

## ON COMPACT PERTURBATIONS AND COMPACT RESOLVENTS OF NONLINEAR $m$ -ACCRETIVE OPERATORS IN BANACH SPACES

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**ABSTRACT.** Several mapping results are given involving compact perturbations and compact resolvents of accretive and  $m$ -accretive operators. A simple and straightforward proof is given to an important special case of a result of Morales who has recently improved and/or extended various results by the author and Hirano. Improved versions of results of Browder and Morales are shown to be possible by studying various homotopies of compact transformations.

### 1. INTRODUCTION-PRELIMINARIES

In what follows, the symbol  $X$  stands for a real Banach space with norm  $\|\cdot\|$  and (normalized) duality mapping  $J$ . An operator  $T : X \supset D(T) \rightarrow 2^X$  is called "accretive" if for every  $x, y \in D(T)$  there exists  $j \in J(x-y)$  such that

$$\langle u - v, j \rangle \geq 0$$

for all  $u \in Tx, v \in Ty$ . An accretive operator  $T$  is " $m$ -accretive" if  $R(T + \lambda I) = X$  for all  $\lambda \in (0, \infty)$ . We denote by  $B_r(0)$  the open ball of  $X$  with center at zero and radius  $r > 0$ .

One of our purposes in this paper is to prove the following theorem:

**Theorem 1.** *Let  $T : X \supset D(T) \rightarrow 2^X$  be  $m$ -accretive and  $C : \overline{D(T)} \rightarrow X$  compact and uniformly continuous on bounded sets. Assume that there exist positive constants  $b, r$  such that, given  $p \in \overline{B_r(0)}$  and  $x \in D(T)$  with  $\|x\| \geq b$ , there exists  $j = j(p, x) \in Jx$  such that*

$$(*) \quad \langle u + Cx - p, j \rangle \geq 0$$

for all  $u \in Tx$ . Then  $\overline{B_r(0)} \subset \overline{(T + C)(B_b(0) \cap D(T))}$ .

This result improves, when  $C$  is uniformly continuous on bounded sets, a relevant result of Hirano [6, Theorem 2], a result of the author [8, Theorem 3] and the following theorem of Morales in [13].

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**Theorem 2** (Morales [13]). *Let  $X$  be a Banach space,  $T : D(T) \subset X \rightarrow 2^X$   $m$ -accretive, and  $C : \overline{D(T)} \rightarrow X$  compact. Suppose that there exist positive constants  $b, r$  such that for every  $x \in D(T)$  with  $\|x\| \geq b$  there exists  $j \in Jx$  satisfying*

$$(**) \quad \langle u + Cx, j \rangle \geq r\|x\|$$

for all  $u \in Tx$ . Then  $\overline{B_r(0)} \subset \overline{(T + C)(\overline{B_b(0)} \cap D(T))}$ .

It is easy to see that condition  $(**)$  implies condition  $(*)$  but with the same  $j \in Jx$  working for all  $p \in \overline{B_r(0)}$ . We can also show that Theorem 1 can be proved by using Theorem 2, but our intention here is to give a simple and straightforward proof of Theorem 1. Hirano [6, Theorem 2] gave a rather involved proof of his result, and several other results of his in [6] are dependent on that proof. The same is true for Morales's results in [13] in connection with Theorem 2 above. Our proof of Theorem 1 actually involves only the verification of the fact that a certain mapping is a homotopy of compact transformations. The author showed in [8, Theorem 3] that Hirano's result [6, Theorem 2] is true, with a much simpler proof, under the additional assumption that  $D(T)$  is absorbing (i.e.,  $x \in D(T)$  implies  $\lambda x \in D(T)$  for every  $\lambda \in (0, 1)$ ). Although the author considered single-valued mappings in [8], his results hold equally well for multivalued operators.

For an  $m$ -accretive operator  $T$ , the "resolvents"  $J_\lambda : X \rightarrow D(T)$  of  $T$  are defined by  $J_\lambda = (I + \lambda T)^{-1}$  for all  $\lambda \in (0, \infty)$ . The "Yosida approximants"  $T_\lambda : X \rightarrow X$  of  $T$  are defined by  $T_\lambda = \frac{1}{\lambda}(I - J_\lambda)$ . For  $x \in X$ , we define  $|Tx|$  by  $|Tx| = \lim_{\lambda \rightarrow 0} \|T_\lambda x\|$ . Some of the main properties of  $J_\lambda$  and  $T_\lambda$  are:

1.  $\|J_\lambda x - J_\lambda y\| \leq \|x - y\|$  for all  $x, y \in X$ .
2.  $\|J_\lambda x - x\| = \lambda \|T_\lambda x\| \leq \lambda \inf\{\|y\|; y \in Tx\}$  for all  $x \in D(T)$ .
3.  $T_\lambda$  is  $m$ -accretive on  $X$  and  $\|T_\lambda x - T_\lambda y\| \leq \frac{2}{\lambda} \|x - y\|$  for all  $\lambda > 0$ ,  $x, y \in X$ .
4.  $T_\lambda x \in TJ_\lambda x$  for all  $x \in X$ .
5.  $\|T_\lambda x\| \leq |Tx|$  for all  $x \in D(T)$ .

