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ADDITIVE DERIVATIONS OF NEST ALGEBRAS

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ABSTRACT. In this paper we prove that every additive derivation of a nest algebra acting on an infinite-dimensional Hilbert space is inner. This extends the relative result for linear derivations of nest algebras.

1. INTRODUCTION

The study of additive derivations was initiated by Johnson and Sinclair in [6] where they proved the following result: Let $\mathscr{A}$ be a semisimple Banach algebra and $D: \mathscr{A} \to \mathscr{A}$ be an additive derivation. Then $\mathscr{A}$ contains a central idempotent e such that $e\mathscr{A}$ and $(I-e)\mathscr{A}$ are invariant for D, $D_{(I-e)\mathscr{A}}$ is continuous, and $e\mathscr{A}$ is finite dimensional. This result implies that every additive derivation from $L(H)$ into $L(H)$ is inner, provided that H is an infinite-dimensional Hilbert space. Motivated by this result, we consider the additive derivation problem on nest algebras.

First we recall some definitions. Let $L(H)$ be the algebra of all linear bounded operators on Hilbert space H. A nest $\mathscr{N}$ is a complete totally ordered family of selfadjoint projections on H that contains 0 and I. The algebra $\operatorname{alg}\mathscr{N} = \{T \in L(H): P^{\perp}TP = 0 \text{ for all } P \in \mathscr{N}\}$ is called the nest algebra with respect to the nest $\mathscr{N}$.

For a subalgebra $\mathscr{A}$ of $L(H)$, an additive (linear) map $D: \mathscr{A} \to L(H)$ is said to be an additive (linear) derivation if $D(AB) = AD(B) + D(A)B$ holds for all $A, B \in \mathscr{A}$. D is said to be inner if there exists an operator T in $L(H)$ such that $D(A) = AT - TA$ holds for all A in $\mathscr{A}$.

It is well known that if $\mathscr{A}$ is a nest algebra on H, then every linear derivation from $\mathscr{A}$ into $L(H)$ is inner [2]. A natural question is: Is every additive derivation of a nest algebra inner? If $\dim H < \infty$, then there exist additive derivations of the nest algebra which are not inner (see [4, 7]). In [4] we proved that every additive derivation of a triangular operator algebra acting on an infinite-dimensional Hilbert space is inner. The purpose of this paper is to show that this is true for any nest algebra acting on an infinite-dimensional Hilbert space.

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2. The main part

First we give some definitions and state some basic facts.

Let K_1 and K_2 be two linear spaces over $\mathbb{C}$. A map T from K_1 into K_2 is called additive if $T(x+y) = Tx + Ty$ holds for all $x, y \in K_1$.

The following Lemma 2.1 is taken from [4].

Lemma 2.1. *Let $\mathcal{N}$ be a nest in $L(H)$ and D an additive derivation from $\operatorname{alg}\mathcal{N}$ into $L(H)$. Then there exists an additive map T from K into H such that $D(A) = AT - TA$ holds on K for all $A \in \operatorname{alg}\mathcal{N}$, where $K = H$ if $I_- \leq I$ and $K = \bigcup\{PH : P \in \mathcal{N}$ and $P < I\}$ if $I_- = I$. Here $I_- = \bigvee\{P : P \in \mathcal{N}, P < I\}$.*

Lemma 2.2. *Let $\mathcal{N}$ be a nest in $L(H)$ and $D\colon \operatorname{alg}\mathcal{N} \to L(H)$ be a linear derivation. Assume that $T\colon K \to H$ is an additive map such that $D(A) = AT - TA$ for all $A \in \operatorname{alg}\mathcal{N}$. Here K is as in Lemma 2.1. Then T is linear on K.*

Proof. By Christensen's result [2], there exists an operator $S \in L(H)$ such that $D(A) = AS - SA$ holds for all $A \in \operatorname{alg}\mathcal{N}$. Thus $(T-S)A = A(T-S)$ on K for all $A \in \operatorname{alg}\mathcal{N}$.

