BEST APPROXIMATIONS IN $L^1$ ARE NEAR BEST IN $L^p$, $p < 1$

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Abstract. We show that any best $L^1$ polynomial approximation to a function $f$ in $L^p$, $0 < p < 1$, is near best in $L^p$.

Let $I = [0, 1]^d$ and let $\mathcal{P}_r$ be the set of all polynomials in $d$ variables of total degree less than $r$. It is known that for each $f$ in $L^1(I)$ a (not necessarily unique) best approximation $E_1 f$ to $f$ exists in $\mathcal{P}_r$ and that $E_1 f = q$ if and only if

$$\int_{E_+} s(x) dx - \int_{E_-} s(x) dx \leq \int_{E_0} |s(x)| dx$$

for all $s \in \mathcal{P}_r$, where $E_+ = \{x \in I \mid q(x) > f(x)\}$, $E_- = \{x \in I \mid q(x) < f(x)\}$, and $E_0 = \{x \in I \mid q(x) = f(x)\}$. Condition (1) makes sense even if $f$ is not in $L^1(I)$, and when (1) is satisfied, we call $q$ a best $L^1(I)$ approximation to $f$ and denote $q$ by $E_1 f$. It is easy to show that for constant approximations ($r = 1$) the extended $E_1$, which is the median operator, is defined for all measurable $f$ and bounded on $L^p(I)$ for any $p > 0$ (see [2] for a discussion of medians). It is also easy to show that the similarly extended $L^2$ best projection operator onto polynomials is bounded on $L^p$ for $p \geq 1$ and any $r > 0$. These facts motivate us to prove the following theorem.

Theorem. For each $f$ in $L^p(I)$, $0 < p < 1$, and for all $r > 0$, a best $L^1(I)$ approximation $E_1 f$ exists in $\mathcal{P}_r$. Moreover, for all choices of $E_1 f$,

$$\|f - E_1 f\|_p \leq (1 + 2K)^{1/p} \sup_{q \in \mathcal{P}_r} \|f - q\|_p,$$

where

$$K = \sup_{q \in \mathcal{P}_r} \frac{\|q\|_{\infty}}{\|q\|_1}.$$

This theorem provides a method to find near-best polynomial approximations in $L^p(I)$ for $0 < p < 1$. Such approximations are useful in atomic decompositions of the Besov spaces $B^o_p(L^p(I))$, $p < 1$, which are the regularity spaces.
for nonlinear approximation in $L^q(I)$, $q^{-1} = p^{-1} - \alpha/d$, by wavelets and free-knot splines. Theory and applications to image and surface compression can be found in [2-6]. Of course, such near-best approximations are known to exist; the new thing here is that $L^1(I)$ projections provide them.

We make several remarks about the theorem. First, a more careful argument shows that

$$
\|f - E_1 f\|_p \leq (2K)^{1/p - 1} \inf_{q \in \mathcal{P}, \|q\|_1} \|f - q\|_p
$$

for $0 < p < 1$; see [1]. Second, our proof extends to approximation by any finite-dimensional subspace of $L^1(\Omega) \cap L^\infty(\Omega)$ for any finite measure space $(\Omega, d\mu)$. (A measure space with atoms must first be embedded into a continuous measure space.) Generalizations to the behavior of the best $L^p(I)$ approximation in $L^{p-1}(I)$ for $p > 1$ and the fact that the best $L^1(I)$ approximation is defined for any measurable $f$ are contained in [1].

We first prove the following lemma, which contains the main argument.

**Lemma.** If there exists a best $L^1(I)$ approximation $E_1 f \in \mathcal{P}$, to $f \in L^p(I)$, then

$$
\|E_1 f\|_p \leq \left( 2 \sup_{q \in \mathcal{P}} \frac{\|q\|_\infty}{\|q\|_1} \right)^{1/p} \|f\|_p.
$$

**Proof.** Condition (1) implies, in particular, that for $q = E_1 f$,

$$
\left| \int_{E_+} q(x) \, dx - \int_{E_-} q(x) \, dx \right| \leq \int_{E_0} |q(x)| \, dx.
$$

Our approach is to bound $\|f\|_p$ from below among all $f$ satisfying (2) for a particular $q$.