For these facts the reader is referred to Barbu [1] and Lakshmikantham and Leela [11].

In what follows, "continuous" means "strongly continuous" and the symbol " $\rightarrow$ " (" $\dashrightarrow$ ") means strong (weak) convergence. The symbol  $R$  ( $R_+$ ) stands for the set  $(-\infty, \infty)$  ( $[0, \infty)$ ) and the symbols  $\partial D$ ,  $\overline{D}$  denote the strong boundary and the strong closure of the set  $D$ , respectively. An accretive operator  $T$  is called "strongly accretive" if there exists a constant  $\alpha > 0$  such that for each  $x, y \in D(T)$  there exists  $j \in J(x - y)$  such that  $\langle u - v, j \rangle \geq \alpha \|x - y\|^2$  for all  $u \in Tx, v \in Ty$ . An operator  $T : X \supset D(T) \rightarrow X$  is "bounded" if it maps bounded subsets of  $D(T)$  onto bounded sets. It is "compact" if it is continuous and maps bounded subsets of  $D(T)$  onto relatively compact sets. It is called "demicontinuous" ("completely continuous") if it is strong-weak (weak-strong) continuous on  $D(T)$ .

**Definition 1.** Let  $U : [0, 1] \times Q \rightarrow X$ , where  $Q$  is a bounded subset of  $X$ . We say that the mapping  $U(t, x)$  is  $Q$ -continuous at  $t_0$  if for every  $\epsilon > 0$  there exists  $\delta(\epsilon, t_0) > 0$  with the property:  $\|U(t, x) - U(t_0, x)\| < \epsilon$  for every  $t \in [0, 1]$  with  $|t - t_0| < \delta(\epsilon, t_0)$  and every  $x \in Q$ . We say that  $U$  is  $Q$ -continuous on  $[0, 1]$  if for every  $\epsilon > 0$  there exists  $\delta(\epsilon) > 0$  with the property:  $\|U(t_1, x) - U(t_2, x)\| < \epsilon$  for every  $t_1, t_2 \in [0, 1]$  with  $|t_1 - t_2| < \delta(\epsilon)$  and every  $x \in Q$ .

We now state a fundamental lemma that is an easy consequence of the Leray-Schauder theory. It can be found in Lloyd [12, Theorem 4.4.11].

**Lemma 1.** Let  $X$  be a Banach space, and consider the equation

$$(1.1) \quad u - U(t, u) = 0,$$

for which we assume the following:

- (i)  $U : [0, 1] \times Q \rightarrow X$  is compact in its second variable for each  $t \in [0, 1]$ , where  $Q$  is a closed, convex, and bounded subset of  $X$  containing the origin in its interior. Furthermore,  $U(t, x)$  is  $Q$ -continuous on  $[0, 1]$ .
- (ii)  $U(0, \partial Q) \subset Q$ .
- (iii)  $U(t, x) \neq x, t \in [0, 1), x \in \partial Q$ .

Then (1.1) has a solution in  $Q$  for  $t = 1$ .

## 2. MAIN RESULTS

Theorem 1 is just a trivial corollary of the more informative Theorem 3 below.

**Theorem 3.** Let  $T : X \supset D(T) \rightarrow 2^X$  be  $m$ -accretive and  $C : \overline{D(T)} \rightarrow X$  compact and uniformly continuous on bounded sets. Let  $p \in X$ , and assume that there exists a positive constant  $b$  such that  $x \in D(T)$  and  $\|x\| \geq b$  imply that there exists  $j \in Jx$  such that

$$(*) \quad \langle u + Cx - p, j \rangle \geq 0$$

for all  $u \in Tx$ . Then  $p \in \overline{(T + C)(B_b(0) \cap D(T))}$ .