(1) If $I_- < I$, then we can choose $y \in (I - I_-)H$ such that $\|y\| = 1$. For any $x \in H$, define $A \in \operatorname{alg}\mathcal{N}$ as

$$Az = \langle z, y\rangle x, \qquad z \in H.$$

Then $(T-S)Ay = A(T-S)y$, i.e., $(T-S)x = \langle (T-S)y, y\rangle x$. Since x is arbitrary, we obtain that $(T-S) = \lambda I$ for some $\lambda \in \mathbb{C}$. Thus T is linear on H.

(2) Suppose that $I_- = I$. Take $P_n \in \mathcal{N}$ such that $P_n < I$ and $P_n \uparrow I$ in the strong operator topology. Then $K = \bigcup P_n H$ is dense in H. For any $x \in K$, there is some n such that $x \in P_n H$. Take $y \in (P_{n+1} - P_n)H$ such that $\|y\| = 1$. Define $A \in \operatorname{alg}\mathcal{N}$ as

$$Az = \langle z, y\rangle x, \qquad z \in H.$$

Since $(T-S)Ay = A(T-S)y$, we obtain

$$(T-S)x = \langle (T-S)y, y\rangle x;$$

thus $(T-S)x = \lambda_x x$ for some $\lambda_x \in \mathbb{C}$, which implies that $T - S = \lambda I$ on K for some $\lambda \in \mathbb{C}$. Therefore, T is linear on K.

Remark. From the above proof, we have that, if D and T satisfy the conditions in the lemma, then T can be extended uniquely to an operator $T \in L(H)$ such that $D(A) = AT - TA$ for all $A \in \operatorname{alg}\mathcal{N}$. Thus we may suppose that $T \in L(H)$.

Let $\mathcal{N}$ be a nest in $L(H)$ and $P, Q \in \mathcal{N}$ such that $Q < P < I$. Define $\widetilde{D}(RAR) = RD(RAR)R$ for all $A \in \operatorname{alg}\mathcal{N}$, where $R = P - Q$ and D is an additive derivation from $\operatorname{alg}\mathcal{N}$ into $L(H)$. Then $\widetilde{D}$ can be regarded as an additive derivation from $R(\operatorname{alg}\mathcal{N})R \subseteq L(RH)$ into $L(RH)$.

Lemma 2.3. *Let $\mathcal{N}$, P, Q, D, and $\widetilde{D}$ be as above. If $\widetilde{D}$ is linear, then so is D.*

Proof. From Lemma 2.1 there exists an additive map T from K into H such that $D(A) = AT - TA$ holds on K for all $A \in \operatorname{alg}\mathcal{N}$.

It is a routine exercise that T can be decomposed as $T = (T_{ij})_{3\times 3}$ with respect to $K = QH + RH + P^{\perp}H$. The additive mappings $T_{13}\colon P^{\perp}H \to QH$, $T_{23}\colon P^{\perp}H \to RH$, and $T_{33}\colon P^{\perp}H \to P^{\perp}H$ are defined on a dense subspace $M \subseteq P^{\perp}H$, where $M = \bigcup\{(E-P)H\colon E \in \mathcal{N},\ P < E < I\}$ when $I_- = I$ and $M = P^{\perp}H$ when $I_- < I$.

Let

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & RAR & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A \in \operatorname{alg}\mathcal{N}.$$

Then $B \in \operatorname{alg}\mathcal{N}$, and $\widetilde{D}(RAR) = RD(RBR)R = RART_{22} - T_{22}RAR$ for all $A \in \operatorname{alg}\mathcal{N}$. Since $\widetilde{D}$ is linear, by Lemma 2.2 we have that T_{22} is linear on a dense subspace N of RH.

Let

$$A = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

respectively. Since $D(A) = AT - TA$ on K and $D(A) \in L(H)$, we can obtain that T_{ij} is linear when $i \neq j$.