We introduce the function $g(x) = \begin{cases} 0, & x \in E_+, \; q(x) > 0, \\ q(x), & x \in E_+, \; q(x) \leq 0, \\ 0, & x \in E_-, \; q(x) < 0, \\ q(x), & x \in E_-, \; q(x) \geq 0, \\ q(x), & x \in E_0. \end{cases}$

Clearly, $\|g\|_p \leq \|f\|_p$ and the sets $\tilde{E}_+ \subset E_+$, $\tilde{E}_- \subset E_-$, and $\tilde{E}_0 \supset E_0$ for $g$ and $q$ satisfy

$$
\left| \int_{\tilde{E}_+} q(x) \, dx - \int_{\tilde{E}_-} q(x) \, dx \right| \leq \int_{\tilde{E}_0} |q(x)| \, dx.
$$

Now $\int_{\tilde{E}_0} |q|^p = \int_{E_0} |q|^p$, and to further bound $\|f\|_p$ from below we minimize $\int_{\tilde{E}_0} |q(x)|^p \, dx$ among all partitions $(\tilde{E}_+, \tilde{E}_-, \tilde{E}_0)$ with $q > 0$ on $\tilde{E}_+$, $q < 0$ on $\tilde{E}_-$, and (3) holding. These conditions imply that

$$
\int_{\tilde{E}_+ \cup \tilde{E}_-} |q(x)| \, dx \leq \int_{E_0} |q(x)| \, dx;
$$

i.e.,

$$
\int_{\tilde{E}_0} |q(x)| \, dx \geq \frac{1}{2} \|q\|_1.
$$
We claim that the best choice of $\tilde{E}_0$ satisfies

$$\inf_{x \in \tilde{E}_0} |q(x)| \geq \sup_{x \notin \tilde{E}_0} |q(x)| \quad \text{and} \quad \int_{\tilde{E}_0} |q(x)| \, dx = \frac{1}{2} \|q\|_1.$$ 

Suppose $\tilde{E}_0'$ is any other choice. We can assume $\int_{\tilde{E}_0} |q(x)| \, dx = \frac{1}{2} \|q\|_1$, because otherwise we could make $\tilde{E}_0'$, and, a fortiori, $\int_{\tilde{E}_0} |q(x)|^p \, dx$, smaller.

Let $a$ be any number between $\sup_{x \notin \tilde{E}_0} |q(x)|$ and $\inf_{x \in \tilde{E}_0} |q(x)|$, and let $A = \tilde{E}_0 \setminus \tilde{E}_0'$ and $B = \tilde{E}_0 \setminus \tilde{E}_0$. Then $\int_A |q(x)| \, dx = \int_B |q(x)| \, dx$ and

$$\int_A |q(x)| \, dx = \int_A |q(x)|^p |q(x)|^{1-p} \, dx \geq a^{1-p} \int_A |q(x)|^p \, dx,$$

$$\int_B |q(x)| \, dx = \int_B |q(x)|^p |q(x)|^{1-p} \, dx \leq a^{1-p} \int_B |q(x)|^p \, dx.$$ 

Therefore,

$$\int_A |q(x)|^p \, dx \leq \int_B |q(x)|^p \, dx$$

and

$$\int_{\tilde{E}_0} |q(x)|^p \, dx \leq \int_{\tilde{E}_0} |q(x)|^p \, dx.$$ 

So

$$\frac{\int_{\tilde{E}_0} |q(x)|^p \, dx}{\int_{\tilde{E}_0} |f(x)|^p \, dx} \leq \frac{\int_{\tilde{E}_0} |q(x)|^p \, dx}{\int_{\tilde{E}_0} |q(x)|^p \, dx}.$$ 