*Proof.* We may (and do) assume that  $0 \in Tx_0$  for some  $x_0 \in D(T)$ . Otherwise, we pick  $x_0 \in D(T)$  and consider instead the mappings  $T_1x \equiv Tx - v, C_1x \equiv Cx + v$ , where  $v$  is some point in  $Tx_0$ . These mapping have exactly the same properties as  $T, C$ . We fix  $p \in \overline{B_r(0)}$ , and we consider the approximate problem

$$(2.1) \quad Tx + Cx + \frac{1}{n}x \ni p$$

for  $n = 1, 2, \dots$ . We observe that this equation can be rewritten as

$$u + C(T + \frac{1}{n}I)^{-1}u - p = 0$$

or

$$(2.2) \quad u + (C(nT + I)^{-1})(nu) - p = 0.$$

In order to apply Lemma 1, we consider the equation

$$(2.3) \quad u + t[(C(tnT + I)^{-1})(nu) - p] = 0,$$

with  $t \in [0, 1]$ , and the mapping

$$U(t, u) \equiv t[(C(tnT + I)^{-1})(nu) - p].$$

We define  $U(0, x) = 0$ ,  $x \in X$ , and observe that  $U(t, x)$  is compact in  $x$  for all  $t \in [0, 1]$ . Let  $Q$  denote any bounded subset of  $X$ . We show that the mapping  $U(t, u)$  is  $Q$ -continuous on  $[0, 1]$ . To this end, we first let  $t_0 \in (0, 1]$ , and we show that  $U(t, u)$  is  $Q$ -continuous at  $t_0$ . In fact, for  $t \in (0, 1]$ ,  $u \in Q$ , let

$$y_{t_0}(u) = (t_0 n T + I)^{-1} n u, \quad y_t(u) = (t n T + I)^{-1} n u.$$

Then

$$y_{t_0}(u) = n u - t_0 n T_{t_0 n}(n u), \quad y_t(u) = n u - t n T_{t n}(n u),$$

and

$$y_t(u) - y_{t_0}(u) = -n(t T_{t n}(n u) - t_0 T_{t_0 n}(n u)).$$

Choosing  $j \in J(y_t(u) - y_{t_0}(u))$  properly, we obtain

$$\begin{aligned} \|y_t(u) - y_{t_0}(u)\|^2 &= -n \langle t T_{t n}(n u) - t_0 T_{t_0 n}(n u), j \rangle \\ &= -n t \langle T_{t n}(n u) - T_{t_0 n}(n u), j \rangle \\ &\quad - n \langle t T_{t_0 n}(n u) - t_0 T_{t_0 n}(n u), j \rangle \\ &\leq n |t - t_0| \|T_{t_0 n}(n u)\| \|y_t(u) - y_{t_0}(u)\|, \end{aligned} \tag{2.4}$$

where we have used the fact that  $T_{t n}(n u) \in T J_{t n}(n u) = T y_t(u)$ ,  $T_{t_0 n}(n u) \in T J_{t_0 n}(n u) = T y_{t_0}(u)$  and the accretiveness of  $T$ . To estimate  $\|T_{t_0 n}(n u)\|$ , we observe that

$$\|T_{t_0 n}(n u)\| = \|T_{t_0 n}(n u) - T_{t_0 n}(x_0)\| \leq \frac{2}{t_0 n} \|n u - x_0\| \tag{2.5}$$

because

$$t_0 n \|T_{t_0 n} x_0\| = \|x_0 - J_{t_0 n} x_0\| \leq \inf\{\|y\|; y \in T x_0\} = 0.$$

Combining (2.4) and (2.5), we obtain

$$\|y_t(u) - y_{t_0}(u)\| \leq \frac{2|t - t_0|}{t_0} \|n u - x_0\|,$$

which shows the  $Q$ -continuity of the function  $y_t(u)$  at each  $t_0 > 0$ .

The boundedness of the function  $y_t(u)$  on  $(0, 1] \times Q$  follows from

$$y_t(u) = J_{t n}(n u) = [J_{t n}(n u) - J_{t n} x_0] + J_{t n} x_0$$

and

$$\|y_t(u)\| \leq \|J_{t n}(n u) - J_{t n} x_0\| + \|J_{t n} x_0\| \leq \|n u - x_0\| + \|x_0\|.$$

It is easy to see now that  $U(t, u) \equiv t[Cy_t(u) - p]$  is  $Q$ -continuous at each  $t_0 \in (0, 1]$ . This follows from

$$\begin{aligned} \|t C y_t(u) - t_0 C y_{t_0}(u)\| &\leq \|t C y_t(u) - t C y_{t_0}(u)\| + \|t C y_{t_0}(u) - t_0 C y_{t_0}(u)\| \\ &\leq t \|C y_t(u) - C y_{t_0}(u)\| + |t - t_0| \|C y_{t_0}(u)\|. \end{aligned}$$

Using the compactness of  $C$ , we can also show that  $\|U(t, u)\| = t \|C y_t(u) - p\| \rightarrow 0$  as  $t \rightarrow 0^+$  uniformly with respect to  $u \in Q$ .

Thus, the mapping  $U(t, u)$  is  $Q$ -continuous at any point  $t \in [0, 1]$ . Since the interval  $[0, 1]$  is compact, a simple covering argument shows that  $U(t, u)$  is actually  $Q$ -continuous on  $[0, 1]$ .

