ADDITIVE DERIVATIONS OF NEST ALGEBRAS

HAN DEGUANG

(Communicated by Palle E. T. Jorgensen)

Abstract. In this paper we prove that every additive derivation of a nest algebra acting on an infinite-dimensional Hilbert space is inner. This extends the relative result for linear derivations of nest algebras.

1. Introduction

The study of additive derivations was initiated by Johnson and Sinclair in [6] where they proved the following result: Let $\mathcal{A}$ be a semisimple Banach algebra and $D: \mathcal{A} \to \mathcal{A}$ be an additive derivation. Then $\mathcal{A}$ contains a central idempotent $e$ such that $e\mathcal{A}$ and $(I-e)\mathcal{A}$ are invariant for $D$, $D((I-e)\mathcal{A})$ is continuous, and $e\mathcal{A}$ is finite dimensional. This result implies that every additive derivation from $L(H)$ into $L(H)$ is inner, provided that $H$ is an infinite-dimensional Hilbert space. Motivated by this result, we consider the additive derivation problem on nest algebras.

First we recall some definitions. Let $L(H)$ be the algebra of all linear bounded operators on Hilbert space $H$. A nest $\mathcal{N}$ is a complete totally ordered family of selfadjoint projections on $H$ that contains 0 and I. The algebra $\text{alg}\mathcal{N} = \{T \in L(H): P^\perp TP = 0 \text{ for all } P \in \mathcal{N}\}$ is called the nest algebra with respect to the nest $\mathcal{N}$.

For a subalgebra $\mathcal{A}$ of $L(H)$, an additive (linear) map $D: \mathcal{A} \to L(H)$ is said to be an additive (linear) derivation if $D(AB) = AD(B) + D(A)B$ holds for all $A, B \in \mathcal{A}$. $D$ is said to be inner if there exists an operator $T$ in $L(H)$ such that $D(A) = AT - TA$ holds for all $A$ in $\mathcal{A}$.

It is well known that if $\mathcal{A}$ is a nest algebra on $H$, then every linear derivation from $\mathcal{A}$ into $L(H)$ is inner [2]. A natural question is: Is every additive derivation of a nest algebra inner? If $\dim H < \infty$, then there exist additive derivations of the nest algebra which are not inner (see [4, 7]). In [4] we proved that every additive derivation of a triangular operator algebra acting on an infinite-dimensional Hilbert space is inner. The purpose of this paper is to show that this is true for any nest algebra acting on an infinite-dimensional Hilbert space.
2. The main part

First we give some definitions and state some basic facts. Let $K_1$ and $K_2$ be two linear spaces over $\mathbb{C}$. A map $T$ from $K_1$ into $K_2$ is called additive if $T(x + y) = Tx + Ty$ holds for all $x, y \in K_1$.

The following Lemma 2.1 is taken from [4].

**Lemma 2.1.** Let $\mathcal{N}$ be a nest in $L(H)$ and $D$ an additive derivation from $\text{alg}\mathcal{N}$ into $L(H)$. Then there exists an additive map $T$ from $K$ into $H$ such that $D(A) = AT - TA$ holds on $K$ for all $A \in \text{alg}\mathcal{N}$, where $K = H$ if $\mathcal{I}_- \quad \mathcal{I}$ and $K = \bigcup \{PH: P \in \mathcal{N} \text{ and } P \ll \mathcal{I}\}$ if $\mathcal{I}_- = \mathcal{I}$. Here $\mathcal{I}_- = \bigcup \{P: P \in \mathcal{N}, P \ll \mathcal{I}\}$.

**Lemma 2.2.** Let $\mathcal{N}$ be a nest in $L(H)$ and $D: \text{alg}\mathcal{N} \to L(H)$ be a linear derivation. Assume that $T: K \to H$ is an additive map such that $D(A) = AT - TA$ holds on $K$ for all $A \in \text{alg}\mathcal{N}$. Here $K$ is as in Lemma 2.1. Then $T$ is linear on $K$.

**Proof.** By Christensen's result [2], there exists an operator $S \in L(H)$ such that $D(A) = AS - SA$ holds for all $A \in \text{alg}\mathcal{N}$. Thus $(T - S)A = A(T - S)$ on $K$ for all $A \in \text{alg}\mathcal{N}$.

(1) If $\mathcal{I}_- < \mathcal{I}$, then we can choose $y \in (I - I_-)H$ such that $\|y\| = 1$. For any $x \in H$, define $A \in \text{alg}\mathcal{N}$ as

$$Az = (z, y)x, \quad z \in H.$$  

Then $(T - S)y = (T - S)x = (T - S)y, y)x$. Since $x$ is arbitrary, we obtain that $(T - S) = \lambda I$ for some $\lambda \in \mathbb{C}$. Thus $T$ is linear on $H$.

