PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 119, Number 4, December 1993

A NOTE ON A TRANSPLANTATION THEOREM OF KANJIN AND MULTIPLE LAGUERRE EXPANSIONS

S. THANGAVELU

(Communicated by J. Marshall Ash)

ABSTRACT. By applying a transplantation theorem of Kanjin, a multiplier theorem and a Cesàro summability result are proved for multiple Laguerre expansions. In the one-dimensional case an improved version of the multiplier theorem is obtained.

1

Consider the normalised Laguerre functions $\mathscr{L}_k^\alpha$, $\alpha > -1$, on $\mathbb{R}_+ = (0, \infty)$ defined by

$$\mathscr{L}_k^\alpha(t) = \left(\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}\right)^{1/2} L_k^\alpha(t) e^{-t/2} t^{\alpha/2} \tag{1.1}$$

where $L_k^\alpha(t)$ are the Laguerre polynomials of type α. The functions $\{\mathscr{L}_k^\alpha\}$ form a complete orthonormal system for $L^2(\mathbb{R}_+)$. Recently, in [4] Kanjin studied the mapping properties of the operator T_α^β, which is defined as

$$T_\alpha^\beta f = \sum_{k=0}^\infty (f, \mathscr{L}_k^\beta) \mathscr{L}_k^\alpha \tag{1.2}$$

where (f, g) stands for the inner product in $L^2(\mathbb{R}_+)$. For the operator T_α^β he has proved the following result.

Theorem 1.1 (Kanjin). *Let $\alpha, \beta > -1$ and $\gamma = \min\{\alpha, \beta\}$. If $\gamma \geq 0$ then*

$$\|T_\alpha^\beta f\|_p \leq C \|f\|_p \quad \text{for } 1 < p < \infty. \tag{1.3}$$

If $-1 < \gamma < 0$ then (1.3) is valid for p in the interval $(1+\gamma/2)^{-1} < p < -2/\gamma$.

The above theorem is called a transplantation theorem for the following reason. Given a bounded sequence $\lambda(k)$ we can define an operator M_λ^α by setting

$$M_\lambda^\alpha f = \sum_{k=0}^\infty \lambda(k) (f, \mathscr{L}_k^\alpha) \mathscr{L}_k^\alpha \tag{1.4}$$

Received by the editors February 25, 1992.
1991 *Mathematics Subject Classification*. Primary 42C15; Secondary 42C10, 43A55.

whenever f has the Laguerre expansion

$$f = \sum_{k=0}^{\infty} (f, \mathscr{L}_k^\alpha) \mathscr{L}_k^\alpha. \tag{1.5}$$

From the theorem, we can deduce the norm inequality

$$\|M_\lambda^\alpha f\|_p \le C \|f\|_p \tag{1.6}$$

for any α if we know (1.6) for a particular α_0. This follows from the identity

$$T_\beta^\alpha M_\lambda^\alpha T_\alpha^\beta f = M_\lambda^\beta f. \tag{1.7}$$

As an application Kanjin proves the following result concerning M_λ^α.

Theorem 1.2 (Kanjin). *Let $\lambda(t)$ be a four times differentiable function on $(0, \infty)$ and satisfy*

$$\sup_{t>0} |t^k \lambda^{(k)}(t)| \le c_k \tag{1.8}$$

for $k = 0, 1, 2, 3, 4$. Then (1.6) is true for $1 < p < \infty$ if $\alpha \ge 0$ and for $(1 + \alpha/2)^{-1} < p < -2/\alpha$ if $-1 < \alpha < 0$.

Theorem 1.2 is deduced by applying the transplantation theorem to the particular case $\alpha = 0$, which is proved by Dlugosz in [1]. Now the aim of this note is to prove an improved version of the above multiplier theorem and also to give applications to higher-dimensional Laguerre expansions.

2

Let $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_j \ge 0 \text{ for all } j\}$, and consider for every $\alpha \in \mathbb{R}_+^n$ and a multi-index $m = (m_1, m_2, \ldots, m_n)$, the normalised Laguerre functions $\mathscr{L}_m^\alpha$ on $\mathbb{R}_+^n$ defined by

$$\mathscr{L}_m^\alpha(x) = \prod_{j=1}^{n} \mathscr{L}_{m_j}^{\alpha_j}(x_j). \tag{2.1}$$

They form a complete orthonormal system for $L^2(\mathbb{R}_+^n)$, and the Laguerre expansion of a function f in $L^p(\mathbb{R}_+^n)$ can be written as

$$f = \sum_{m=0}^{\infty} (f, \mathscr{L}_m^\alpha) \mathscr{L}_m^\alpha \tag{2.2}$$

where the sum is extended over all the multi-indices. Expansions of the above type have been studied by Dlugosz [1] when α is a multi-index.

