) = (*, y)$ ,  $\phi([x, t]) = (\langle x, 2t \rangle, *)$ ,  $0 \leq t \leq 1/2$ ,  $\phi([x, t]) = (*, [x, 2t - 1])$ ,  $1/2 \leq t \leq 1$ .*

**Lemma 3.3.** *Let  $g_k : C_f \rightarrow A_k$  be maps,  $k = 1, 2$ , and  $g : C_f \rightarrow A_1 \vee A_2$  a map such that  $p_kjgi \sim g_ki$ ,  $k = 1, 2$ , where  $j : A_1 \vee A_2 \rightarrow A_1 \times A_2$  is the inclusion and  $p_k : A_1 \times A_2 \rightarrow A_k$ ,  $k = 1, 2$ , are projections. Then there is a map  $h : C_f \rightarrow A_1 \vee A_2$  such that  $gi = hi$  and  $p_kjh \sim g_k$ ,  $k = 1, 2$ .*

*Proof.* By Lemma 3.2, there are maps  $\gamma_k : \Sigma X \rightarrow A_k$  such that  $g_k \sim \nabla(\gamma_k \vee p_kjg)\phi$ ,  $k = 1, 2$ . Let  $\gamma = (\gamma_1 \vee \gamma_2)\mu : \Sigma X \rightarrow A_1 \vee A_2$ , where  $\mu : \Sigma X \rightarrow \Sigma X \vee \Sigma X$  is the co- $H$ -structure. Consider the map  $h = \nabla(\gamma \vee g)\phi : C_f \rightarrow A_1 \vee A_2$ . Then  $hi = gi$  follows from the fact  $\phi i = (1 \vee i)i_2$ . Moreover,

$$\begin{aligned} p_kjh &= p_kj\nabla(\gamma \vee g)\phi = \nabla(p_k \vee p_k)(j \vee j)(\gamma \vee g)\phi \\ &= \nabla(p_kj\gamma \vee p_kjg)\phi \sim \nabla(\gamma_k \vee p_kjg)\phi \sim g_k. \end{aligned}$$

This proves the lemma.

Now we have the following theorem which is the dual of Theorem 2.3.

**Theorem 3.4.** *Let  $g : Y \rightarrow A$  be cocyclic, that is, there is a map  $\theta : Y \rightarrow Y \vee A$  such that  $j\theta \sim (1 \times g)\Delta$ , where  $j : Y \vee A \rightarrow Y \times A$  is the inclusion and  $\Delta : Y \rightarrow Y \times Y$  is the diagonal map. Then there exists a map  $\hat{\theta} : C_f \rightarrow C_f \vee A$  such that  $p_1j'\hat{\theta} \sim 1_{C_f}$  and the diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\theta} & Y \vee A \\ i \downarrow & & i \vee 1 \downarrow \\ C_f & \xrightarrow{\hat{\theta}} & C_f \vee A \end{array}$$

*commutes if and only if  $(i \vee 1)\theta f \sim *$ , where  $j' : C_f \vee A \rightarrow C_f \times A$  is the inclusion and  $p_1 : C_f \times A \rightarrow C_f$  is the projection.*

*Proof.* If such a  $\hat{\theta}$  exists, we have, from Lemma 3.1, that  $(i \vee 1)\theta f \sim *$ . Conversely, suppose  $(i \vee 1)\theta f \sim *$ . By Lemma 3.1, there is a map  $\theta' : C_f \rightarrow C_f \vee A$  such that  $\theta'i = (i \vee 1)\theta$ . Then  $p_1j'\theta'i = p_1j'(i \vee 1)\theta = p_1(i \times 1)j\theta \sim p_1(i \times g)\Delta = 1_{C_f}i$ . Thus we have, from Lemma 3.3, that there is a map  $\hat{\theta} : C_f \rightarrow C_f \vee A$  such that  $\hat{\theta}i = \theta'i = (i \vee 1)\theta$ ,  $p_1j'\hat{\theta} \sim 1_{C_f}$ , and  $p_2j'\hat{\theta} \sim p_2j'\theta'$ . This proves the theorem.

