

CANONICAL SYSTEM ON ELLIPTIC CURVES

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ABSTRACT. We deduce a canonical algebraic complete integrable system using the representation of the Heisenberg group. This system is shown to have solutions equivalent to those of the rigid body motion on $SO(3)$ (Euler Top).

Let A be an abelian variety embedded into some projective space and $\mathcal{L} = \mathcal{L}(D)$ the invertible sheaf associated with the divisor D cut out by the hyperplane section at infinity. The Heisenberg group is isomorphic to $\mathcal{G}(\mathcal{L}) = \{ \text{set of pairs } (x, \varphi) \text{ such that } x \in A \text{ and } \varphi: \mathcal{L} \approx T^*\mathcal{L} \text{ is an isomorphism} \}$, and, since it contains the geometric information of an abelian variety, it plays a fundamental role in the description of the equations defining abelian varieties [Mu]. This group can also be used to represent vector fields on abelian varieties. Indeed, the Wronskian with respect to the vector field X on A (in short, $W(f, g) = fXg - gXf$) is invariant under the action of the Heisenberg group [P]. This allows us to represent holomorphic vector fields on abelian varieties in a canonical fashion by means of some free parameters related to the equations defining the variety.

We realize a system by presenting it as the data (A_α, D_α, X) where A_α is an abelian variety, D_α is a divisor on A_α , and X is a vector field on A_α , parametrized by the integrals of the motion. Such parameters are seen to be (rationally) related to modular forms of a certain level.

The aim of this note is to provide an analysis leading to the following system of differential equations:

$$(*) \quad \dot{y}_0 = -\gamma y_1 y_2, \quad \dot{y}_1 = -\beta y_0 y_2, \quad \dot{y}_2 = -\alpha y_0 y_1$$

which defines (under certain restrictions on α, β, γ) a holomorphic vector field on each elliptic curve in \mathbb{P}^3 . We prove the following

Theorem. *The system (*) associated with the canonical family of elliptic curves in \mathbb{P}^3 and the Euler Top system (4) define in general equivalent vector fields on an elliptic curve. The linear map in \mathbb{P}^3 preserving the elliptic curves and the vector fields up to a constant is given by the map (7).*

The basic technique used here is the Schrödinger representation of the Heisenberg group. This can be useful in describing algebraic complete integrable systems linearizing on higher-dimensional abelian varieties (such as those discussed

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in [AvM]), in particular, in finding their canonical equations and invariants. See [Ba] for a different approach to this problem. Our analysis applied to elliptic curves leads essentially to the Euler Top. Thus, we have

Corollary. *The only holomorphic vector field on the family of elliptic curves in \mathbb{P}^3 is (up to linear equivalence) given by the Euler-Arnold equations of the rigid body in $SO(3)$.*

1. REPRESENTATIONS OF THE HEISENBERG GROUP $\mathcal{G}(4)$

In the sequel we use the notation and definitions of Mumford [Mu].

The Heisenberg group $\mathcal{G}(4)$ is the set $\mathbb{C}^* \times K(4) \times \widehat{K(4)}$, $K(4) \approx \widehat{K(4)} \approx \mathbb{Z}/4$. Its group law is

$$(\alpha, \sigma, \chi) \cdot (\alpha', \sigma', \chi') = (\alpha \cdot \alpha' \cdot \chi'(\sigma), \sigma + \sigma', \chi + \chi').$$

If $V(4)$ denotes the vector space of \mathbb{C} -valued functions in $K(4)$, then the action of $\mathcal{G}(4)$ on this space is given by

$$((\alpha, \sigma, \chi) \cdot f)(y) = \alpha \cdot \chi(y) \cdot f(\sigma + y).$$