For the above series (2.2) we define the Cesàro means σ_N^δ of order δ by the equation

$$\sigma_N^\delta f = \frac{1}{A_N^\delta} \sum_{k=0}^{N} A_{N-k}^\delta \sum_{|m|=k} (f, \mathscr{L}_m^\alpha) \mathscr{L}_m^\alpha \tag{2.3}$$

where $A_k^\delta = \Gamma(k+\delta+1)/\Gamma(k+1)$ are the binomial coefficients. Given a function λ on $(0, \infty)$ we also define the multiplier operator M_λ^α as

$$M_\lambda^\alpha f = \sum_{k=0}^{\infty} \lambda(2k+n) \sum_{|m|=k} (f, \mathscr{L}_m^\alpha) \mathscr{L}_m^\alpha. \tag{2.4}$$

For the operators (2.3) and (2.4) we prove the following two theorems.

Theorem 2.1. *Let $\delta > \frac{1}{2}$. Then the uniform estimates*

$$\|\sigma_N^\delta f\|_p \le C\|f\|_p \tag{2.5}$$

are valid iff $4n/(2n+1+2\delta) < p < 4n/(2n-1-2\delta)$.

Theorem 2.2. *Assume that the function λ satisfies the conditions*

$$\sup_{t>0} |t^k \lambda^{(k)}(t)| \le c_k \tag{2.6}$$

for $k = 0, 1, 2, \ldots, \nu$ where $\nu = n+1$ if n is odd and $\nu = n+2$ if n is even. Then for $1 < p < \infty$ we have

$$\|M_\lambda^\alpha f\|_p \le C\|f\|_p. \tag{2.7}$$

In the case $n = 1$ we can take $\nu = 1$ in the hypothesis and (2.7) *is valid for* $\frac{4}{3} < p < 4$.

A slightly weaker form of Theorem 2.2 is proved in [1] when α is a multi-index. In that version one has $\nu = n+3$ for all n. Theorem 2.1 is known when $n = 1$ and is due to Gorlich and Markett [3, 5].

For the Laguerre series (2.2) we also define the Riesz transforms R_j, $j = 1, 2, \ldots, n$, by the formula

$$R_j f = \sum_{m=0}^{\infty} (2m_j + 1)(2|m| + n)^{-1} (f, \mathscr{L}_m^\alpha) \mathscr{L}_m^\alpha. \tag{2.8}$$

Riesz transforms for the Hermite and special Hermite expansions have been studied by the author in [9, 12]. For the above Riesz transforms (2.8) we prove

Theorem 2.3. *For $1 < p < \infty$ all the Riesz transforms R_j are bounded on* $L^p(\mathbb{R}_+^n)$.

All three theorems will be proved by appealing to the n-dimensional version of Kanjin's transplantation Theorem 1.1. For α, β in $\mathbb{R}_+^n$ we define T_α^β by

$$T_\alpha^\beta f = \sum_{m=0}^{\infty} (f, \mathscr{L}_m^\beta) \mathscr{L}_m^\alpha. \tag{2.9}$$

Then, for f in $C_0^\infty(\mathbb{R}_+^n)$ and $1 < p < \infty$,

$$\|T_\alpha^\beta f\|_p \le C\|f\|_p. \tag{2.10}$$

This follows from Theorem 1.1 by iteration.

In view of (2.10) Theorems 2.1, 2.2, and 2.3 will follow once we show that they are true in the particular case $\alpha = 0$. It will be shown in the next section that the case $\alpha = 0$ follows from known results on special Hermite expansions as a special case. The one-dimensional case of Theorem 2.2 when $\alpha = \frac{1}{2}$ will be deduced from the corresponding result on the Hermite expansions. This will be done in the last section.

