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ON THE FREDHOLM THEORY OF MULTIPLIERS

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Dedicated to George Maltese on the occasion of his sixtieth birthday

ABSTRACT. Multipliers that are Fredholm operators on certain commutative semisimple Banach algebras may be characterized by means of a quotient algebra of multipliers. Some spectral properties of multipliers of these algebras are considered

1. INTRODUCTION

Although the general theory of multipliers for abstract Banach algebras has been widely developed by several authors (see [12] for an exhaustive treatment of the subject), it seems that the various aspects of the spectral theory of these operators have been rarely studied.

In a recent paper [1] we have investigated some general spectral properties of a multiplier defined on a commutative semisimple Banach algebra. These properties are similar to those of a normal operator defined on a complex Hilbert space.

The aim of this note is to investigate some aspects concerning the Fredholm theory of multipliers. The main result states that the multipliers of certain commutative semisimple Banach algebras may be characterized by replacing the so-called Calkin algebra by a quotient algebra of multipliers. Moreover, some spectral properties of a multiplier and some characterizations of Riesz multipliers are given for commutative semisimple Banach algebras that have a dense socle or, more generally, a discrete maximal ideal space. In the last part of the work we consider some applications to group algebras and other algebras.

For the basic details we refer to the book of Larsen [12] and to the monograph [5] for the general theory of multipliers and the abstract Fredholm theory, respectively. All the algebras here considered will be over the complex field.

2. MULTIPLIERS AND FREDHOLM THEORY

We recall that if A is any Banach algebra with or without a unit then a

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mapping $T : A \to A$ is said to be a *multiplier* if $(Tx)y = x(Ty)$ holds for each $x, y \in A$. In the sequel we shall always suppose that A is a commutative semisimple Banach algebra. Then the ideal $\operatorname{soc} A$, the *socle* of A, does exist [5], and denoting by E_A the set of all minimal idempotents of A (i.e., $eAe = \mathbb{C}e$ for all $e \in E_A$), we have

$$\operatorname{soc} A = \left\{ \sum_{k=1}^{n} e_k A : e_k \in E_A , \ \ n \in \mathbb{N} \right\} = \operatorname{span}\{E_A\} .$$

Let $M(A)$ denote the set of all multipliers of A. $M(A)$ is a closed commutative subalgebra of $L(A)$, the Banach algebra of all bounded linear operators of A, which contains the identity operator I. Moreover, $M(A)$ is semisimple [12, Corollary 1.4.2].

Let $K(A)$ denote the closed ideal of all compact operators on A, and let us denote by $K_M(A)$ the closed ideal $M(A) \cap K(A)$ of $M(A)$. Since $M(A)$ is semisimple, the socle of $M(A)$ does exist. Moreover, for each $T \in K_M(A)$ the spectrum $\sigma_{M(A)}(T) = \sigma(T)$ has 0 as unique accumulation point, so we have (see [5, Theorem R.2.6])

$$K_M(A) \subseteq k(h(\operatorname{soc} M(A))) . \tag{1}$$

Hence it is possible to develop an abstract Fredholm theory of $M(A)$ relative to $K_M(A)$ [5, Chapter F].

Let $\Phi_M(A)$ denote the class of all Fredholm elements of $M(A)$ relative to $K_M(A)$, i.e., those elements of $M(A)$ invertible modulo $K_M(A)$, and let $\Phi(A)$ denote the set of all Fredholm operators, i.e., the elements of $L(A)$ invertible modulo $K(A)$. It is well known, denoting by $\alpha(T)$ the dimension of $\ker(T)$, the kernel of T, and by $\beta(T)$ the codimension of $T(A)$, the range of $T(A)$, that $T \in \Phi(A)$ if and only if $\alpha(T)$ and $\beta(T)$ are both finite. Trivially, we always have $\Phi_M(A) \subseteq \Phi(A) \cap M(A)$. This inclusion may be proper, as the following example shows.