Consider the following diagram where each square homotopy commutes and

each column is the Puppe sequence of the fibration:

$$\begin{array}{ccccc}
 C_f \vee A & \xleftarrow{i \vee 1} & Y \vee A & & \\
 \uparrow 1 \vee \varepsilon & & \uparrow 1 \vee \varepsilon & & \\
 C_f \vee PY & \xleftarrow{i \vee 1} & Y \vee PA & & \\
 \uparrow & & \uparrow & & \\
 F_2 & \xleftarrow{i} & F_1 & & \\
 \uparrow q & & \uparrow q & & \\
 \Omega(C_f \vee A) & \xleftarrow{\Omega(i \vee 1)} & \Omega(Y \vee A) & \xleftarrow{\Omega \theta} & \Omega Y \xleftarrow{\Omega f} \Omega X
 \end{array}$$

where  $\hat{i}$  is induced by  $i \vee 1$ ,  $F_1, F_2$  are fibres of maps  $1 \vee \varepsilon : Y \vee PA \rightarrow Y \vee A$ ,  $1 \vee \varepsilon : C_f \vee PA \rightarrow C_f \vee A$  respectively.

**Corollary 3.5** [4, Theorem 6]. *Let  $X$  be a co- $H$ -space and  $g : Y \rightarrow A$  cocyclic, that is, there is a map  $\theta : Y \rightarrow Y \vee A$  such that  $j\theta \sim (1 \times g)\Delta$ , where  $j : Y \vee A \rightarrow Y \times A$  is the inclusion and  $\Delta : Y \rightarrow Y \times Y$  is the diagonal map. Then there exists a map  $\hat{\theta} : C_f \rightarrow C_f \vee A$  such that  $p_1 j' \hat{\theta} \sim 1_{C_f}$  and  $\hat{\theta} i = (i \vee 1)\theta$  if and only if  $\hat{i} q \Omega(\theta f) \sim *$ , where  $j' : C_f \vee A \rightarrow C_f \times A$  is the inclusion and  $p_1 : C_f \times A \rightarrow C_f$  is the projection.*

*Proof.* From Theorem 3.4, it is sufficient to show that  $\hat{i} q \Omega(\theta f) \sim *$  if and only if  $(i \vee 1)\theta f \sim *$ . If  $(i \vee 1)\theta f \sim *$ , then  $\hat{i} q \Omega(\theta f) \sim q \Omega((i \vee 1)\theta f) \sim *$ . Now suppose  $\hat{i} q \Omega(\theta f) \sim *$ . Since  $F_2 \rightarrow C_f \vee PA \xrightarrow{(1 \vee \varepsilon)} C_f \vee A$  is a fibration, there is an exact sequence  $\rightarrow [\Omega X, \Omega(C_f \vee PA)] \xrightarrow{\Omega(1 \vee \varepsilon)^*} [\Omega X, \Omega(C_f \vee A)] \xrightarrow{q_*} [\Omega X, F_2] \rightarrow [\Omega X, C_f \vee PA] \rightarrow$ . Since  $q_*(\Omega((i \vee 1)\theta f)) = \hat{i} q \Omega(\theta f) \sim *$  and  $\text{Ker } q_* = \text{Im } \Omega(1 \vee \varepsilon)_*$ , there is a map  $\beta : \Omega X \rightarrow \Omega(C_f \vee PA)$  such that  $\Omega(1 \vee \varepsilon)\beta \sim \Omega((i \vee 1)\theta f)$ . Since  $X$  is a co- $H$ -space, there is a map  $s_X : X \rightarrow \Sigma \Omega X$  such that  $e_X s_X \sim 1_X$ , where  $e_X = \tau^{-1}(1_{\Omega X})$ . Let  $\nu = \tau^{-1}(\beta) s : X \rightarrow C_f \vee PA$ . Since  $\tau^{-1}(\beta) = e_{(C_f \vee PA)} \Sigma \beta$  and  $\Omega e_{(C_f \vee PA)} \Omega \Sigma \beta = \beta \Omega e_X$ ,  $\Omega \nu = \Omega(e_{(C_f \vee PA)} \Sigma \beta s_X) = \beta \Omega(e_X s_X) \sim \beta$ . Thus  $\Omega((i \vee 1)\theta f) \sim \Omega(1 \vee \varepsilon)\beta \sim \Omega((1 \vee \varepsilon)\nu) : \Omega X \rightarrow \Omega(C_f \vee A)$ . Since  $X$  is a co- $H$ -space, the function  $\Omega : [X, C_f \vee A] \rightarrow [\Omega X, \Omega(C_f \vee A)]$ , given by  $f \mapsto \Omega f$ , is injective. Thus  $(i \vee 1)\theta f \sim (1 \vee \varepsilon)\nu$ . Let  $l : C_f \vee PA \rightarrow (C_f \vee A)^I$  be given by  $l(z, *) = \text{constant path at } (z, *)$  and by  $l(*, \eta) = i_2 \eta$ , where  $i_2 : A \rightarrow C_f \vee A$  is the inclusion. Consider the map  $h = l\nu : X \rightarrow (C_f \vee A)^I$ . Then  $h : X \rightarrow (C_f \vee A)^I$  gives rise to a map  $H : X \times I \rightarrow C_f \vee A$  with  $H(x, t) = h(x)(t)$ . Then  $H(\cdot, 0) = (1 \vee *)\nu$  and  $H(\cdot, 1) = (1 \vee \varepsilon)\nu$ , where  $*$  :  $PA \rightarrow A$  is the constant map. Thus  $(i \vee 1)\theta f \sim (1 \vee \varepsilon)\nu \sim (1 \vee *)\nu = i_1 p_1 j(1 \vee \varepsilon)\nu \sim i_1 p_1 j(i \vee 1)\theta f \sim i_1 i f$ , where  $i_1 : C_f \rightarrow C_f \vee A$ ,  $j : C_f \vee A \rightarrow C_f \times A$  are natural inclusions and  $p_1 : C_f \times A \rightarrow C_f$  is the projection. Thus we know, from Lemma 3.1, that  $(i \vee 1)\theta f \sim *$ .