Since for any ball  $Q = \overline{B_q(0)}$  we have  $U(0, \partial B_q(0)) = \{0\} \subset Q$ , Lemma 1 (with  $U$  replaced by  $-U$ ) will be applicable here for all large balls  $B_q(0)$

if we show that all possible solutions of (2.3) are bounded independently of  $t \in [0, 1)$ . Assume that this is not true. Since the only solution of (2.3) for  $t = 0$  is  $u = 0$ , there exists an unbounded sequence of solutions  $\{u_m\}_{m=1}^\infty$  of (2.3) with  $t = t_m \in (0, 1)$ . Let

$$x_m = (t_m nT + I)^{-1}(nu_m) = J_{t_m n}(nu_m) \in D(T).$$

Then we have that  $u_m = t_m v_m + \frac{1}{n}x_m$  for some  $v_m \in Tx_m$  and

$$(2.6) \quad t_m(v_m + Cx_m - p) + \frac{1}{n}x_m = 0.$$

We note that if  $\{\|x_m\|\}_{m=1}^\infty$  is bounded, then the compactness of  $C$  and (2.6) imply that  $\{t_m v_m + \frac{1}{n}x_m\}_{m=1}^\infty$  is bounded, which contradicts the unboundedness of  $\{u_m\}_{m=1}^\infty$ . Consequently,  $\{\|x_m\|\}_{m=1}^\infty$  is unbounded and  $\|x_m\| \geq b$  for some  $m$ . This implies that there exists some  $j \in J(x_m)$  such that

$$(2.7) \quad \langle v_m + Cx_m - p, j \rangle \geq 0.$$

This and (2.6) yield the contradiction:

$$(2.8) \quad \frac{1}{n}\|x_m\|^2 \leq t_m \langle v_m + Cx_m - p, j \rangle + \frac{1}{n}\|x_m\|^2 = 0.$$

Lemma 1 implies that (2.1) is solvable for each  $n = 1, 2, \dots$ . It is easy to see, as above, that all solutions of (2.1) lie in the ball  $B_b(0)$ . Thus, we have our conclusion,  $p \in \overline{(T + C)(B_b(0) \cap D(T))}$ .

Since Morales considers in [13] only the possibility  $\overline{B_b(0)} \subset \overline{R(T + C)}$  or  $\overline{B_b(0)} \subset R(T + C)$ , our comments on his results refer to the conditions which are implied by his results to conclude that a single point  $p$  belongs to  $\overline{R(T + C)}$  or  $R(T + C)$ . Theorem 4 below provides an improvement of Corollary 1 and the Proposition in Morales [13]. In the Proposition of [13]  $T : X \rightarrow B(X)$  is continuous, where  $B(X)$  is the space of closed and bounded subsets of  $X$  associated with the Hausdorff metric. Also, a condition stronger than ours holds there on  $\partial B_b(0)$ . Our proof is also different because we introduce a new homotopy  $U(t, x)$ .

**Theorem 4.** *Let  $T : X \supset D(T) \rightarrow 2^X$  be  $m$ -accretive with  $0 \in D(T)$  and  $C : \overline{D(T)} \cap \overline{B_b(0)} \rightarrow X$  compact. Let  $p \in X$ , and assume that there exists a constant  $b > 0$  such that  $x \in D(T) \cap \partial B_b(0)$  implies that there exists  $j \in Jx$  such that  $(*)$  is satisfied for all  $u \in Tx$ . Then  $p \in \overline{(T + C)(D(T) \cap \overline{B_b(0)})}$ . If, moreover,  $T$  is strongly accretive, then  $p \in R(T + C)$ .*

*Proof.* We may (and do) assume that  $0 \in T0$ ; otherwise we can use the operators  $T_1, C_1$  as in the proof of Theorem 3 with  $x_0 = 0$ . By Lemma 31 in Rothe's book [14] ("completely continuous" means "compact" in that book), we consider the compact extension of  $C$  from the closed and bounded set  $\overline{D(T)} \cap \overline{B_b(0)}$  to the whole space  $X$ . We denote this extension also by  $C$ . We now make use of equation (2.1), which we rewrite as

$$x = (T + \frac{1}{n}I)^{-1}(-Cx - p) = (nT + I)^{-1}(-n(Cx - p)).$$

This equation leads to the homotopy equation

$$(2.9) \quad x - U(t, x) = 0,$$

where

$$U(t, x) \equiv (tnT + I)^{-1}(-tn(Cx - p)), \quad (t, x) \in [0, 1] \times X,$$

and we observe that  $U(0, x) \equiv 0$  and that  $U(t, x)$  is compact in  $x$  for all  $t \in [0, 1]$ . Let  $Q = \overline{B_b(0)}$ . The  $Q$ -continuity of  $U(t, x)$  at any  $t_0 \in (0, 1]$  follows from