Example. Let $\mathscr{A}(D)$ denote the *disc algebra*, i.e., the Banach algebra of all complex-valued continuous functions on the closed unit disc D of $\mathbb{C}$ that are analytic on the interior of D. Since $\mathscr{A}(D)$ has an indentity, each multiplier of $\mathscr{A}(D)$ is a multiplication operator T_f for some $f \in \mathscr{A}(D)$. It is obvious that an idempotent of $\mathscr{A}(D)$ is minimal if and only if it is the characteristic function of an isolated point of D. Hence $E_{\mathscr{A}(D)}$ is empty and, therefore, $\operatorname{soc}(M(\mathscr{A}(D))) = \{0\}$. By (1) it turns out that

$$K_M(A) = k(h(\operatorname{soc} M(\mathscr{A}(D)))) = \{0\}$$

and thus

$$\Phi_M(A) = \{T_f : f \text{ is an invertible element of } \mathscr{A}(D)\} .$$

Now, let us denote by T_g the multiplication operator by $g(z) = z$, $z \in D$. Clearly T_g is injective. It is easy to check that the range of T_g is the maximal ideal $\{h \in \mathscr{A}(D) : h(0) = 0\}$, and thus its codimension is 1. Hence $T_g \in \Phi(A) \cap M(A)$ while $T_g \notin \Phi_M(A)$. Moreover, we note that the index of T_g is equal to -1.

Question. Under what conditions on A do we have $\Phi_M(A) = \Phi(A) \cap M(A)$?

The above question has a special interest since the Calkin algebra $L(A)/K(A)$ is not commutative; however, $M(A)$ and $K_M(A)$ are commutative. Moreover, as we shall see in the sequel, in several applications there are concrete models of $M(A)$ and $K_M(A)$, so in this case, it is easier to characterize all Fredholm multipliers.

The first part of the following theorem is a particular case of a result given in [4]. The terminology that a Banach algebra is regular is the classical one given in [12].

Theorem 1. *Let A be a commutative semisimple Banach algebra and suppose $M(A)$ is regular. We have*

(i) $\Phi_M(A) = \Phi(A) \cap M(A)$.

(ii) *If $T \in \Phi(A) \cap M(A)$ then* $\operatorname{ind}(T) = 0$ *(*$\operatorname{ind}(T)$ *is the index of* T*).*

Moreover, properties (i) *and* (ii) *hold if A is a C^*-algebra.*

Proof. (i) By hypothesis, $M(A)$ is regular; moreover, as we remarked before, $M(A)$ is also semisimple. Hence $M(A)$ verifies condition (III) of [4] (taking $R = M(A)$ and $\alpha =$ the identity operator of $M(A)$ onto $M(A)$). By Corollary 3(i) of [4], $\Phi_M(A) = \Phi(A) \cap M(A)$.

(ii) See Corollary 3(ii) of [4].

To prove the last assertion, we recall the well-known fact that any C^*-algebra is regular. Hence it suffices to prove that $M(A)$ is a C^*-algebra.

Let $T^*(x) = (Tx^*)^*$. Then it follows easily that $T^* \in M(A)$. Moreover, the mapping $T \to T^*$ is an involution. Let U denote the closed unit ball of A. U is selfadjoint, thus

$$\|T^*\| = \sup_{x\in U} \|T^*(x^*)\| = \sup_{x\in U} \|(T^*x)^*\| = \sup_{x\in U} \|Tx\| = \|T\|$$

and, therefore, $\|TT^*\| \le \|T\|^2$. The C^*-condition $\|TT^*\| = \|T\|^2$ follows by observing that if $z \in A$ then $\|z\| = \sup_{y\in U} \|yz\|$ and hence

$$\begin{aligned}\|TT^*\| &= \sup_{x\in U} \|(TT^*)x\| = \sup_{x\in U}\sup_{y\in U} \|TT^*(xy)\| \ge \sup_{x\in U} \|TT^*(xx^*)\| \\ &= \sup_{x\in U} \|(Tx)(T^*x^*)\| = \sup_{x\in U} \|(Tx)(Tx)^*\| = \sup_{x\in U} \|Tx\|^2 = \|T\|^2. \quad \square\end{aligned}$$