Let σ be the generator of $K(4)$ and χ the generator of $\widehat{K(4)}$ such that $\chi(\sigma) = i = \sqrt{-1}$. A basis for $V(4)$ has the form $X_i = \delta_{\sigma^i}$, $i = 0, 1, 2, 3$, where $\delta_{\sigma}(\sigma^j) = \delta_{ij}$, and the action of $\sigma = (1, \sigma, 1)$ and $\chi = (1, 1, \chi)$ is

$$\sigma \cdot X_i = X_{i-1}, \quad \chi \cdot X_i = (\sqrt{-1})^i X_i.$$

Analogously, there is a “squared action” of $\mathcal{G}(4)$ on $V(8) =$ vector space of \mathbb{C} -valued functions on $\mathbb{Z}/8 = \langle \delta_k = \delta_{\tau^k}, k = 0, 1, \dots, 7 \rangle$ (see [Mu, p. 316]). This is described by the formula

$$((\alpha, \sigma^u, \chi^v) \cdot \delta_k)(\tau^s) = \alpha^2 \chi(\sigma)^{v \cdot s} \delta_{k-2u}(\tau^s), \quad \chi(\sigma) = \sqrt{-1}.$$

However, instead of X_i and δ_j , we shall use a basis that distinguishes between odd and even sections with respect to the -1 involution $\iota (if(u) = f(-u))$:

$$\{Y_0 = X_0 + X_2, Y_1 = X_0 - X_2, Y_2 = X_1 + X_3, Y_3 = X_1 - X_3\} \quad \text{for } V(4)$$

and

$$\{Z_0 = \delta_0 + \delta_4, Z_1 = \delta_0 - \delta_4, Z_2 = \delta_2 + \delta_6, Z_3 = \delta_2 - \delta_6, \\ Z_4 = \delta_1 + \delta_5, Z_5 = \delta_1 - \delta_5, Z_6 = \delta_3 + \delta_7, Z_7 = \delta_3 - \delta_7\} \quad \text{for } V(8).$$

Thus, the actions of $\mathcal{G}(4)$ on $V(4)$ and $V(8)$ are described in Tables I and II.

TABLE I. Action of $\mathcal{G}(4)$ on $V(4)$.

	Y_0	Y_1	Y_2	Y_3
σ	Y_2	$-Y_3$	Y_0	Y_1
χ	Y_1	Y_0	iY_3	iY_2
σ^2	1	-1	1	-1
χ^2	1	1	-1	-1

TABLE II. Action of $\mathcal{G}(4)$ on $V(8)$.

	Z_0	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7
σ	Z_2	$-Z_3$	Z_0	Z_1	Z_6	$-Z_7$	Z_4	Z_5
χ	1	1	-1	-1	i	i	$-i$	$-i$
σ^2	1	-1	1	-1	1	-1	1	-1
χ^2	1	1	1	1	-1	-1	-1	-1

2. NORMAL EQUATIONS FOR ELLIPTIC CURVES IN \mathbb{P}^3

For the sake of completeness we shall deduce normal equations for elliptic curves in \mathbb{P}^3 .

The invertible sheaf $\mathcal{L} = \mathcal{L}(4e)$ gives a projectively normal embedding of the elliptic curve A in projective space \mathbb{P}^3 via a basis of sections $s_i \in \Gamma(A, \mathcal{L})$, $i = 1, 2, 3$, $A \rightarrow \mathbb{P}(T(A, \mathcal{L}))$. In this case $\mathcal{L} = i^*(\mathcal{O}(1))$ (or, in other terms, the divisor $4e$ is cut out by a hyperplane in \mathbb{P}^3).

Consider the sequence

$$0 \rightarrow \mathcal{I}(2) \rightarrow \mathcal{O}_{\mathbb{P}^3}(2) \xrightarrow{\text{res}} \mathcal{O}_A(2) \rightarrow 0$$

where \mathcal{I} is the ideal sheaf. This leads to the exact sequence

$$(1) \quad 0 \rightarrow \Gamma(\mathbb{P}^3, \mathcal{I}(2)) \rightarrow \Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \rightarrow \Gamma(A, \mathcal{O}_A(2)) \rightarrow 0.$$

But $\Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \cong S^2(\Gamma(A, \mathcal{L}))$ is the space of homogeneous polynomials of degree 2 in s_0, s_1, s_2, s_3 (or 4×4 symmetric matrices). Therefore,

$$\dim \Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) = 10$$

and

$$\dim \Gamma(A, \mathcal{O}_A(2)) = \dim \Gamma(A, \mathcal{L}^{\otimes 2}) = 2 \dim \Gamma(A, \mathcal{L}) = 8.$$

Hence, there are two linearly independent quadratic equations defining the embedded curve.