3

Consider the functions $\psi_m(z)$ on $\mathbb{C}^n$ defined by

$$\psi_m(z) = \prod_{j=1}^{n} L_{m_j}(\tfrac{1}{2}|z_j|^2) e^{-|z_j|^2/4} \tag{3.1}$$

where $L_k(t)$ are the Laguerre polynomials of type 0. The functions $\psi_m(z)$ are called special Hermite functions since they are related to the Hermite function $\Phi_m(x)$ on $\mathbb{R}^n$. This terminology is due to Strichartz [6]. In fact, one has

$$\psi_m(z) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\Phi_m\left(\xi + \frac{y}{2}\right)\Phi_m\left(\xi - \frac{y}{2}\right) d\xi \tag{3.2}$$

where $z = x + iy$, $x, y \in \mathbb{R}^n$ (see [2]). Given f on $\mathbb{C}^n$ we have the special Hermite expansion

$$f(z) = (2\pi)^{-n}\sum_{m=0}^{\infty} f \times \psi_m(z) \tag{3.3}$$

where the twisted convolution $f \times g$ of two functions is defined by

$$f \times g(z) = \int_{\mathbb{C}^n} f(z-w)g(w)e^{(i/2)\operatorname{Im} z\cdot\overline{w}}\, dw. \tag{3.4}$$

We can also write (3.3) in the form

$$f(z) = (2\pi)^{-n}\sum_{k=0}^{\infty} f \times \varphi_k^{n-1}(z) \tag{3.5}$$

where $\varphi_k^{n-1}(z) = L_k^{n-1}(\frac{1}{2}|z|^2)e^{-|z|^2/4}$. For all these facts we refer to [11].

For the special Hermite expansion let C_N^δ be the Cesàro means defined by

$$C_N^\delta f = \frac{1}{A_N^\delta}\sum_{k=0}^{N} A_{N-k}^\delta \sum_{|m|=k}(f \times \psi_m). \tag{3.6}$$

Given a function λ on $(0, \infty)$ we also define a multiplier transform T_λ by

$$T_\lambda f = \sum_{k=0}^{\infty}\lambda(2k+n)\sum_{|m|=k}(f \times \psi_m). \tag{3.7}$$

In [11] we proved

Theorem 3.1. *Let* $\delta > \frac{1}{2}$. *Then for* f *in* $L^p(\mathbb{C}^n)$

$$\|C_N^\delta f\|_p \le C\|f\|_p$$

holds if and only if $4n/(2n+1+2\delta) < p < 4n/(2n-1-2\delta)$.

Regarding T_λ we have proved the following multiplier theorem in [10].

Theorem 3.2. *Let* λ *satisfy the hypothesis of Theorem* 2.2. *Then for* $1 < p < \infty$ *one has* $\|T_\lambda f\|_p \le C\|f\|_p$.

The case $\alpha = 0$ of Theorems 2.1 and 2.2 will be deduced from the above theorems in the following way. When f is a radial function the twisted convolution $f \times \varphi_k^{n-1}$ becomes

$$f \times \varphi_k^{n-1}(z) = \frac{k!(n-1)!}{(k+n-1)!}\left(\int_0^\infty f(r)\varphi_k^{n-1}(r)r^{2n-1}\, dr\right)\varphi_k^{n-1}(z) \tag{3.8}$$

where $\varphi_k^{n-1}(r) = \varphi_k^{n-1}(z)$ with $|z| = r$. If f is a polyradial function, i.e., $f(z_1, \dots, z_n) = f(r_1, \dots, r_n)$, $r_j = |z_j|$, then in view of (3.8) and (3.1) one has

$$f \times \psi_m = \left\{ \int_{\mathbb{R}_+^n} f(r_1, \dots, r_n) \left(\prod_{j=1}^n \mathscr{L}_{m_j}(\tfrac{1}{2} r_j^2) \right) r_1 \cdots r_n \, dr_1 \cdots dr_n \right\} \psi_m. \tag{3.9}$$

Therefore, one sees that

$$f \times \psi_m(\sqrt{2} z) = (g, \mathscr{L}_m) \mathscr{L}_m(r) \tag{3.10}$$

where $g(r_1, \dots, r_n) = f(\sqrt{2r_1}, \dots, \sqrt{2r_n})$. Therefore, $C_N^\delta f$ becomes $\sigma_N^\delta g$ and $T_\lambda f$ becomes $M_\lambda^0 g$; hence, the case $\alpha = 0$ of Theorems 2.1 and 2.2 follow.

The case $\alpha = 0$ of Theorem 3.3 follows from the fact (see [12]) that the Riesz transforms

$$S_j f = \sum_{m=0}^{\infty} (2m_j + 1)(2|m| + n)^{-1} f \times \psi_m \tag{3.11}$$

for the special Hermite expansions are bounded on $L^p(\mathbb{C}^n)$, $1 < p < \infty$.