We can also obtain the following proposition. The proof of Proposition 3.6 is similar to that of Theorem 3.4. So we will omit the proof.

**Proposition 3.6.** *Let  $g : C_f \rightarrow A$  be a map such that  $gi : Y \rightarrow A$  is cocyclic, that is, there is a map  $\theta : Y \rightarrow Y \vee A$  such that  $j\theta \sim (1 \times gi)\Delta$ , where  $j : Y \vee A \rightarrow Y \times A$  is the inclusion. Then there exists a map  $\hat{\theta} : C_f \rightarrow C_f \vee A$  such that  $p_1 j' \hat{\theta} \sim 1_{C_f}$ ,  $p_2 j' \hat{\theta} \sim g$ , and  $\hat{\theta}i = (i \vee 1)\theta$  if and only if  $(i \vee 1)\theta f \sim *$ , where  $j' : C_f \vee A \rightarrow C_f \times A$  is the inclusion and  $p_1 : C_f \times A \rightarrow C_f$ ,  $p_2 : C_f \times A \rightarrow A$  are projections.*

**Corollary 3.7.** *Let  $[Y, A] = 0$ . Then, for any map  $f : X \rightarrow Y$ ,  $[C_f, A] = DG(C_f, A)$ .*

*Proof.* Let  $\theta : Y \rightarrow Y \vee A$  be given by  $\theta(y) = (y, *)$ . Then  $(i \vee 1)\theta = i_1 i$ , where  $i_1 : C_f \rightarrow C_f \vee A$  is the inclusion. That is,  $(i \vee 1)\theta : Y \rightarrow C_f \vee A$  can be extended to the map  $i_1 : C_f \rightarrow C_f \vee A$ . Thus we know, from Lemma 3.1, that  $(i \vee 1)\theta f \sim *$ . Let  $[g] \in [C_f, A]$ . Since  $i^*([g]) \in [Y, A] = 0$ ,  $gi \sim *$ . Thus  $j\theta \sim (1 \times *)\Delta \sim (1 \times gi)\Delta$  and  $gi$  is cocyclic. From the fact  $(i \vee 1)\theta f \sim *$  and Proposition 3.6,  $[g] \in DG(C_f, A)$ .

**Remark 3.8.** Consider the map  $\phi : C_f \rightarrow \Sigma X \vee C_f$  given by  $\phi(y) = (*, y)$ ,  $\phi([x, t]) = (\langle x, 2t \rangle, *)$ ,  $0 \leq t \leq 1/2$ ,  $\phi([x, t]) = (*, [x, 2t - 1])$ ,  $1/2 \leq t \leq 1$ . Then the quotient map  $q : C_f \rightarrow \Sigma X$  is cocyclic. Thus the induced map  $q^*$  maps  $[\Sigma X, A]$  into  $DG(C_f, A)$  [8]. Therefore, Corollary 3.7 can also be obtained from the Puppe sequence of the cofibration  $Y \rightarrow C_f \rightarrow \Sigma X$ .

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