(2) Suppose that $\mathcal{I}_- = \mathcal{I}$. Take $P_n \in \mathcal{N}$ such that $P_n < I$ and $P_n \uparrow I$ in the strong operator topology. Then $K = \bigcup P_n H$ is dense in $H$. For any $x \in K$, there is some $n$ such that $x \in P_n H$. Take $y \in (P_n + 1 - P_n)H$ such that $\|y\| = 1$. Define $A \in \text{alg}\mathcal{N}$ as

$$Az = (z, y)x, \quad z \in H.$$  

Since $(T - S)y = A(T - S)y$, we obtain

$$(T - S)x = ((T - S)y, y)x;$$

thus $(T - S)x = \lambda_x x$ for some $\lambda_x \in \mathbb{C}$, which implies that $T - S = \lambda I$ on $K$ for some $\lambda \in \mathbb{C}$. Therefore, $T$ is linear on $K$.

**Remark.** From the above proof, we have that, if $D$ and $T$ satisfy the conditions in the lemma, then $T$ can be extended uniquely to an operator $T \in L(H)$ such that $D(A) = AT - TA$ for all $A \in \text{alg}\mathcal{N}$. Thus we may suppose that $T \in L(H)$.

Let $\mathcal{N}$ be a nest in $L(H)$ and $P, Q \in \mathcal{N}$ such that $Q \ll P \ll I$. Define $\tilde{D}(RAR) = RD(RAR)R$ for all $A \in \text{alg}\mathcal{N}$, where $R = P - Q$ and $D$ is an additive derivation from $\text{alg}\mathcal{N}$ into $L(H)$. Then $\tilde{D}$ can be regarded as an additive derivation from $R(\text{alg}\mathcal{N})R \subseteq L(RH)$ into $L(RH)$.

**Lemma 2.3.** Let $\mathcal{N}, P, Q, D$, and $\tilde{D}$ be as above. If $\tilde{D}$ is linear, then so is $D$.

**Proof.** From Lemma 2.1 there exists an additive map $T$ from $K$ into $H$ such that $D(A) = AT - TA$ holds on $K$ for all $A \in \text{alg}\mathcal{N}$.
It is a routine exercise that \( T \) can be decomposed as \( T = (T_{ij})_{3 \times 3} \) with respect to \( K = QH + RH + P^\perp H \). The additive mappings \( T_{13} : P^\perp H \to QH \), \( T_{23} : P^\perp H \to RH \), and \( T_{33} : P^\perp H \to P^\perp H \) are defined on a dense subspace \( M \subseteq P^\perp H \), where \( M = \bigcup \{(E - P)H : E \in \mathcal{N}, P < E < I\} \) when \( I_- = I \) and \( M = P^\perp H \) when \( I_- < I \).

Let

\[
B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & RAR & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A \in \text{alg} \mathcal{N}.
\]

Then \( B \in \text{alg} \mathcal{N} \), and \( \tilde{D}(RAR) = RD(RBR)R = RART_{22} - T_{22}RAR \) for all \( A \in \text{alg} \mathcal{N} \). Since \( \tilde{D} \) is linear, by Lemma 2.2 we have that \( T_{22} \) is linear on a dense subspace \( N \) of \( RH \).

Let

\[
A = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}
\]

respectively. Since \( D(A) = AT - TA \) on \( K \) and \( D(A) \in L(H) \), we can obtain that \( T_{ij} \) is linear when \( i \neq j \).

Let

\[
A = \begin{bmatrix} 0 & C & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C \in L(RH, QH).
\]

Then \( A \in \text{alg} \mathcal{N} \), and

\[
D(A) = AT - TA = \begin{bmatrix} CT_{21} & CT_{22} - T_{11}C & CT_{23} \\ 0 & -T_{21}C & 0 \\ 0 & -T_{31}C & 0 \end{bmatrix}
\]

holds on \( K \). Since \( D(A) \in L(H) \), we have that \( CT_{22} - T_{11}C \) is linear. For any \( x \in QH \) and \( t \in C \), take \( y \in N \) such that \( \|y\| = 1 \). Let \( Cz = \langle z, y \rangle x, \ z \in RH \). Then, from \( (CT_{22} - T_{11}C)y = t(CT_{22} - T_{11}C)y \) and \( C_{22}ty = tCT_{22}y \), it follows that \( T_{11}tx = T_{11}Cty = tT_{11}x \). Thus, \( T_{11} \) is linear.

Now we show that \( T_{33} \) is linear. Let

\[
A = \begin{bmatrix} 0 & C & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C \in L(P^\perp H, QH).
\]

Then \( A \in \text{Alg} \mathcal{N} \). Since \( D(A) \in L(H) \) and \( D(A) = AT - TA \) on \( K \), we have that \( CT_{33} - T_{11}C \) is linear on \( M \); thus, \( CT_{33} \) is linear on \( M \). Fix any \( x \in M \) and \( t \in C \). For any \( y \in P^\perp H \), define \( Cz = \langle z, y \rangle u, \ z \in P^\perp H \), where \( u \) is a fixed element in \( QH \) such that \( \|u\| = 1 \). Then

\[
CT_{33}tx = \langle T_{33}tx, y \rangle u.
\]

On the other hand,

\[
CT_{33}tx = tCT_{33}x = \langle tT_{33}x, y \rangle u.
\]