4

Consider the normalised Hermite functions $h_k(x)$ on $\mathbb{R}$. We also consider the Laguerre function φ_k^α of another type defined by, for α real,

$$\varphi_k^\alpha(x) = \mathscr{L}_k^\alpha(x^2)(2x)^{1/2}, \qquad x \in \mathbb{R}_+. \tag{4.1}$$

Then the Hermite functions h_k and φ_k^α are related by (see [7])

$$h_{2k}(x) = (-1)^k \frac{1}{\sqrt{2}} \varphi_k^{-1/2}(x), \qquad h_{2k+1}(x) = (-1)^k \frac{1}{\sqrt{2}} \varphi_k^{1/2}(x). \tag{4.2}$$

Consider a multiplier transform M for the Hermite series defined by

$$Mf(x) = \sum_{k=0}^{\infty} \lambda(k)(f, h_k) h_k(x). \tag{4.3}$$

In [8] we proved

Theorem 4.1. *Assume that λ is bounded and satisfies $|t\lambda'(t)| \le C$ for all $t > 0$. Then M is bounded on $L^p(\mathbb{R})$, $1 < p < \infty$.*

Since h_{2k} is even and h_{2k+1} is odd, by considering f to be odd we see that

$$Mf(x) = \sum_{k=0}^{\infty} \lambda(2k+1)(f, \varphi_k^{1/2}) \varphi_k^{1/2}(x), \tag{4.4}$$

and this is related to $M_\lambda^{1/2}$ in the following way. An easy calculation shows that

$$(f, \varphi_k^{1/2}) = \frac{1}{\sqrt{2}} (g, \mathscr{L}_k^{1/2}) \tag{4.5}$$

where $f(\sqrt{x})x^{-1/4} = g(x)$. Therefore,

$$Mf(\sqrt{x})x^{-1/4} = 2\sum_{k=0}^{\infty} \lambda(2k+1)(g, \mathscr{L}_k^{1/2})\mathscr{L}_k^{1/2}(x). \tag{4.6}$$

If we know that for $\frac{4}{3} < p < 4$

$$\int_0^\infty |Mf(x)|^p x^{-p/2+1}\, dx \le C \int_0^\infty |f(x)|^p x^{-p/2+1}\, dx \tag{4.7}$$

then it follows that

$$\int_0^\infty |M_\lambda^{1/2} g(x)|^p\, dx \le C \int_0^\infty |g(x)|^p\, dx; \tag{4.8}$$

hence, the case $n = 1$, $\alpha = \frac{1}{2}$ of Theorem 2.2 follows. We claim that (4.7) is true.

To prove the claim we recall the proof of Theorem 4.1. Let T^t be the semigroup on $L^p(\mathbb{R})$ defined by

$$T^t f = \sum_{k=0}^{\infty} e^{-(2k+1)t}(f, h_k)h_k. \tag{4.9}$$

For this semigroup we defined the g and g^* functions in the following way:

$$(g(f, x))^2 = \int_0^\infty t|\partial_t T^t f(x)|^2\, dt, \tag{4.10}$$

$$(g^*(f, x))^2 = \int_{-\infty}^{\infty} \int_0^\infty t^{1/2}(1 + t^{-1/2}|x-y|)^{-2}|\partial_t T^t f(y)|^2\, dy\, dt. \tag{4.11}$$

For the g and g^* functions we proved that

$$C_1\|f\|_p \le \|g(f)\|_p \le C_2\|f\|_p, \qquad 1 < p < \infty, \tag{4.12}$$

$$\|g^*(f)\|_p \le C\|f\|_p, \qquad p > 2. \tag{4.13}$$

Under the assumption that $|t\lambda'(t)|$ is bounded we verified that

$$g(Mf, x) \le Cg^*(f, x), \tag{4.14}$$

and in view of (4.12) and (4.13) this proved Theorem 4.1.

Therefore, in order to prove the weighted version we need to check that

$$C_1\|f\|_{p,w} \le \|g(f)\|_{p,w} \le C_2\|f\|_{p,w}, \qquad \tfrac{4}{3} < p < 4, \tag{4.12$'$}$$

$$\|g^*(f)\|_{p,w} \le C\|f\|_{p,w}, \qquad 2 < p < 4, \tag{4.13$'$}$$

where $\|f\|_{p,w}$ stands for the norm

$$\|f\|_{p,w} = \left(\int_{-\infty}^{\infty} |f(x)|^p |x|^{-p/2+1}\, dx\right)^{1/p}.$$

Thus we need weighted norm inequalities for the g and g^* functions.