Since \mathcal{L} is very ample, the second arrow of (1) amounts to the canonical map $S^2(\Gamma(A, \mathcal{L})) \rightarrow \Gamma(A, \mathcal{L}^{\otimes 2})$ given by tensoring sections. Now, a symmetric theta structure for \mathcal{L} [Mu, pp. 317-318] induces isomorphisms (unique up to scalar multiple) $V(4) \cong \Gamma(A, \mathcal{L})$, $V(8) \cong \Gamma(A, \mathcal{L}^{\otimes 2})$, and we get a sequence of $\mathcal{G}(4)$ -modules

$$(2) \quad 0 \rightarrow K \rightarrow S^2(V(4)) \rightarrow V(8) \rightarrow 0,$$

which is isomorphic to (1).

On the space of 4×4 symmetric matrices, the action of $\mathcal{G}(4)$ leaves invariant the following spaces marked with A, B, C, D :

$$\begin{array}{ccccc} & Y_0 & Y_1 & Y_2 & Y_3 \\ Y_0 & A & B & C & D \\ Y_1 & & A & D & C \\ Y_2 & & & A & B \\ Y_3 & & & & A \end{array}$$

The space $K \cong \Gamma(\mathbb{P}^3, \mathcal{I}(2))$ cannot be contained in B, D , or C . If so, K must be either B or C or D . But such quadrics cannot be zero on an

TABLE III

	Y_0	Y_1	Y_2	Y_3	invariant line
1	1	1	1	1	-
$\sigma_1 = \sigma^2$	1	-1	1	-1	$\{Y_1 = 0, Y_3 = 0\}$
$\chi_1 = \chi^2$	1	1	-1	-1	$\{Y_2 = 0, Y_3 = 0\}$
$\sigma_1 \cdot \chi_1$	1	-1	-1	1	$\{Y_1 = 0, Y_2 = 0\}$

elliptic curve, or else they contain lines which are fixed under the action of the 2-torsion translation group (Table III).

Therefore $\Gamma(\mathbb{P}^3, \mathcal{S}(2))$ is contained in the four-dimensional space of diagonal quadrics.

On the quadrics, polynomials σ and χ act as follows:

$$\begin{array}{cccccccc}
 & Y_0^2 & Y_1^2 & Y_2^2 & Y_3^2 & Y_0^2 + Y_1^2 & Y_2^2 + Y_3^2 & Y_0^2 - Y_1^2 & Y_2^2 - Y_3^2 \\
 \sigma & Y_2^2 & Y_3^2 & Y_0^2 & Y_1^2 & Y_2^2 + Y_3^2 & Y_0^2 + Y_1^2 & Y_2^2 - Y_3^2 & Y_0^2 - Y_1^2 \\
 \chi & Y_1^2 & Y_0^2 & -Y_3^2 & -Y_2^2 & 1 & -1 & -1 & 1
 \end{array}$$

and the space of diagonal quadrics decomposes into two isomorphic irreducible representations of dimension two. Generators are

$$q_1 = a(Y_0^2 + Y_1^2) + b(Y_2^2 - Y_3^2), \quad q_2 = b(Y_0^2 - Y_1^2) + a(Y_2^2 + Y_3^2).$$

The space of diagonal quadrics contains nontrivial modules generated by rank-2 quadrics, but such a module does not give equations for an elliptic curve. Therefore, the space $\Gamma(\mathbb{P}^3, \mathcal{S}(2))$ is generated by q_1 and q_2 , and the curve is completely characterized by the parameter $\lambda = b/a$.

3. THE CANONICAL VECTOR FIELD

We want to determine the equations of a globally defined vector field X in \mathbb{P}^3 whose flow is linear on each curve.

If X is a vector field on an abelian variety A and s and t are sections of the invertible sheaf \mathcal{L} , we define the Wronskian along X by

$$W_X(s, t) = W_X \left(\frac{f_\alpha}{h_\alpha}, \frac{g_\alpha}{h_\alpha} \right) = \frac{f_\alpha \cdot X g_\alpha - g_\alpha \cdot X f_\alpha}{h_\alpha^2} \quad \text{on } U_\alpha$$

where $\{(U_\alpha, h_\alpha)\}$ is the local data of a divisor D such that $\mathcal{L} = \mathcal{L}(D)$. This gives a well-defined pair

$$W_X: \Gamma(A, \mathcal{L}) \otimes \Gamma(A, \mathcal{L}) \rightarrow \Gamma(A, \mathcal{L}^{\otimes 2})$$

which has the following properties:

(1) $t^*W_X(s, t) = -W_X(t^*s, t^*t)$, t = the (-1) involution (on sections $t^*s(u) = s(-u)$).