Thus \( \langle T_{33}tx, y \rangle = \langle tT_{33}x, y \rangle \). Since \( y \) is arbitrary in \( P^\perp H \), we obtain that \( T_{33}tx = tT_{33}x \). Hence \( T_{33} \) is linear on \( M \).
Now we have proven that $T$ is linear on a dense linear manifold $\tilde{K}$ of $H$. For any $t \in \mathbb{C}$ and $x \in H$, take $x_n \in \tilde{K}$ such that $x_n \to x$. Then
$$D(tI)x_n = (tI)Tx_n - T(tI)x_n = 0.$$ Therefore, $D(tI)x = 0$, which implies that
$$D(tI) = 0 = tD(I).$$
For any $A \in \text{alg}\mathcal{N}$ and $t \in \mathbb{C}$,
$$D(tA) = tD(A) + D(tI)A = tD(A).$$ Thus $D$ is linear, and hence $D$ is inner.

**Lemma 2.4.** Let $\mathcal{N}$ be a nest in $L(H)$ and $D$ an additive derivation from $\text{alg}\mathcal{N}$ into $L(H)$. If $D$ is not inner, then for any $M > 0$ there exists $t \in \mathbb{C}$ such that $|t| < 1$ and $\|D(tI)\| \geq M$.

**Proof.** Since $D$ is not inner, $D$ is not linear. It is easy to verify that $D(tI)$ is not linear in $t$, which implies that there is a continuous functional $g$ on $L(H)$ such that $\|g\| = 1$ and $g(D(tI))$ is not linear in $t$. Let $f(t) = g(D(tI))$. Then $f$ is additive and is not continuous. By the result in [1], there exists $t \in \mathbb{C}$ such that $|t| < 1$ and $|f(t)| \geq M$. Thus $\|D(tI)\| \geq \|g(D(tI))\| \geq M$.

**Theorem 2.5.** Let $\mathcal{N}$ be a nest acting on an infinite-dimensional Hilbert space $H$. Then every additive derivation from $\text{alg}\mathcal{N}$ into $L(H)$ is inner.

**Proof.** $\mathcal{N}$ must satisfy one of the following three conditions:

1. $\mathcal{N}$ is a finite nest. Then the conclusion follows from Proposition 2.4 in [4].

2. There exist $P_n \in \mathcal{N}$ such that $P_n < P_{n+1}$, $n = 1, 2, \ldots$. Let $R_n = P_n - P_{n-1}$, and define
$$D_n(R_nAR_n) = R_nD_n(R_nAR_n)R_n, \quad A \in \text{alg}\mathcal{N}.$$ Then $D_n$ can be regarded as an additive derivation from $R_n(\text{alg}\mathcal{N})R_n$ into $L(R_nH)$.

We claim that there exists $n$ such that $D_n$ is linear, and thus by Lemma 2.3 $D$ is inner.

In fact, if every $D_n$ is not inner, then by Lemma 2.4 there exist $t_n \in \mathbb{C}$ such that $|t_n| < 1$ and $\|D_n(t_nR_n)\| \geq n + 2\|R_nD_n(R_n)R_n\| = n$, $n = 1, 2, \ldots$.

Let $S = \sum_{n=1}^{\infty} t_nR_n$. Then $S \in \text{alg}\mathcal{N}$ and $R_nS = SR_n = t_nR_n$. Since $R_nD_n(SR_n)R_n = R_nD_n(t_nR_n)R_n = D_n(t_nR_n)$ and
$$R_nD_n(SR_n)R_n = R_nD_n(SR_n)R_n + R_nD_n(R_n)SR_n = R_nD(S)R_n + R_nSD(R_n)R_n + R_nD_n(R_n)SR_n = R_nD(S)R_n + 2t_nR_nD_n(R_n)R_n = R_nD(S)R_n,$$ we have that
$$\|R_nD(S)R_n\| \geq \|D_n(t_nR_n)\| - 2|t_n||\|R_nD_n(R_n)R_n\| \geq n, \quad n = 1, 2, \ldots.$$ This contradicts the fact that
$$\|R_nD(S)R_n\| \leq \|D(S)\|.$$

3. There exists a sequence $\{Q_n\} \subset \mathcal{N}$ such that $Q_{n+1} < Q_n$, $n = 1, 2, \ldots$. By considering $(\text{alg}\mathcal{N})^* = \text{alg}\mathcal{N}^\perp$ and $\overline{D}(A) = D(A^*)^*$ for all $A \in \text{alg}\mathcal{N}^\perp$, where $\mathcal{N}^\perp = \{P: P^\perp \in \mathcal{N}\}$, from case (2) $\overline{D}$ is inner, and thus $D$ is inner.
Corollary 2.6. Let $\mathcal{N}$ be a nest acting on an infinite-dimensional Hilbert space and $\mathcal{M}$ be an ultraweakly closed bimodule of $\mathcal{A}$ which contains the nest algebra. Then every additive derivation from $\mathcal{A}$ into $\mathcal{M}$ is inner.

Proof. This follows from Theorem 6 in [5] and Theorem 2.5.

Acknowledgment

The author expresses his thanks to the referee for several useful suggestions.

References

4. Han Deguang, Additive derivations of triangular operator algebras, (preprint)