In [8] we proved the L^p boundedness of g by applying singular integral theory. We identified g with a singular integral operator whose kernel takes values in the Hilbert space $L^2(\mathbb{R}_+, t\,dt)$. When the weight function w is in the Muckenhoupt class A_p (see [13]) then we also have

$$\int_{-\infty}^{\infty} |g(f)|^p w(x)\,dx \le C \int_{-\infty}^{\infty} |f(x)|^p w(x)\,dx. \tag{4.15}$$

When $\frac{4}{3} < p < 4$, $w(x) = |x|^{-p/2+1}$ is in A_p; hence, the right-hand side inequality of $(4.12)'$ is valid. We will now show that the reverse inequality is also valid.

From [8] we recall that we have the partial isometry

$$\|g(f)\|_2 = \tfrac{1}{2}\|f\|_2; \tag{4.16}$$

from this, by polarisation, we obtain

$$\left|\int_{-\infty}^{\infty} f_1(x)\overline{f}_2(x)\,dx\right| = 4\int_{-\infty}^{\infty}\int_0^{\infty} t\partial_t T^t f_1(x)\overline{\partial_t T^t f_2(x)}\,dt\,dx. \tag{4.17}$$

This gives the inequality

$$\left|\int_{-\infty}^{\infty} f_1(x)\overline{f}_2(x)\,dx\right| \le 4\int_{-\infty}^{\infty} g(f_1, x)g(f_2, x)\,dx. \tag{4.18}$$

Let us now take $h(x) = f_2(x)|x|^{-1/2+1/p}$ so that

$$\begin{aligned} &\left|\int_{-\infty}^{\infty} f_1(x)|x|^{-1/2+1/p}\overline{f}_2(x)\,dx\right| \\ &\qquad \le 4\int_{-\infty}^{\infty} g(f_1, x)|x|^{-1/2+1/p} g(h, x)|x|^{-1/2+1/q}\,dx \end{aligned} \tag{4.19}$$

where q is the index conjugate to p. An application of Holder's inequality gives

(4.20)

$$\begin{aligned} &\int_{-\infty}^{\infty} g(f_1, x)|x|^{-1/2+1/p} g(h, x)|x|^{-1/2+1/q}\,dx \\ &\qquad \le \left(\int_{-\infty}^{\infty} |g(f_1, x)|^p|x|^{-p/2+1}\,dx\right)^{1/p}\left(\int_{-\infty}^{\infty} |g(h, x)|^q|x|^{-q/2+1}\,dx\right)^{1/q}. \end{aligned}$$

Applying the direct inequality $(4.12)'$ to the second factor we get

$$\begin{aligned} \int_{-\infty}^{\infty} |g(h, x)|^q|x|^{-q/2+1}\,dx &\le C\int_{-\infty}^{\infty} |f_2(x)|^q|x|^{-q/2+q/p+1-q/2}\,dx \\ &\le C\int_{-\infty}^{\infty} |f_2(x)|^q\,dx. \end{aligned} \tag{4.21}$$

In view of (4.20) and (4.21) the inequality (4.19) becomes

$$\left|\int_{-\infty}^{\infty} f_1(x)|x|^{-1/2+1/p}\overline{f}_2(x)\,dx\right| \le C\|g(f_1)\|_{p,w}\|f_2\|_q. \tag{4.22}$$

Taking the supremum over all f with $\|f_2\|_q \le 1$ we obtain

$$\int_{-\infty}^{\infty} |f_1(x)|^p|x|^{-p/2+1}\,dx \le C\|g(f_1)\|_{p,w}. \tag{4.23}$$

This completes the proof of $(4.12)'$.