If $M(A)$ is regular then A is regular [12, Theorem 1.4.4], but unfortunately the converse generally does not need to be true. A classical counterexample is $A = L_1(G)$, G a nondiscrete locally compact abelian group. Then $M(A) \cong M(G)$, the convolution algebra of all bounded regular complex Borel measures on G [15]. So we cannot apply Theorem 1 to such group algebras. However, the next theorem shows that the equality $\Phi_M(A) = \Phi(A) \cap M(A)$ holds for a wide class of Banach algebras, which includes the case $A = L_1(G)$, G compact and abelian.

First we need to specify some notations.

Let us denote by $\Delta(A)$ the regular maximal ideal space of A and by $x^\wedge$ the Gelfand transform of $x \in A$, defined by $x^\wedge(m) = m(x)$ for each multiplicative functional $m \in \Delta(A)$. In [14] Wang showed that, if $\Delta(A)$ is endowed with the Gelfand topology, then for each $T \in M(A)$ there exists a unique continuous function φ_T on $\Delta(A)$ such that the equation $(Tx)^\wedge(m) = \varphi_T(m)x^\wedge(m)$ holds for all $x \in A$ and $m \in \Delta(A)$.

Now let us denote by $\Phi_+(A)$ the class of all *upper semi-Fredholm operators* and by $\Phi_-(A)$ the class of all *lower semi-Fredholm operators*, defined as

$$\Phi_+(A) = \{T \in L(A) : \alpha(T) < \infty \text{ and } T(A) \text{ is closed}\},$$
$$\Phi_-(A) = \{T \in L(A) : \beta(T) < \infty\}.$$

It is well known that $\Phi(A) = \Phi_+(A) \cap \Phi_-(A)$ [8]. We recall that a bounded operator T on a Banach space is said to be a *Riesz operator* if $\lambda I - T$ is a Fredholm operator for each $\lambda \neq 0$.

Lemma. *Let A be a commutative semisimple Banach algebra and $T \in M(A)$. Then* $\operatorname{soc} A \subseteq T(A) \oplus \ker(T)$.

Proof. Suppose $x \in T(A) \cap \ker(T)$. We have $x = Ty$ for some $y \in A$ and hence

$$x^2 = x(Ty) = (Tx)y = 0.$$

Since A is commutative and semisimple, it follows that $x = 0$. Now assume that $e \in E_A$. The element e is an eigenvector of T, in fact, denoting by $\{m_0\}$ the support of $e^\wedge$, we have $Te = \varphi_T(m_0)e$, thus $e \in T(A) \oplus \ker(T)$. Therefore $\operatorname{soc} A \subseteq T(A) \oplus \ker(T)$.

Theorem 2. *Let A be a commutative semisimple Banach algebra such that $A = \overline{\operatorname{soc} A}$ and $T \in M(A)$. Then the following conditions are equivalent:*

(i) $T \in \Phi_+(A)$.
(ii) $T \in \Phi_-(A)$.
(iii) $T \in \Phi(A)$.
(iv) $T \in \Phi(A)$ *and* $\operatorname{ind} T = 0$.

Moreover, we have $\Phi_M(A) = \Phi(A) \cap M(A)$.

Proof. Suppose $T \in \Phi_+(A)$. Then $T(A)$ is closed and $\ker(T)$ is finite dimensional and hence $T(A) \oplus \ker(T)$ is closed. By the above lemma we have $\operatorname{soc} A \subseteq T(A) \oplus \ker(T)$ and hence $A = T(A) \oplus \ker(T) = \overline{\operatorname{soc} A}$, i.e., $\alpha(T) = \beta(T) < \infty$. From this it easily follows that (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