Let

$$A = \begin{bmatrix} 0 & C & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad C \in L(RH, QH).$$

Then $A \in \operatorname{alg}\mathcal{N}$, and

$$D(A) = AT - TA = \begin{bmatrix} CT_{21} & CT_{22} - T_{11}C & CT_{23} \\ 0 & -T_{21}C & 0 \\ 0 & -T_{31}C & 0 \end{bmatrix}$$

holds on K. Since $D(A) \in L(H)$, we have that $CT_{22} - T_{11}C$ is linear. For any $x \in QH$ and $t \in \mathbb{C}$, take $y \in N$ such that $\|y\| = 1$. Let $Cz = \langle z, y\rangle x$, $z \in RH$. Then, from $(CT_{22} - T_{11}C)ty = t(CT_{22} - T_{11}C)y$ and $CT_{22}ty = tCT_{22}y$, it follows that $T_{11}tx = T_{11}Cty = tT_{11}Cy = tT_{11}x$. Thus, T_{11} is linear.

Now we show that T_{33} is linear. Let

$$A = \begin{bmatrix} 0 & 0 & C \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad C \in L(P^{\perp}H, QH).$$

Then $A \in \operatorname{Alg}\mathcal{N}$. Since $D(A) \in L(H)$ and $D(A) = AT - TA$ on K, we have that $CT_{33} - T_{11}C$ is linear on M; thus, CT_{33} is linear on M. Fix any $x \in M$ and $t \in \mathbb{C}$. For any $y \in P^{\perp}H$, define $Cz = \langle z, y\rangle u$, $z \in P^{\perp}H$, where u is a fixed element in QH such that $\|u\| = 1$. Then

$$CT_{33}tx = \langle T_{33}tx, y\rangle u.$$

On the other hand,

$$CT_{33}tx = tCT_{33}x = \langle tT_{33}x, y\rangle u.$$

Thus $\langle T_{33}tx, y\rangle = \langle tT_{33}x, y\rangle$. Since y is arbitrary in $P^{\perp}H$, we obtain that $T_{33}tx = tT_{33}x$. Hence T_{33} is linear on M.

Now we have proven that T is linear on a dense linear manifold $\widetilde{K}$ of H. For any $t \in \mathbb{C}$ and $x \in H$, take $x_n \in \widetilde{K}$ such that $x_n \to x$. Then

$$D(tI)x_n = (tI)Tx_n - T(tI)x_n = 0.$$

Therefore, $D(tI)x = 0$, which implies that

$$D(tI) = 0 = tD(I).$$

For any $A \in \operatorname{alg}\mathscr{N}$ and $t \in \mathbb{C}$,

$$D(tA) = tID(A) + D(tI)A = tD(A).$$

Thus D is linear, and hence D is inner.

Lemma 2.4. *Let $\mathscr{N}$ be a nest in $L(H)$ and D an additive derivation from $\operatorname{alg}\mathscr{N}$ into $L(H)$. If D is not inner, then for any $M > 0$ there exists $t \in \mathbb{C}$ such that $|t| < 1$ and $\|D(tI)\| \geq M$.*

Proof. Since D is not inner, D is not linear. It is easy to verify that $D(tI)$ is not linear in t, which implies that there is a continuous functional g on $L(H)$ such that $\|g\| = 1$ and $g(D(tI))$ is not linear in t. Let $f(t) = g(D(tI))$. Then f is additive and is not continuous. By the result in [1], there exists $t \in \mathbb{C}$ such that $|t| < 1$ and $|f(t)| \geq M$. Thus $\|D(tI)\| \geq |g(D(tI))| \geq M$.

Theorem 2.5. *Let $\mathscr{N}$ be a nest acting on an infinite-dimensional Hilbert space H. Then every additive derivation from $\operatorname{alg}\mathscr{N}$ into $L(H)$ is inner.*

Proof. $\mathscr{N}$ must satisfy one of the following three conditions:

(1) $\mathscr{N}$ is a finite nest. Then the conclusion follows from Proposition 2.4 in [4].

