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BEST APPROXIMATIONS IN L^1 ARE NEAR BEST IN L^p, $p < 1$

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ABSTRACT. We show that any best L^1 polynomial approximation to a function f in L^p, $0 < p < 1$, is near best in L^p.

Let $I = [0, 1]^d$ and let $\mathscr{P}_r$ be the set of all polynomials in d variables of total degree less than r. It is known that for each f in $L^1(I)$ a (not necessarily unique) best approximation $E_1 f$ to f exists in $\mathscr{P}_r$ and that $E_1 f = q$ if and only if

$$(1) \qquad \left| \int_{E_+} s(x)\,dx - \int_{E_-} s(x)\,dx \right| \le \int_{E_0} |s(x)|\,dx \quad \text{for all } s \text{ in } \mathscr{P}_r,$$

where $E_+ = \{x \in I \mid q(x) > f(x)\}$, $E_- = \{x \in I \mid q(x) < f(x)\}$, and $E_0 = \{x \in I \mid q(x) = f(x)\}$. Condition (1) makes sense even if f is not in $L^1(I)$, and when (1) is satisfied, we call q a best $L^1(I)$ approximation to f and denote q by $E_1 f$. It is easy to show that for constant approximations ($r = 1$) the extended E_1, which is the median operator, is defined for all measurable f and bounded on $L^p(I)$ for any $p > 0$ (see [2] for a discussion of medians). It is also easy to show that the similarly extended L^2 best projection operator onto polynomials is bounded on L^p for $p \ge 1$ and any $r > 0$. These facts motivate us to prove the following theorem.

Theorem. *For each f in $L^p(I)$, $0 < p < 1$, and for all $r > 0$, a best $L^1(I)$ approximation $E_1 f$ exists in $\mathscr{P}_r$. Moreover, for all choices of $E_1 f$,*

$$\|f - E_1 f\|_p \le (1 + 2K)^{1/p} \inf_{q \in \mathscr{P}_r} \|f - q\|_p,$$

where

$$K = \sup_{q \in \mathscr{P}_r} \frac{\|q\|_\infty}{\|q\|_1}.$$

This theorem provides a method to find near-best polynomial approximations in $L^p(I)$ for $0 < p < 1$. Such approximations are useful in atomic decompositions of the Besov spaces $B_p^\alpha(L^p(I))$, $p < 1$, which are the regularity spaces

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for nonlinear approximation in $L^q(I)$, $q^{-1} = p^{-1} - \alpha/d$, by wavelets and free-knot splines. Theory and applications to image and surface compression can be found in [2–6]. Of course, such near-best approximations are known to exist; the new thing here is that $L^1(I)$ projections provide them.

We make several remarks about the theorem. First, a more careful argument shows that

$$\|f - E_1 f\|_p \le (2K)^{1/p-1} \inf_{q\in\mathscr{P}_r} \|f - q\|_p$$

for $0 < p < 1$; see [1]. Second, our proof extends to approximation by any finite-dimensional subspace of $L^1(\Omega) \cap L^\infty(\Omega)$ for any finite measure space $(\Omega, d\mu)$. (A measure space with atoms must first be embedded into a continuous measure space.) Generalizations to the behavior of the best $L^p(I)$ approximation in $L^{p-1}(I)$ for $p > 1$ and the fact that the best $L^1(I)$ approximation is defined for any measurable f are contained in [1].

We first prove the following lemma, which contains the main argument.

Lemma. *If there exists a best $L^1(I)$ approximation $E_1 f \in \mathscr{P}_r$ to $f \in L^p(I)$, then*

$$\|E_1 f\|_p \le \left(2 \sup_{q\in\mathscr{P}_r} \frac{\|q\|_\infty}{\|q\|_1}\right)^{1/p} \|f\|_p.$$

Proof. Condition (1) implies, in particular, that for $q = E_1 f$,

$$\left| \int_{E_+} q(x)\,dx - \int_{E_-} q(x)\,dx \right| \le \int_{E_0} |q(x)|\,dx. \tag{2}$$

Our approach is to bound $\|f\|_p$ from below among all f satisfying (2) for a particular q.

We introduce the function

$$g(x) = \begin{cases} 0, & x \in E_+,\ q(x) > 0, \\ q(x), & x \in E_+,\ q(x) \le 0, \\ 0, & x \in E_-,\ q(x) < 0, \\ q(x), & x \in E_-,\ q(x) \ge 0, \\ q(x), & x \in E_0. \end{cases}$$

Clearly, $\|g\|_p \le \|f\|_p$ and the sets $\widetilde{E}_+ \subset E_+$, $\widetilde{E}_- \subset E_-$, and $\widetilde{E}_0 \supset E_0$ for g and q satisfy

$$\left| \int_{\widetilde{E}_+} q(x)\,dx - \int_{\widetilde{E}_-} q(x)\,dx \right| \le \int_{\widetilde{E}_0} |q(x)|\,dx. \tag{3}$$

Now $\int_I |g|^p = \int_{\widetilde{E}_0} |q|^p$, and to further bound $\|f\|_p$ from below we minimize $\int_{\widetilde{E}_0} |q(x)|^p\,dx$ among all partitions $(\widetilde{E}_+, \widetilde{E}_-, \widetilde{E}_0)$ with $q > 0$ on $\widetilde{E}_+$, $q < 0$ on $\widetilde{E}_-$, and (3) holding. These conditions imply that

$$\int_{\widetilde{E}_+ \cup \widetilde{E}_-} |q(x)|\,dx \le \int_{\widetilde{E}_0} |q(x)|\,dx;$$

i.e.,

$$\int_{\widetilde{E}_0} |q(x)|\,dx \ge \frac{1}{2}\|q\|_1.$$

We claim that the best choice of $\widetilde{E}_0$ satisfies

$$\inf_{x\in\widetilde{E}_0}|q(x)|\geq\sup_{x\notin\widetilde{E}_0}|q(x)|\quad\text{and}\quad\int_{\widetilde{E}_0}|q(x)|\,dx=\frac{1}{2}\|q\|_1.$$