To prove the equivalence (ii) $\Leftrightarrow$ (iii) we need only show that if $T \in \Phi_-(A)$ then $T \in \Phi(A)$. Let us suppose $\beta(T) < \infty$. By a well-known theorem of Kato (see [7, Proposition 36.3]), the range $T(A)$ is closed. We observe that if $\{e_1, \dots, e_n\}$ is any finite set of distinct elements of E_A, then the elements $e_1, \dots, e_n$ are linearly independent (in fact, if $\lambda_1 e_1 + \cdots + \lambda_n e_n = 0$, we have for $k = 1, \dots, n$, $\lambda_k e_k = e_k(\lambda_1 e_1 + \cdots + \lambda_n e_n) = 0$, which implies $\lambda_k = 0$). Hence, since $T(A)$ is finite codimensional, there exists a finite set of elements $\Lambda = \{e_1, \dots, e_n\}$ such that

$$T(A) = \overline{\operatorname{span}\{x : x \in E_A \backslash \Lambda\}}.$$

Now, let $e = e_1 + \cdots + e_n$ and let us denote by T_e the multiplier defined by

$$T_e x = ex \quad \text{for each } x \in A.$$

Clearly T_e is idempotent and it projects A onto the finite-dimensional ideal eA.

It is easy to check that $T(A) \subseteq \ker T_e$. We want to show that actually $T(A) = \ker T_e$. To prove the opposite inclusion first we observe that $e_k^\wedge$ has a one point support for each $k = 1, \dots, n$. Let us denote by $\{m_k\}$ the support of

$e_k^\wedge$ for each $k = 1, \ldots, n$. If $x \in \ker T_e$ then $ex = 0$ and hence $x^\wedge(m_k) = 0$ for each m_k. So if $x \neq 0$ then $x \notin \operatorname{span}\{e_1, \ldots, e_n\}$ and, since $\operatorname{span}\{E_A\}$ is norm dense in A, we have that $x \in \overline{\operatorname{span}\{x : x \in E_A \backslash \Lambda\}} = T(A)$.

Next we show that $\ker T = eA$. By the lemma we have $\ker T \cap T(A) = \{0\}$ and this trivially implies $\ker T \subseteq eA$. To prove the opposite inclusion, first we observe that the equality $T(A) = \ker T_e$ implies $e(Tx) = 0$ for each $x \in A$, so if $y = ex \in eA$ we have $Ty = T(ex) = e(Tx) = 0$ and therefore $eA \subseteq \ker T$. Then $\ker T = eA$ and hence, since eA is finite dimensional, we have $\alpha(T) < \infty$.

To prove the equality $\Phi(A) \cap M(A) = \Phi_M(A)$, we need only to prove the inclusion $\Phi(A) \cap M(A) \subseteq \Phi_M(A)$. Assume that $T \in \Phi(A)$ and let T_e be as above. Since T_e is finite dimensional, we have that $T + T_e \in \Phi(A)$. We want to prove that $T + T_e$ is invertible in $L(A)$. In fact, as shown above, $\operatorname{ind}(T + T_e) = 0$, so it suffices to prove that $\ker(T + T_e) = \{0\}$. Let us suppose $(T + T_e)x = 0$. Then $Tx = -ex \in eA = \ker(T)$. By the lemma then, we have $0 = Tx = -ex$ and, therefore, $x = ex = 0$. Hence $T + T_e$ is invertible in $M(A)$ [12] and that clearly implies $T \in \Phi_M(A)$. □

Remark 1. Generally for each semisimple commutative Banach algebra, if $T \in \Phi(A) \cap M(A)$ then we have $\operatorname{ind}(T) \leq 0$. In fact, by Theorem 3 of [1], we have $\operatorname{ascent}(T) \leq 1$ for each $T \in M(A)$. Hence if $T \in \Phi(A) \cap M(A)$, by Theorem 51.1 of [8] there are exactly the following two possibilities:

(a) T has index zero and $\operatorname{ascent}(T) = \operatorname{descent}(T) \leq 1$.

(b) T has index < 0 and $\operatorname{ascent}(T) \leq 1$, $\operatorname{descent}(T) = \infty$.

The previously considered example of the operator T_g defined on the disc algebra $\mathscr{A}(D)$ shows that (b) may occur if A does not satisfy the hypothesis of Theorem 1 or the hypothesis of Theorem 2. In fact, in this case $\operatorname{ind}(T_g) < 0$.