$$\begin{aligned} \|U(t, x) - U(t_0, x)\| &\leq \|(tnT + I)^{-1}(-tn(Cx - p)) - (t_0nT + I)^{-1}(-t_0n(Cx - p))\| \\ &\quad + \|(t_0nT + I)^{-1}(-t_0n(Cx - p)) - (t_0nT + I)^{-1}(-t_0n(Cx - p))\| \\ &\leq \frac{2|t - t_0|}{t_0} \|tn(Cx - p)\| + n|t - t_0| \|Cx - p\| \end{aligned}$$

for all  $x \in Q$ , where we have used our estimates from the proof of Theorem 3 for  $x_0 = 0$ . The  $Q$ -continuity of  $U(t, x)$  at  $t = 0$  follows from

$$\|U(t, x)\| \leq tn\|Cx - p\|$$

for all  $t > 0$  and all  $x \in Q$ . As in the proof of Theorem 3,  $U(t, x)$  is  $Q$ -continuous on  $[0, 1]$ . To apply Lemma 1 with  $Q = \overline{B_b(0)}$ , we need to show that (2.9) has no solution on  $\partial B_b(0)$  for any  $t \in [0, 1]$ . Obviously, this is true for  $t = 0$ . Let  $t \in (0, 1)$  and  $x_t \in \partial B_b(0)$  solve (2.9). Then  $x_t \in D(T)$ . Let  $j \in J(x_t)$  be such that (2.7) holds with  $x_m$  replaced by  $x_t$ , and  $v_m$  by  $v_t$ . Then (2.8) holds, with  $t_m$  replaced by  $t$ ,  $x_m$  by  $x_t$ , and  $v_m$  by  $v_t$ , and we have a contradiction. Thus (2.1) is solvable for each  $n$  with each solution  $x_n \in D(T) \cap \overline{B_b(0)}$ . Now, assume that  $T$  is strongly accretive. Then the compactness of  $C$  implies that  $Tx_{n_k}$  is convergent, for some subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$ . The strong accretiveness of  $T$  implies easily that  $x_{n_k} \rightarrow$  (some)  $x_0 \in D(T) \cap \overline{B_b(0)}$ . Since  $C$  is continuous and  $T$  is closed, we obtain that  $Tx_0 + Cx_0 \ni p$ . This completes the proof.

For compact resolvents of the  $m$ -accretive operator  $T$  we have the following result which improves Theorem 7 of Morales [13].

**Theorem 5.** *Let  $T : D(T) \subset X \rightarrow 2^X$  be  $m$ -accretive with  $J_1 = (T + I)^{-1}$  compact. Let  $C : \overline{D(T)} \rightarrow X$  be continuous and bounded and  $p \in X$ . Assume that there exists  $b > 0$  such that for every  $x \in D(T)$  with  $\|x\| \geq b$  there exists  $j \in Jx$  such that (\*) holds for all  $u \in Tx$ . Then  $p \in R(T + C)$ .*

*Proof.* We just note here that the proof follows the steps of the proof of Theorem 3 because the mapping  $U(t, x)$  is still compact for all  $t \in [0, 1]$  and its  $Q$ -continuity on  $[0, 1]$  (for a bounded set  $Q \subset X$ ) follows now from the continuity and the boundedness of the mapping  $C$ . Thus, (2.1) is solvable with solutions  $x_n$  lying inside  $B_b(0)$  as before. Now, we observe that

$$x_n = (T + dI)^{-1}((d - \frac{1}{n})x_n - Cx_n + p),$$

for a fixed  $d > 0$ , and the compactness of  $(T + dI)^{-1}$  along with the boundedness of  $C$  implies that  $\{x_n\}$  lies inside a compact set. Thus, there exist a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  and a point  $\bar{x} \in \overline{D(T)} \cap \overline{B_b(0)}$  such that  $x_{n_k} \rightarrow \bar{x}$  as  $k \rightarrow \infty$ . Since  $C$  is continuous, we have  $Cx_{n_k} \rightarrow C\bar{x}$  as  $k \rightarrow \infty$ . Since  $T$  is closed, we finally obtain  $\bar{x} \in D(T) \cap \overline{B_b(0)}$  and  $T\bar{x} + C\bar{x} \ni p$ . The proof is complete.

**Theorem 6.** Let  $T : D(T) \subset X \rightarrow 2^X$  be  $m$ -accretive with  $0 \in D(T)$  and  $C : \overline{B_b(0)} \rightarrow X$  compact. Let  $p \in X$ , and assume that for every  $x \in \partial B_b(0)$  there exists  $j \in Jx$  with

$$(**) \quad \langle Cx + v - p, j \rangle \geq 0,$$

where  $v$  is a fixed element of  $T0$ . Then  $p \in \overline{R(T + C)}$ .