Suppose $\widetilde{E}_0'$ is any other choice. We can assume $\int_{\widetilde{E}_0'}|q(x)|\,dx=\frac{1}{2}\|q\|_1$, because otherwise we could make $\widetilde{E}_0'$, and, *a fortiori*, $\int_{\widetilde{E}_0'}|q(x)|^p\,dx$, smaller.

Let a be any number between $\sup_{x\notin\widetilde{E}_0}|q(x)|$ and $\inf_{x\in\widetilde{E}_0}|q(x)|$, and let $A=\widetilde{E}_0\backslash\widetilde{E}_0'$ and $B=\widetilde{E}_0'\backslash\widetilde{E}_0$. Then $\int_A|q(x)|\,dx=\int_B|q(x)\,dx$ and

$$\int_A|q(x)|\,dx=\int_A|q(x)|^p|q(x)|^{1-p}\,dx\geq a^{1-p}\int_A|q(x)|^p\,dx,$$
$$\int_B|q(x)|\,dx=\int_B|q(x)|^p|q(x)|^{1-p}\,dx\leq a^{1-p}\int_B|q(x)|^p\,dx.$$

Therefore,

$$\int_A|q(x)|^p\,dx\leq\int_B|q(x)|^p\,dx$$

and

$$\int_{\widetilde{E}_0}|q(x)|^p\,dx\leq\int_{\widetilde{E}_0'}|q(x)|^p\,dx.$$

So

$$\frac{\int_I|q(x)|^p\,dx}{\int_I|f(x)|^p\,dx}\leq\frac{\int_I|q(x)|^p\,dx}{\int_{\widetilde{E}_0}|q(x)|^p\,dx}.$$

Since $\int_{\widetilde{E}_0}|q(x)|\,dx=\frac{1}{2}\|q\|_1$, we have

$$\|q\|_\infty|\widetilde{E}_0|\geq\frac{1}{2}\|q\|_1\quad\text{or}\quad|\widetilde{E}_0|\geq\frac{1}{2}\frac{\|q\|_1}{\|q\|_\infty}.$$

Now

$$\int_{I\backslash\widetilde{E}_0}|q(x)|^p\,dx\leq a^p(1-|\widetilde{E}_0|)$$

and

$$\int_{\widetilde{E}_0}|q(x)|^p\,dx\geq a^p|\widetilde{E}_0|.$$

So

$$\frac{\int_I|q(x|)^p\,dx}{\int_{\widetilde{E}_0}|q(x)|^p\,dx}=1+\frac{\int_{I\backslash\widetilde{E}_0}|q(x)|^p\,dx}{\int_{\widetilde{E}_0}|q(x)|^p\,dx}\leq1+\frac{1-|\widetilde{E}_0|}{|\widetilde{E}_0|}=\frac{1}{|\widetilde{E}_0|}\leq2\frac{\|q\|_\infty}{\|q\|_1}.$$

Therefore, $\|q\|_p\leq(2K)^{1/p}\|f\|_p$, where $K=\sup_{q\in\mathscr{P}_r}(\|q\|_\infty/\|q\|_1)$. □

Corollary. *For each f in $L^p(I)$, $0<p<1$, a best $L^1(I)$ polynomial approximation E_1f exists.*

Proof. For each positive integer n, define f_n by

$$f_n(x)=\begin{cases}n, & f(x)>n,\\ f(x), & |f(x)|\leq n,\\ -n, & f(x)<-n.\end{cases}$$

Then $f_n \in L^\infty(I)$ and $\|f_n\|_p \le \|f\|_p$. Best $L^1(I)$ approximations q_n to f_n exist for all n and

$$\|q_n\|_p \le C\|f_n\|_p \le C\|f\|_p.$$

However, for each p and r there is a constant C_1 such that for all $q \in \mathscr{P}_r$, $\|q\|_\infty \le C_1\|q\|_p$. Therefore for all n we have $\|q_n\|_\infty \le CC_1\|f\|_p$, so that for some n we have $\|q_n\|_\infty < n$. This implies that we can choose $E_1 f = q_n$ because the sets E_+, E_-, and E_0 are the same for f as for f_n. □

Proof of the main theorem. The corollary shows that $E_1 f$ exists. Now E_1 is linear with respect to addition of polynomials in $\mathscr{P}_r$: If we let $g = f + q$, $q \in \mathscr{P}_r$, and $E_1 g = E_1 f + q$, then clearly condition (1) is satisfied because E_+, E_-, and E_0 are the same for g and $E_1 g$ as for f and $E_1 f$. So $E_1 f + q$ is a best $L^1(I)$ approximation to $f + q$ according to our definition.

Finally, because $\|f + g\|_p^p \le \|f\|_p^p + \|g\|_p^p$, we have for all $q \in \mathscr{P}_r$,

$$\begin{aligned}\|f - E_1 f\|_p^p &\le \|f - q\|_p^p + \|q - E_1 f\|_p^p \\ &= \|f - q\|_p^p + \|E_1(q - f)\|_p^p \\ &\le (1 + 2K)\|f - q\|_p^p\end{aligned}$$

by the lemma. □

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