Remark 2. Observe that if $A = \overline{\operatorname{soc} A}$ then A is a Riesz algebra, and hence $\Delta(A)$ is discrete [5, Corollary R.3.5].

Remark 3. Let $F_M(A) = \{T \in M(A) : T \text{ is finite dimensional}\}$. Clearly $F_M(A)$ is an ideal and we can consider the Fredholm elements of $M(A)$ relative to $F_M(A)$. Let $\Phi_M^F(A)$ be such a set of operators and let us suppose that $A = \overline{\operatorname{soc} A}$. If U denotes the inverse of $T+T_e$, T_e as in the proof of Theorem 2, we have $(T+T_e)U = TU+T_eU = I$ and this trivially implies, since $T_eU \in F_M(A)$, that $T \in \Phi_M^F(A)$. Hence, if $A = \overline{\operatorname{soc} A}$, $\Phi_M(A) = \Phi_M^F(A) = \Phi(A) \cap M(A)$.

Let $\omega_M(T)$, $W_M(T)$, and $\beta_M(T)$ denote the *essential spectrum*, the *Weyl spectrum*, and the *Riesz spectrum* of T relative to the algebra $M(A)$ and to the ideal $K_M(A)$, respectively (see [5] for definitions). By $\omega(T)$, $W(T)$, and $\beta(T)$, we denote the corresponding sets of the operator T relative to $L(A)$ and $K(A)$. Generally we have $\omega_M(T) \subseteq W_M(T) \subseteq \beta_M(T)$ and $\omega(T) \subseteq W(T) \subseteq \beta(T)$. Let $\Omega(T)$ denote the set

$$\Omega(T) = \{\lambda \in \mathbb{C} : \lambda \text{ is an isolated point of } \sigma(T) \text{ and } \alpha(\lambda I - T) = \infty\}.$$

Theorem 3. *Let A be a commutative semisimple Banach algebra that satisfies either $M(A)$ regular or $A = \overline{\operatorname{soc} A}$. If $T \in M(A)$ then*

$$\omega(T) = W(T) = \beta(T) = \{\lambda \in \mathbb{C} : \lambda \textit{ is a limit point of } \sigma(T)\} \cup \Omega(T).$$

Proof. $M(A)$ is commutative and contains the identity, hence $\omega_M(T) = \beta_M(T)$ [5, Theorem R.5.1], which implies $\omega_M(T) = W_M(T) = \beta_M(T)$. By Theorem 1, if $M(A)$ is regular, or by Theorem 2 if $A = \overline{\operatorname{soc} A}$, then $\omega_M(T) = \omega(T)$. By Lemma A.1.6 of [5] we also have $\beta_M(T) = \beta(T)$ and thus $\omega(T) = W(T) = \beta(T)$. Moreover, by Theorem 1 or by Theorem 2, any $T \in \Phi(A) \cap M(A)$ has index 0, thus by Theorem 6 of [1] we also have $\omega(T) = \{\lambda \in \mathbb{C} : \lambda$ is a limit point of $\sigma(T)\} \cup \Omega(T)$. □

In the sequel we shall denote by $\sigma_{\mathrm{p}}(T)$, $\sigma_{\mathrm{r}}(T)$, $\sigma_{\mathrm{c}}(T)$, $\sigma_{\mathrm{ap}}(T)$, the *point spectrum*, the *residual spectrum*, the *continuous spectrum*, and the *approximate-point spectrum* of T, respectively.

Theorem 4. *Suppose that A is a commutative semisimple Banach algebra such that $\Delta(A)$ is discrete. Then for each $T \in M(A)$ we have*

(i) $\sigma_{\mathrm{p}}(T) = \varphi_T(\Delta(A))$, *the range of* φ_T.

Moreover, if $A = \overline{\operatorname{soc} A}$ we have

(ii) $\sigma_{\mathrm{r}}(T)$ *is empty,*

(iii) $\sigma(T) = \sigma_{\mathrm{ap}}(T)$.