*Proof.* As in [8, Theorem 6], we look at the approximation problem

$$(2.10) \quad \tilde{T}_n x + \tilde{C}x + \alpha x = p.$$

We have made the following conventions:  $\tilde{T}x \equiv Tx - v$ ,  $\tilde{C}x \equiv Cx + v$ ,  $\alpha$  is a positive constant,  $\tilde{T}_n = n(I - \tilde{J}_n)$ , and  $\tilde{J}_n = (I + \frac{1}{n}\tilde{T})^{-1}$ . Let  $x \in \partial B_b(0)$ ,  $j \in Jx$  be such that  $(**)$  holds. Then, since  $\tilde{T}_n(0) = 0$ , we have

$$\langle \tilde{T}_n x + \tilde{C}x + \alpha x - p, j \rangle = \langle \tilde{T}_n x + Cx + v + \alpha x - p, j \rangle \geq 0.$$

Since the operator  $\tilde{T}_n + \alpha I$  is strongly accretive and uniformly continuous on bounded subsets of  $X$  and  $\tilde{C}$  is compact, Theorem 13.21 of Browder [3] says that (2.10) is solvable for each  $\alpha > 0$  with a solution  $x_n$  lying inside the closed ball  $\overline{B_b(0)}$ . Since  $T + \alpha I$  is strongly accretive and  $\tilde{C}$  is compact, we obtain, as in the proof of Theorem 8 of [13], that  $x_{n_k} \rightarrow$  (some)  $x_0 \in \overline{B_b(0)}$ . Since  $\tilde{T}_{n_k} x_{n_k} \rightarrow -\tilde{C}x_0 - \alpha x_0 + p$ , we have (see Barbu [1, Proposition 3.4])  $x_0 \in D(T)$  and  $-\tilde{C}x_0 - \alpha x_0 + p \in \tilde{T}x_0$ . It follows that the problem

$$Tx + Cx + \alpha_n x \ni p$$

is solvable for a positive sequence  $\{\alpha_n\}$  with  $\alpha_n \rightarrow 0^+$  and solutions  $x_n$  lying inside  $\overline{B_b(0)}$ . The proof is finished.

Theorem 6 is an improvement of Morales' Theorem 8 in [13]. Morales considered single-valued operators there and a stronger condition on the boundary of  $B_b(0)$ .

We are now going to establish a new result involving compact perturbations of continuous and demicontinuous accretive operators. This result improves a fundamental result of Browder [3, Theorem 13.21] dealing with continuous accretive operators in spaces  $X$ , with  $X^*$  uniformly convex, or uniformly continuous accretive operators in general Banach spaces.

**Theorem 7.** Let  $G$  be a bounded open subset of  $X$  with  $0 \in D$ , and let  $C : \overline{G} \rightarrow X$  be compact. Moreover, fix  $p \in X$ , and assume one of the following:

(i)  $X^*$  is uniformly convex and  $T : \overline{G} \rightarrow X$  is demicontinuous, accretive, and such that

$$(A) \quad \langle Tx + Cx - p, Jx \rangle \geq 0, \quad x \in \partial G.$$

(ii)  $T : \overline{G} \rightarrow X$  is continuous, accretive, and for every  $x \in \partial G$  condition  $(*)$  is satisfied with  $u = Tx$  for all  $j \in Jx$ . Then  $p \in \overline{R(T + C)}$ .

*Proof.* We assume (i) and use the approximating problem

$$(2.11) \quad Tx + Cx + \frac{1}{n}x = p.$$

As before, we may (and do) assume that  $T0 = 0$ . We first remark that the operator  $\tilde{T} : x \rightarrow Tx + \frac{1}{n}x$  is demicontinuous and strongly accretive on  $\overline{G}$ . Because

of this,  $\tilde{T}G$  is open in  $X$  and  $\tilde{T}\bar{G}$  is closed in  $X$  by the author's invariance of domain result [7, Theorem 1]. We also have  $\overline{\tilde{T}G} = \tilde{T}G \cup \partial\tilde{T}G$  and  $\tilde{T}\bar{G} = \tilde{T}G \cup \tilde{T}(\partial G)$ . Since  $\tilde{T}G = \overline{\tilde{T}G} \supset \tilde{T}\bar{G}$  and  $\tilde{T}$  is injective, we obtain that  $\partial\tilde{T}G \subset \tilde{T}(\partial G)$  and  $\tilde{T}^{-1}$  is defined on  $\partial\tilde{T}G$  and  $\tilde{T}^{-1}(\partial\tilde{T}G) \subset \partial G$ . Consequently, the homotopy mapping  $f_t(x) = x + t(C\tilde{T}^{-1}x - p)$  is well defined on  $\overline{\tilde{T}G}$ , and the Leray-Schauder degree  $d(f_t, \tilde{T}G, 0)$  is also well defined for all  $t \in [0, 1]$ , provided that  $0 \notin f_t(\partial\tilde{T}G)$ , because  $0 \in \tilde{T}G$  and the range of the mapping  $x \rightarrow C\tilde{T}^{-1}x - p$  on  $\overline{\tilde{T}G}$  is a relatively compact subset of  $X$ . The reader should note that we have not assumed that  $\tilde{T}G$  is a bounded set and refer to Browder [3, p. 183] and Lloyd [12, Remarks, p. 59] for further information on this degree. To show that (2.11) is solvable, it suffices to show that  $f_t$  has no zero on  $\partial\tilde{T}G$  for any  $t \in [0, 1]$ . This is certainly true for  $t = 0$ . Assume that  $x_t \in \partial\tilde{T}G$ , for some  $t \in (0, 1)$ , and let  $u_t = (T + \frac{1}{n}I)^{-1}x_t \in \partial G$ . Then