(2) W_X is invariant under the action of the Heisenberg group. Namely, if $(x, \varphi) \in \mathcal{G}(\mathcal{L})$, $(x, \varphi^{\otimes 2}) \in \mathcal{G}(\mathcal{L}^{\otimes 2})$, and $U_{(x, \varphi)}: \Gamma(A, \mathcal{L}) \rightarrow \Gamma(A, \mathcal{L})$ is the action of $(x, \varphi): U_{(x, \varphi)}(s) = T_{-x}^*(\varphi(s))$, then

$$U_{(x, \varphi^{\otimes 2})} \cdot W_X(s, t) = W_X(U_{(x, \varphi)} \cdot s, U_{(x, \varphi)} \cdot t).$$

We use this invariance property and Tables I and II to get the Wronskian matrix

$(W_X(Y_i, Y_j))$. Here α, β, γ are free parameters:

$$(W_X(Y_i, Y_j)) = \begin{bmatrix} 0 & -\alpha \cdot Z_3 & \beta \cdot (Z_4 - Z_6) & \gamma \cdot (Z_5 - Z_7) \\ \alpha \cdot Z_3 & 0 & \gamma \cdot (Z_5 + Z_7) & \beta \cdot (Z_4 + Z_6) \\ -\beta \cdot (Z_4 - Z_6) & -\gamma \cdot (Z_5 + Z_7) & 0 & \alpha \cdot Z_1 \\ -\gamma \cdot (Z_5 - Z_7) & -\beta \cdot (Z_4 + Z_6) & -\alpha \cdot Z_1 & 0 \end{bmatrix}.$$

Furthermore, we can use the multiplication formula [Mu, p. 330] to obtain the description

$$(W_X(Y_i, Y_j)) = \begin{bmatrix} 0 & -\alpha Y_2 Y_3 & \beta Y_1 Y_3 & \gamma Y_1 Y_2 \\ \alpha Y_2 Y_3 & 0 & \gamma Y_0 Y_3 & \beta Y_0 Y_2 \\ -\beta Y_1 Y_3 & -\gamma Y_0 Y_3 & 0 & \alpha Y_0 Y_1 \\ -\gamma Y_1 Y_2 & -\beta Y_0 Y_2 & -\alpha Y_0 Y_1 & 0 \end{bmatrix}.$$

The equations of the vector field X can be obtained from this description of W . In the coordinates $y_i = Y_i/Y_3$ we have

$$(3) \quad X \left(\frac{Y_i}{Y_3} \right) = \frac{Y_3 X Y_i - Y_i X Y_3}{Y_3^2} = \frac{W(Y_3, Y_i)}{Y_3^2}, \quad \begin{cases} \dot{y}_0 = -\gamma y_1 y_2, \\ \dot{y}_1 = -\beta y_0 y_2, \\ \dot{y}_2 = -\alpha y_0 y_1. \end{cases}$$

To determine the integrals, we look for quadratic polynomials in $y_i, i = 1, 2, 3$. These are the forms killed by the vector field X . There is a basis of such forms invariant under the subgroup of $\mathcal{G}(\mathcal{L})$ that fixes the chosen section $Y_3 = 0$. The action of this group is given by:

$$\begin{matrix} & y_0 & y_1 & y_2 \\ \sigma^2 & -1 & 1 & -1 \\ \chi^2 & -1 & -1 & 1 \end{matrix}$$