Proof. (i) The inclusion $\sigma_{\mathrm{p}}(T) \subseteq \varphi_T(\Delta(A))$ has been proved in [1, Theorem 3]. Since $\Delta(A)$ is discrete, by the Silov idempotent theorem [14], A is regular. Hence if m_0 is a fixed multiplicative functional, there exists an element $x \in A$ such that $x^\wedge(m_0) = 1$ and $x^\wedge$ vanishes identically in the set $\Delta(A)\backslash\{m_0\}$. Then we have

$$([\varphi_T(m_0)I - T]x)^\wedge(m) = (\varphi_T(m_0) - \varphi_T(m) \cdot x^\wedge(m)) = 0$$

for each $m \in \Delta(A)$. This implies $(\varphi_T(m_0)I - T)x = 0$, and since $x \neq 0$, we have $\varphi_T(m_0) \in \sigma_{\mathrm{p}}(T)$. Thus $\sigma_{\mathrm{p}}(T) = \varphi_T(\Delta(A))$.

(ii) Let us suppose $A = \overline{\operatorname{soc} A}$ and $\sigma_{\mathrm{r}}(T) \neq \varnothing$. Then if $\lambda \in \sigma_{\mathrm{r}}(T)$, by (i) and by Remark 2 we have $\lambda \notin \varphi_{\mathrm{p}}(T) = \varphi_T(\Delta(A))$ and, therefore, $\lambda \neq \varphi_T(m)$ for each $m \in \Delta(A)$. If $x \in E_A$ there exists a multiplicative functional $m_0 \in \Delta(A)$ such that $x^\wedge(m_0) = 1$ and $x^\wedge$ vanishes identically on the set $\Delta(A)\backslash\{m_0\}$. Taking $z = (\lambda - \varphi_T(m_0))^{-1}x$ we have

$$[(\lambda I - T)z]^\wedge(m) = x^\wedge(m) \quad \text{for each } m \in \Delta(A),$$

and hence $[(\lambda I - T)z] = x$, i.e., $E_A \subseteq (\lambda I - T)(A) \subseteq A$. Since $A = \overline{\operatorname{span}\{E_A\}}$, we also have $\overline{(\lambda I - T)(A)} = A$, which implies $\lambda \notin \sigma_{\mathrm{r}}(T)$, a contradiction. Thus $\sigma_{\mathrm{r}}(T) = \varnothing$.

(iii) Since $\sigma_{\mathrm{p}}(T) \cup \sigma_{\mathrm{c}}(T) \subseteq \sigma_{\mathrm{ap}}(T)$ holds for each bounded operator on a Banach space, we have for each $T \in M(A)$

$$\sigma_{\mathrm{ap}}(T) \subseteq \sigma(T) \subseteq \sigma_{\mathrm{p}}(T) \cup \sigma_{\mathrm{c}}(T) \cup \sigma_{\mathrm{r}}(T) = \sigma_{\mathrm{p}}(T) \cup \sigma_{\mathrm{c}}(T) \subseteq \sigma_{\mathrm{ap}}(T),$$

thus $\sigma_{\mathrm{ap}}(T) = \sigma(T)$. □

We conclude by giving some characterizations of a Riesz multiplier on a Banach algebra that has discrete $\Delta(A)$. Some of these characterizations have been obtained in [1] for a class of Banach algebras, which includes $L_1(G)$, G compact and abelian. First we recall that a bounded operator on a Banach space is said to be *meromorphic* if its nonzero spectral points are all poles of the resolvent $R(\lambda, T) = (\lambda I - T)^{-1}$.

Theorem 5. *Let A be a commutative semisimple Banach algebra. Suppose $\Delta(A)$ is discrete and $T \in M_0(A)$. Then the following conditions are equivalent:*

(i) *T is a Riesz operator.*
(ii) *T is meromorphic.*
(iii) *$\sigma(T)$ is a finite set or a sequence that converges to zero.*
(iv) $\sigma(T) = \varphi_T(\Delta(A)) \cup \{0\}$.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are well known for each bounded operator on a Banach space (cf. [8]).