$$(2.12) \quad Tu_t + tCu_t + \frac{1}{n}u_t = tp.$$

We now show that

$$(2.13) \quad \langle Tu_t + t(Cu_t - p), Ju_t \rangle \geq 0$$

for all  $t \in (0, 1)$ . In fact, we have  $\langle Tu_t, Ju_t \rangle \geq 0$ . If  $\langle Cu_t - p, Ju_t \rangle \geq 0$ , our assertion is trivially true. Let  $\langle Cu_t - p, Ju_t \rangle < 0$ . Then condition (A) implies

$$\langle Tu_t, Ju_t \rangle \geq -\langle Cu_t - p, Ju_t \rangle > -t\langle Cu_t - p, Ju_t \rangle$$

and  $\langle Tu_t + t(Cu_t - p), Ju_t \rangle > 0$ . Applying (2.13) to (2.12), we get the contradiction:

$$\frac{1}{n}\|u_t\|^2 \leq \frac{1}{n}\|u_t\|^2 + \langle Tu_t + t(Cu_t - p), Ju_t \rangle = 0.$$

Thus, (2.11) is solvable with solution  $u_n$ ,  $n = 1, 2, \dots$ , lying in  $\bar{G}$ . Since  $\bar{G}$  is bounded, we have  $p \in \overline{R(T + C)}$ .

The proof of our conclusion under (ii) is almost identical to the above in view of the invariance of domain result of Deimling [5, Theorem 3]. It is therefore omitted.

The condition that (\*) be satisfied "for all  $j \in Jx$ " in Theorem 7 can be reduced to the same condition but "for some  $j \in Jx$ " under one of the following additional assumptions:

- (a)  $\langle Tx - Ty, j \rangle \geq 0$  for all  $x, y \in D(T)$ ,  $j \in J(x - y)$ .
- (b)  $T : X \supset G_1 \rightarrow X$  is continuous and accretive, where  $G_1 \supset \bar{G}$  is open.

Actually, (b) implies (a), by Theorem 9.4 of Browder [3], because the Cauchy problem  $x' + Tx = 0$ ,  $x(0) = v$ , is solvable for all  $v \in G_1$  (cf. Deimling [5, proof of Theorem 3]).

### 3. DISCUSSION-EXAMPLE

We let  $\Omega$  denote a bounded domain in  $R^n$  with smooth boundary and consider the problem

$$(P) \quad -\Delta\rho(u(x)) + \phi(\|u\|_{L^1})g(x, u(x)) = p(x), \quad \text{a.e. } x \in \Omega.$$



**Example 1.** Consider (P) with the following assumptions:

(i)  $\rho \in C(R) \cap C^1(R \setminus \{0\})$  is nondecreasing and such that  $\rho(0) = 0$  and, for some constants  $K > 0, \alpha \geq 1,$

$$\rho'(t) \geq K|t|^{\alpha-1}, \quad t \in R \setminus \{0\}.$$

(ii)  $g : \Omega \times R \rightarrow R$  is continuous and such that

$$|g(x, u)| \leq q(x) + q_1|u|,$$

where  $q : \Omega \rightarrow R_+$  is in  $L^1(\Omega)$  and  $q_1$  is a positive constant.

(iii)  $\phi : R_+ \rightarrow R_+$  is continuous.