So, such quadratic integrals are $ay_0^2 + by_1^2 + cy_2^2$, with $a\alpha + b\beta + c\gamma = 0$ the relations in the parameters. Thus, according to (2), a basis of the equations of the curve is $a_i y_0^2 + b_i y_1^2 + c_i y_2^2 = h_i, i = 1, 2$, for some constants h_i . For a vector field (3) there are Laurent solutions (in terms of the evolution parameter t) about the divisor at infinity:

$$\begin{aligned} y_0 &= \frac{\delta_1}{\sqrt{\beta\alpha}} \frac{1}{t} \left(1 - (u+v)t^2 - \frac{1}{10}(4uv + (u+v)^2)t^4 + \dots \right), \\ y_1 &= \frac{\delta_2}{\sqrt{\alpha\gamma}} \frac{1}{t} \left(1 + ut^2 + \frac{1}{10}(4v(u+v) - u^2)t^4 + \dots \right), \\ y_2 &= \frac{\delta_1\delta_2}{\sqrt{\gamma\beta}} \frac{1}{t} \left(1 + vt^2 + \frac{1}{10}(4u(u+v) - v^2)t^4 + \dots \right), \end{aligned}$$

where $\delta_1^2 = \delta_2^2 = 1$,

$$\begin{aligned} u &= \frac{1}{6}((h_2 a_1 - h_1 a_2)\gamma + (h_1 c_2 - h_2 c_1)\alpha), \\ v &= \frac{1}{6}((h_1 a_2 - h_2 a_1)\gamma + (h_2 b_1 - h_1 b_2)\beta). \end{aligned}$$

The divisor D at infinity (hyperplane section $Y_3 = 0$) is the formal sum of the points

$$Q(\delta_1, \delta_2) = \left[\frac{\delta_1}{\sqrt{\beta\alpha}} : \frac{\delta_2}{\sqrt{\alpha\gamma}} : \frac{\delta_1\delta_2}{\sqrt{\gamma\beta}} : 0 \right], \quad \delta_1^2 = \delta_2^2 = 1.$$

It coincides with the four points fixed under the involution ι . So the support of D is precisely a two-torsion translation orbit for the origin $e = Q(1, 1)$.

4. THE EULER TOP SYSTEM AS RELATED TO THE CANONICAL SYSTEM

The complexified Euler top system is described as the system of differential equations [Ar]

$$(4) \quad \begin{cases} \dot{z}_1 = (\lambda_3 - \lambda_2)z_2z_3, \\ \dot{z}_2 = (\lambda_1 - \lambda_3)z_1z_3, \\ \dot{z}_3 = (\lambda_2 - \lambda_1)z_1z_2, \end{cases} \quad (z_1, z_2, z_3) \in \mathbb{C}^3.$$

The invariant curves

$$p_1: z_1^2 + z_2^2 = z_3^2 = 0, \quad p_2: \lambda_1 z_1^2 + \lambda_2 z_2^2 + \lambda_3 z_3^2 - h = 0$$

are the affine part of elliptic curves obtained by adding four points at infinity (a divisor D'). Moreover, the invertible sheaf $\mathcal{L} = \mathcal{L}(D') \approx \mathcal{L}(4e')$ determines a projectively normal embedding via the functions $\{1, z_1, z_2, z_3\}$ which blow up at D' . Around D' we have the Taylor expansions in terms of $t =$ time evolution parameter:

$$\begin{aligned} z_1 &= \frac{\varepsilon_1}{\sqrt{\alpha_2\alpha_3}} \frac{1}{t} (1 - (u' + v')t^2 + \dots), & \varepsilon_1^2 = \varepsilon_2^2 = 1, \quad \alpha_1 &= \lambda_3 - \lambda_2, \\ & & \alpha_2 &= \lambda_1 - \lambda_3, \quad \alpha_3 = \lambda_2 - \lambda_1; \\ z_2 &= \frac{\varepsilon_2}{\sqrt{\alpha_3\alpha_1}} \frac{1}{t} (1 + u't^2 + \dots), & u' &= \frac{1}{6}((\lambda_3 - h)\alpha_3 + (h - \lambda_1)\alpha_1); \\ z_3 &= \frac{\varepsilon_1\varepsilon_2}{\sqrt{\alpha_1\alpha_2}} \frac{1}{t} (1 + v't^2 + \dots), & v' &= \frac{1}{6}((h - \lambda_2)\alpha_2 + (\lambda_1 - h)\alpha_1). \end{aligned}$$

Using the invariance of the vector field under translations one gets involutions

$$\begin{aligned} \sigma'_1: (z_1, z_2, z_3) &\rightarrow (-z_1, z_2, -z_3), \\ \tau'_1: (z_1, z_2, z_3) &\rightarrow (-z_1, -z_2, z_3), \\ \sigma'_1\tau'_1: (z_1, z_2, z_3) &\rightarrow (z_1, -z_2, -z_3) \end{aligned}$$

which amount to translations by $\frac{1}{2}$ -periods.