To establish the inequality $(4.13)'$ we observe that

$$\int_{-\infty}^{\infty}(g^*(f,x))^2h(x)\,dx \le \int_{-\infty}^{\infty}(g(f,x))^2\Lambda h(x)\,dx \tag{4.24}$$

for every nonnegative function h where Λh is the Hardy-Littlewood maximal function. If $2 < p < 4$, let $r = p/2$ and s be the conjugate index of r. Setting $h_1(x) = h(x)|x|^{-1+1/r}$ we have

$$\begin{aligned} &\int_{-\infty}^{\infty}(g^*(f,x))^2|x|^{-1+1/r}h(x)\,dx \\ &\le C\int_{\infty}^{\infty}(g(f,x))^2|x|^{-1+1/r}|x|^{1/s}\Lambda h_1(x)\,dx \\ &\le C\left(\int_{-\infty}^{\infty}(g(f,x))^p|x|^{-p/2+1}\,dx\right)^{2/p}\left(\int_{-\infty}^{\infty}|x|(\Lambda h_1(x))^s\,ds\right)^{1/s} \end{aligned} \tag{4.25}$$

by an application of Holder's inequality. Since $s > 2$, $|x| \in A_s$; hence,

$$\begin{aligned} \int_{-\infty}^{\infty}|x|(\Lambda h_1(x))^s\,ds &\le C\int_{-\infty}^{\infty}|h(x)|^s|x|^{-s+s/r+1}\,dx \\ &\le C\int_{-\infty}^{\infty}|h(x)|^s\,dx. \end{aligned} \tag{4.26}$$

Thus we have the inequality

$$\begin{aligned} &\int_{-\infty}^{\infty}(g^*(f,x))^2|x|^{-1+1/r}h(x)\,dx \\ &\le C\left(\int_{-\infty}^{\infty}|f(x)|^p|x|^{-p/2+1}\,dx\right)^{2/p}\|h\|_s. \end{aligned} \tag{4.27}$$

Taking the supremum over all h with $\|h\|_s \le 1$ we obtain

$$\int_{-\infty}^{\infty}(g^*(f,x))^p|x|^{-p/2+1}\,dx \le C\int_{-\infty}^{\infty}|f(x)|^p|x|^{-p/2+1}\,dx. \tag{4.28}$$

This proves the inequality $(4.13)'$.

Therefore, in view of $(4.12)'$, $(4.13)'$, and (4.14) we obtain the weighted inequality

$$\int_{-\infty}^{\infty}|Mf(x)|^p|x|^{-p/2+1}\,dx \le C\int_{-\infty}^{\infty}|f(x)|^p|x|^{-p/2+1}\,dx \tag{4.29}$$

for $\frac{4}{3} < p < 4$, and this proves the multiplier theorem for $\alpha = \frac{1}{2}$. By applying the transplantation theorem we complete the proof of Theorem 2.2 when $n = 1$.

References

1. D. Dlugosz, *L^p multipliers for the Laguerre expansions*, Colloq. Math. **54** (1987), 287–293.
2. G. Folland, *Harmonic analysis in phase space*, Ann. of Math. Stud., vol. 112, Princeton Univ. Press, Princeton, NJ, 1989.
3. E. Gorlich and C. Markett, *Mean Cesaro summability and operator norms for Laguerre expansions*, Comment. Math. Prace Mat. Tomus Specialus II (1979), 139–148.
4. Y. Kanjin, *A transplantation theorem for Laguerre series*, preprint, 1990.

5. C. Markett, *Norm estimates for Cesaro means of Laguerre expansions*, Approximation and Function Spaces (Proc. Conf. Gdansk, 1979), North-Holland, Amsterdam, 1981, pp. 419–435.

6. R. Strichartz, *Harmonic analysis as spectral theory of Laplacians*, J. Funct. Anal. **87** (1989), 51–148.

7. G. Szego, *Orthogonal polynomials*, Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, RI, 1967.

8. S. Thangavelu, *Multipliers for Hermite expansions*, Mat. Ibero Americana **3** (1987), 1–24.

9. ———, *Riesz transforms and the wave equation for the Hermite operator*, Comm. Partial Differential Equations **15** (1990), 1199–1215.

10. ———, *Littlewood-Paley-Stein theory on $\mathbb{C}^n$ and Weyl multipliers*, Rev. Mat. Ibero Americana **6** (1990), 75–90.

11. ———, *Bochner-Riesz means, Weyl multipliers and special Hermite expansions*, Ark. Mat. **29** (1991), 307–321.

12. ———, *Conjugate Poisson integrals and Riesz transforms for Hermite and special Hermite expansions*, Colloq. Math. **64** (1993), 103–113.

13. A. Torchinsky, *Real variable methods in harmonic analysis*, Academic Press, London, 1986.

Tata Institute for Fundamental Research Centre, Post Box No. 1234, Indian Institute of Science Campus, Bangalore, India

E-mail address: veluma@tifrbng.ernet.in