Since $\int_{\tilde{E}_0} |q(x)| \, dx = \frac{1}{2} \|q\|_1$, we have

$$\|q\|_{\infty, \tilde{E}_0} \geq \frac{1}{2} \|q\|_1 \quad \text{or} \quad |\tilde{E}_0| \geq \frac{1}{2} \|q\|_{\infty}.$$ 

Now

$$\int_{I \setminus \tilde{E}_0} |q(x)|^p \, dx \leq a^p (1 - |\tilde{E}_0|)$$

and

$$\int_{\tilde{E}_0} |q(x)|^p \, dx \geq a^p |\tilde{E}_0|.$$ 

So

$$\frac{\int_{I} |q(x)|^p \, dx}{\int_{\tilde{E}_0} |q(x)|^p \, dx} = 1 + \frac{\int_{I \setminus \tilde{E}_0} |q(x)|^p \, dx}{\int_{\tilde{E}_0} |q(x)|^p \, dx} \leq 1 + \frac{1 - |\tilde{E}_0|}{|\tilde{E}_0|} = \frac{1}{\tilde{E}_0} \leq 2 \|q\|_{\infty}.$$ 

Therefore, $\|q\|_p \leq (2K)^{1/p} \|f\|_p$, where $K = \sup_{q \in \mathcal{P}} (\|q\|_{\infty} / \|q\|_1)$. □

**Corollary.** For each $f$ in $L^p(I)$, $0 < p < 1$, a best $L^1(I)$ polynomial approximation $E_1 f$ exists.

**Proof.** For each positive integer $n$, define $f_n$ by

$$f_n(x) = \begin{cases} n, & f(x) > n, \\ f(x), & |f(x)| \leq n, \\ -n, & f(x) < -n. \end{cases}$$
Then \( f_n \in L^\infty(I) \) and \( \|f_n\|_p \leq \|f\|_p \). Best \( L^1(I) \) approximations \( q_n \) to \( f_n \) exist for all \( n \) and
\[
\|q_n\|_p \leq C \|f_n\|_p \leq C \|f\|_p.
\]
However, for each \( p \) and \( r \) there is a constant \( C_1 \) such that for all \( q \in \mathcal{P}_r \), \( \|q\|_\infty \leq C_1 \|q\|_p \). Therefore for all \( n \) we have \( \|q_n\|_\infty \leq CC_1 \|f\|_p \), so that for some \( n \) we have \( \|q_n\|_\infty < n \). This implies that we can choose \( E_1f = q_n \) because the sets \( E_+, E_- \), and \( E_0 \) are the same for \( f \) as for \( f_n \).

**Proof of the main theorem.** The corollary shows that \( E_1f \) exists. Now \( E_1 \) is linear with respect to addition of polynomials in \( \mathcal{P}_r \): If we let \( g = f + q \), \( q \in \mathcal{P}_r \), and \( E_1g = E_1f + q \), then clearly condition (1) is satisfied because \( E_+ \), \( E_- \), and \( E_0 \) are the same for \( g \) and \( E_1g \) as for \( f \) and \( E_1f \). So \( E_1f + q \) is a best \( L^1(I) \) approximation to \( f + q \) according to our definition.

Finally, because \( \|f + q\|_p^p \leq \|f\|_p^p + \|q\|_p^p \), we have for all \( q \in \mathcal{P}_r \),
\[
\|f - E_1f\|_p^p \leq \|f - q\|_p^p + \|q - E_1f\|_p^p
= \|f - q\|_p^p + \|E_1(q - f)\|_p^p
\leq (1 + 2K)\|f - q\|_p^p
\]
by the lemma.

**References**

1. L. G. Brown, *\( L^p \) best approximation operators are bounded on \( L^{p-1} \) (in preparation).

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