The involution flipping the sign of the vector field

$$\iota: (z_1, z_2, z_3) \rightarrow (-z_1, -z_2, -z_3)$$

fixes precisely the points of the divisor D' :

$$P(\varepsilon_1, \varepsilon_2) = \left[0: \frac{\varepsilon_1}{\sqrt{\alpha_2\alpha_3}} : \frac{\varepsilon_2}{\sqrt{\alpha_3\alpha_1}} : \frac{\varepsilon_1\varepsilon_2}{\sqrt{\alpha_1\alpha_2}} \right], \quad \varepsilon_1^2 - \varepsilon_2^2 = 1.$$

Any of these points can be chosen as origin. Let us pick $e' = P(1, 1)$ and consider the 2:1 map $\varphi': A \rightarrow \mathbb{P}^1$ via the basis of sections of $\mathcal{L}(2e')$: $\{1, f = (\sqrt{\alpha_2\alpha_3}z_1 + \sqrt{\alpha_1\alpha_3}z_2)(\sqrt{\alpha_2\alpha_3}z_1 + \sqrt{\alpha_1\alpha_2}z_3)\}$. Analogously, the basis

of sections of $\mathcal{L}(2e)$ given by 1 and $g = (\sqrt{\beta\alpha}y_0 + \sqrt{\gamma\alpha}y_1)(\sqrt{\beta\alpha}y_0 + \sqrt{\gamma\beta}y_2)$ determines a 2:1 cover $\varphi: A \rightarrow \mathbb{P}^1$ branched over four points, the image of the four $\frac{1}{2}$ -periods. In order to relate the Euler Top to our canonical system we want to describe a map that preserves the chosen origins $e \leftrightarrow e'$ and $\frac{1}{2}$ -periods translations via the isomorphism $\sigma \leftrightarrow \sigma' \chi \leftrightarrow \tau$. This induces a linear transformation of \mathbb{P}^1 which goes as follows:

$$\begin{aligned} f(e') &= f(P(1, 1)) = \infty, \\ f(\sigma'_1 e') &= f(P(-1, 1)) = -2(2u' + v') = (\lambda_2 - \lambda_1)(h - \lambda_3), \\ f(\tau'_1 e') &= f(P(-1, -1)) = -2(u' + 2v') = (\lambda_1 - \lambda_3)(\lambda_2 - h), \\ f(\sigma'_1 \tau'_1 e') &= f(P(1, -1)) = 0, \\ g(e) &= g(Q(1, 1)) = \infty, \\ g(\sigma_1 e) &= g(Q(-1, 1)) = -2(2u + v), \\ g(\chi_1 e) &= g(Q(-1, -1)) = -2(u + 2v), \\ g(\sigma_1 \chi_1 e) &= g(Q(1, -1)) = 0. \end{aligned}$$

Hence,

$$(5) \quad f = \frac{2u + v}{2u' + v'} g = c \cdot g,$$

and we get the quantity

$$(6) \quad \omega = \frac{(\lambda_1 - \lambda_3)(\lambda_2 - h)}{(\lambda_2 - \lambda_1)(h - \lambda_3)} = \frac{u + 2v}{2u + v} = \frac{u' + 2v'}{2u' + v'}.$$

This quantity represents the moduli of an elliptic curve A' isogenous to A . The linear transformation (5) extends to a linear mapping in \mathbb{P}^3 . Indeed, $\{1, f, f^2, X_e f\}$ and $\{1, g, g^2, X_c g\}$ are bases of sections of $\mathcal{L}(4e')$ and $\mathcal{L}(4e)$, where $X_e =$ Euler vector field $= c^{1/2} \cdot X_c = c^{1/2}$ (canonical vector field), thus, inducing the obvious linear map. In terms of the variables z_i and y_i we have

$$(7) \quad z_1 = \sqrt{\frac{\beta\alpha c^3}{\alpha_2\alpha_3}} y_0, \quad z_2 = \sqrt{\frac{\alpha\gamma c^3}{\alpha_3\alpha_1}} y_1, \quad z_3 = \sqrt{\frac{\gamma\beta c^3}{\alpha_1\alpha_2}} y_2.$$

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