(iv)  $p \in L^1(\Omega)$  has the following property: there exists a constant  $b > 0$  such that for each  $u \in L^1(\Omega)$  with  $\|u\|_{L^1} \geq b$  and each  $j \in Ju$  we have

$$\int_{\Omega} [\phi(\|u\|_{L^1})g(x, u(x)) - p(x)]j(x)d\mu(x) \geq 0.$$

Then (P) has a solution  $u \in L^1(\Omega)$  with  $\rho(u) \in W_0^{1,1}(\Omega).$

*Proof.* Consider the operator  $T : L^1(\Omega) \supset D(T) \rightarrow L^1(\Omega)$  defined by  $(Tu)(x) \equiv -\Delta\rho(u(x)),$  where  $D(T) = \{u \in L^1(\Omega) ; \rho(u) \in W_0^{1,1}(\Omega), \Delta\rho(u) \in L^1(\Omega)\}.$  Also, consider the operator  $C$  defined by

$$(Cu)(x) = \phi(\|u\|_{L^1})g(x, u(x)).$$

Bénilan has shown in [2] that  $T$  is m-accretive and  $\overline{D(T)} = L^1(\Omega).$  Vrabie has shown in [16, Lemma 2.6.2] that  $T$  generates a compact semigroup on  $\overline{D(T)},$  which implies that  $(T + I)^{-1}$  is a compact operator on  $L^1(\Omega).$  Vainberg's Theorem 19.1 in [15] says that the operator  $C$  is continuous and bounded on all of  $L^1(\Omega).$  Our conclusion follows from Theorem 5 for  $X = L^1(\Omega).$

In the above example we may take  $g(x, u) = (2 + \sin u)u, \phi(t) = \frac{1}{1+t}.$  Then (P) is solvable as in Example 1 for all functions  $p \in L^1(\Omega)$  with  $\|p\|_{L^1} < 1.$  In fact, we first note that  $Ju \subset L^\infty(\Omega)$  is given by

$$Ju = \|u\|_{L^1} \text{SGN}(u),$$

where

$$\text{SGN}(u) = \begin{cases} 1, & u > 0, \\ [-1, 1], & u = 0, \\ -1, & u < 0 \end{cases}$$

(cf. Barbu [1, p. 161]). Let  $j \in Ju$  with  $\langle Tu, j \rangle \geq 0.$  Then we have

$$\begin{aligned} \langle Tu + Cu - p, j \rangle &\geq \int_{\Omega} \left[ \frac{(2 + \sin(u(x)))u(x)}{1 + \|u\|_{L^1}} - p(x) \right] j(x) d\mu(x) \\ &\geq \|u\|_{L^1} \int_{\Omega} \left[ \frac{|u(x)|}{1 + \|u\|_{L^1}} - |p(x)| \right] d\mu(x) \\ &= \|u\|_{L^1} \left[ \frac{\|u\|_{L^1}}{1 + \|u\|_{L^1}} - \|p\|_{L^1} \right], \end{aligned}$$

where we have used the fact that  $u(x)j(x) = \|u\|_{L^1}|u(x)|,$  a.e.  $x \in \Omega,$  for all  $j \in Ju.$  Since  $\frac{1}{1+t} \rightarrow 1$  as  $t \rightarrow \infty,$  our assertion is true. Actually, here we have  $p \in (T + C)(\overline{B_b(0)} \cap D(T)),$  where  $b = \|p\|_{L^1}/(1 - \|p\|_{L^1}).$

Naturally, a large number of examples can now be constructed of functions  $g, \phi$  that satisfy the relevant assumptions of Theorem 5. For the problem  $Tu + Cu \ni p$  with  $T = -\Delta$  or  $m$ -accretive and  $C$  the realization in  $L^p(\Omega)$  of an  $m$ -accretive function  $\beta : R \rightarrow R$ , the reader is referred to Proposition 3.7 and Theorem 3.3 in the book of Barbu [1].

Obviously, the conclusion that " $p \in \overline{D}$ " in the various results of this paper, where  $D$  is some subset of  $\overline{R(T+C)}$ , can be replaced by " $p \in D$ " under various assumptions of strong accretiveness or  $\phi$ -expansiveness (cf. [7]) for the operator  $T$  and/or complete continuity for the operator  $C$ . Also, assumptions might need to be made involving compact resolvents of  $T$  and/or convexity properties of the spaces  $X, X^*$ . A result in this direction was given by the author in [8, Lemma 1]. For the sake of completeness, we give below an easy extension of it to multivalued operators  $T$ .

**Lemma 2.** *Let  $X$  be uniformly convex. Let  $T : X \supset D(T) \rightarrow 2^X, C : \overline{D(T)} \rightarrow X$  be  $m$ -accretive and completely continuous, respectively. Assume that there exists a sequence  $\{\alpha_n\}$  of positive constants and a sequence  $\{x_n\} \subset \overline{B_b(0)} \cap D(T)$  such that  $\alpha_n \rightarrow 0$  and  $x_n \rightarrow 0$  as  $n \rightarrow \infty$  and*

$$u_n + Cx_n + \alpha_n x_n = p$$

for some sequence  $u_n \in Tx_n$ . Then  $x_0 \in D(T)$  and  $u_n \rightarrow u_0 = -Cx_0 + p$ , where  $u_0 \in Tx_0$ .

For a survey article on recent results on accretiveness and compactness, the reader is referred to [9]. For applications to the control of various equations with preassigned responses, we cite the paper [10].

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