(2) There exist $P_n \in \mathscr{N}$ such that $P_n < P_{n+1}$, $n = 1, 2, \ldots$. Let $R_n = P_n - P_{n-1}$, and define

$$D_n(R_nAR_n) = R_nD(R_nAR_n)R_n, \qquad A \in \operatorname{alg}\mathscr{N}.$$

Then D_n can be regarded as an additive derivation from $R_n(\operatorname{alg}\mathscr{N})R_n$ into $L(R_nH)$.

We claim that there exists n such that D_n is linear, and thus by Lemma 2.3 D is inner.

In fact, if every D_n is not inner, then by Lemma 2.4 there exist $t_n \in \mathbb{C}$ such that $|t_n| < 1$ and $\|D_n(t_nR_n)\| \geq n + 2\|R_nD(R_n)R_n\| = n$, $n = 1, 2, \ldots$.

Let $S = \sum_{n=1}^{\infty} t_nR_n$. Then $S \in \operatorname{alg}\mathscr{N}$ and $R_nS = SR_n = t_nR_n$. Since $R_nD(R_nSR_n)R_n = R_nD(t_nR_n)R_n = D_n(t_nR_n)$ and

$$\begin{aligned} R_nD(R_nSR_n)R_n &= R_nD(SR_n)R_n + R_nD(R_n)SR_n \\ &= R_nD(S)R_n + R_nSD(R_n)R_n + R_nD(R_n)SR_n \\ &= R_nD(S)R_n + 2t_nR_nD(R_n)R_n = R_nD(S)R_n, \end{aligned}$$

we have that

$$\|R_nD(S)R_n\| \geq \|D_n(t_nR_n)\| - 2|t_n|\|R_nD(R_n)R_n\| \geq n, \qquad n = 1, 2, \ldots.$$

This contradicts the fact that

$$\|R_nD(S)R_n\| \leq \|D(S)\|.$$

(3) There exists a sequence $\{Q_n\} \subset \mathscr{N}$ such that $Q_{n+1} < Q_n$, $n = 1, 2, \ldots$. By considering $(\operatorname{alg}\mathscr{N})^* = \operatorname{alg}\mathscr{N}^\perp$ and $\overline{D}(A) = D(A^*)^*$ for all $A \in \operatorname{alg}\mathscr{N}^\perp$, where $\mathscr{N}^\perp = \{P: P^\perp \in \mathscr{N}\}$, from case (2) $\overline{D}$ is inner, and thus D is inner.

Corollary 2.6. *Let $\mathscr{N}$ be a nest acting on an infinite-dimensional Hilbert space and $\mathscr{M}$ be an ultraweakly closed bimodule of $\mathscr{A}$ which contains the nest algebra. Then every additive derivation from $\mathscr{A}$ into $\mathscr{M}$ is inner.*

Proof. This follows from Theorem 6 in [5] and Theorem 2.5.

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References

1. J. Aczél, *Lectures on functional equations and their applications,*, Academic Press, New York, 1966.
2. E. Christensen, *Derivations of nest algebras*, Math. Ann. **229** (1977), 155–161.
3. K. Davidson, *Nest algebras*, Pitman Res. Notes Math. Ser., vol. 191, Longman Sci. Tech, Harlow, 1988.
4. Han Deguang, *Additive derivations of triangular operator algebras.* (preprint)
5. ——, *A note on the commutants of CSL algebras modulo bimodules*, Proc. Amer. Math. Soc. **116** (1992), 707–709.
6. B. E. Johnson and A. M. Sinclair, *Continuity of derivations and a problem of Kaplansky*, Amer. J. Math. **90** (1968), 1067–1073.
7. P. Šemrl, *Additive derivations of some operator algebras*, Illinois J. Math. **35** (1991), 234–240.

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