The implication (iii) $\Rightarrow$ (i) was shown in [1, Corollary 2] (see also Theorem 3).

The equivalence (i) $\Leftrightarrow$ (iv) follows by using exactly the same arguments of the proof of Theorem 10 of [1] (where only the properties that $\Delta(A)$ is discrete and that $\sigma_p(T) = \varphi_T(\Delta(A))$ have been used). □

3. Some applications

1. Any semisimple annihilator Banach algebra has dense socle (see [6, §32, Corollary 6]) and, in particular, the dual algebras introduced by Kaplansky [11], (see also [6]), also have this property. Examples of commutative semisimple dual Banach algebras are $L_p(G)$, G a compact abelian group, $1 \le p \le \infty$, or $C(G)$ the algebra of all continuous functions on G with convolution for multiplication [11, Theorem 15]. Thus if T_μ is a convolution operator on $L_p(G)$, defined by $T_\mu(f) = \mu * f$, $f \in L_p(G)$, and $\mu \in M(G)$, the results stated above hold.

Let $A = L_1(G)$, G compact and abelian. Then $M(A) \cong M(G)$ and $K_M(A) \cong L_1(G)$ (see [4]). Hence if $\mu \in M(G)$, the convolution operator T_μ is a Fredholm operator if and only if there exists a $\nu \in M(G)$ and a $\varphi \in L_1(G)$ such that $\mu * \nu = \delta_0 - \varphi$, where δ_0 is the Dirac measure concentrated at the identity. Observe that by Remark 3 we can also choose φ to be a trigonometric polynomial. This result was obtained in a more general situation (the group G not necessarily abelian) in [5, Chapter A]. A similar result, for a locally compact abelian group, was founded in [4] by replacing $M(G)$ by the algebra $\{T_f : f \in L_1(G)\}$.

2. Let A be a commutative Banach algebra with an orthogonal basis $\{e_k\}$ [9]. Clearly $E_A = \{e_k : k \in \mathbb{N}\}$, and thus A has dense socle. Examples of these algebras are l^p, $1 \le p < \infty$, c_0, all with respect to pointwise operations, and other algebras [9]. For each $x \in A$ let $x = \sum_{k=1}^{\infty} \lambda_k(x) e_k$ be its representation. Then $M(A)$ is isomorphic to a subalgebra of l^∞, and for any $T \in M(A)$ we have $Tx = \sum_{k=1}^{\infty} \lambda_k(x)\lambda_k(Te_k)e_k$ [2]. If $\{e_k\}$ is unconditional then $M(A) \cong l^\infty$ [2] and a Fredholm multiplier may be characterized in a simple way. In fact, in such a case, denoting by K_{M_0} the set $\{T \in K(A) : \{\lambda_k(Te_k)\} \in c_0\}$, we have $K_{M_0} \cong c_0$ and [2]

$$F_M(A) \subseteq K_{M_0} \subseteq K_M(A).$$

By Remark 3, denoting by $\Phi_{M_0}(A)$ the set of all Fredholm elements of $M(A)$ relative to K_{M_0}, we have

$$\Phi_{M_0}(A) = \Phi_M^F(A) = \Phi(A) \cap M(A).$$

Hence T is a Fredholm multiplier if and only if there exists a sequence $\{\nu_k\} \in l^\infty$ such that $\lim_{k\to\infty} \nu_k \cdot \lambda_k(Te_k) = 1$.

3. Let A verify the assumption of Theorem 1. Moreover, let us suppose that $\Delta(A)$ contains no isolated points. Then $K_M(A) = \{0\}$ [10]. Thus $T \in M(A)$ is Fredholm if and only if T is invertible. In particular, let $A = C_0(X)$, X any locally compact Hausdorff space that contains no isolated points. Then $M(A) \cong C_b(X)$, the algebra of all complex bounded continuous functions, and any multiplier is a multiplication operator T_f for some $f \in C_b(X)$. Then T_f is a Fredholm operator if and only if f is bounded away from 0.

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