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UNIT GROUPS OF INTEGRAL GROUP RINGS

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ABSTRACT. Let $U(\mathbb{Z}G)$ be the unit group of the integral group ring $\mathbb{Z}G$. A group G satisfies $(*)$ if either the set $T(G)$ of torsion elements of G is a central subgroup of G or, otherwise, if $x \in G$ does not centralize $T(G)$, then for every $t \in T(G)$, $x^{-1}tx = t^{-1}$. This property appears quite frequently while studying $U(\mathbb{Z}G)$. In this paper we investigate why one encounters this property and we have also given a "unified proof" for some known results regarding this property. Further, some additional results have been obtained.

1. INTRODUCTION

Let $\mathbb{Z}G$ be an integral group ring of a group G and let $U(\mathbb{Z}G)$ be its group of units. Denote by $T(G)$ the set of all torsion elements of G. A group G is Hamiltonian if G is nonabelian and every subgroup of G is normal.

We say that a group G satisfies the property $(*)$ if one of the following holds:

(a) $T(G)$ is a central subgroup of G.
(b) $T(G)$ is abelian, noncentral, and if $x \in G$ does not centralize $T(G)$, then $x^{-1}tx = t^{-1}$ for every $t \in T(G)$.
(c) $T(G)$ is a Hamiltonian 2-group and every subgroup of $T(G)$ is normal in G.

The property $(*)$ appears quite frequently while studying $U(\mathbb{Z}G)$. We list the following results:

(1) $U(\mathbb{Z}G)$ is nilpotent if and only if G is nilpotent and satisfies $(*)$ [7, VI.3.23].
(2) $U(\mathbb{Z}G)$ is FC if and only if G is FC and G satisfies $(*)$ [7, VI.5.4].
(3) The commutator subgroup $U(\mathbb{Z}G)'$ of $U(\mathbb{Z}G)$ is torsion if and only if G' is torsion and G satisfies $(*)$ [1, 2].
(4) $U(\mathbb{Z}G)/\zeta(U(\mathbb{Z}G))$ is locally finite if and only if $G/\zeta(G)$ is locally finite and G satisfies $(*)$. Here $\zeta(G)$ denotes the center of G [1].
(5) Let G be such that $G/T(G)$ is right ordered. Then $U(\mathbb{Z}G)$ is Engel (n-Engel) if and only if G is Engel (n-Engel) and satisfies $(*)$ [2].

The similarity in the above results motivates us to look for a "unified proof" for these results to see why G always has the property $(*)$. In this process we

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obtain easier proofs of the above mentioned results (particularly of (2) and (5)) and we also get some new results of a similar nature.

2. Condition $(*)$ is necessary

The necessity of $(*)$ in the above results will be shown by proving that the hypothesis in each of them implies the following statements:

(I) $T(G)$ is either abelian or a Hamiltonian group and every subgroup of $T(G)$ is normal in G.

(II) If $x \in G$ and $t \in T(G)$ are such that $x^{-1}tx \neq t$, then $x^{-1}tx = t^{-1}$.

It is easy to see that (I) and (II) imply that if $T(G)$ is nonabelian, then it is a Hamiltonian 2-group, thus establishing $(*)$ in all cases.

If $T(U(\mathbb{Z}G))$ is a subgroup, then $T(G)$ is either an abelian or a Hamiltonian 2-group with every subgroup of $T(G)$ normal in G [5]. Thus (I) holds if $U(\mathbb{Z}G)$ is one of the types as described in (1)–(4). For (5) we let $x \in G$ and $t \in T(G)$. Then $1+\alpha \in U(\mathbb{Z}G)$, where $\alpha = (1-t)x(1+t+\cdots+t^{O(t)-1})$. Now as $U(\mathbb{Z}G)$ is Engel, for some k,

$$(1+\alpha\,,_k t^{-1}) = (1+\alpha\,, \underbrace{t^{-1}\,, \dots\,, t^{-1}}_{k}) = 1\,,$$

i.e.,

$$(1-t)^{k+1}x(1+t+\cdots+t^{O(t)-1}) = 0\,,$$

thus

$$(1-t)x(1+t+\cdots+t^{O(t)-1}) = 0\,.$$

Since $(1-t)^2\beta = 0$ for any $\beta \in \mathbb{Z}G$ and $t \in T(G)$, it follows that $(1-t)\beta = 0$ (see [7, p. 182]).

This shows that $\langle t\rangle$ is normal in G and thus $T(G)$ is either abelian or a Hamiltonian group.

For (II) first note that if G is nonabelian torsion, then by [4] G will be a Hamiltonian 2-group, otherwise $U(\mathbb{Z}G)$ will have a noncyclic free subgroup.

We assume that G is nontorsion and $T(G)$ is either abelian or a Hamiltonian group and every subgroup of $T(G)$ is normal in G.

Let $t \in T(G)$ be an element of order l, $x \in G$, and $x^{-1}tx \neq t$. We can assume that x is of infinite order.

Now consider the rational group algebra $\mathbb{Q}\langle t\rangle$. By [7, II.2.6],

$$\mathbb{Q}\langle t\rangle = \bigoplus\sum_{d|l}\mathbb{Q}(\xi_d)\,; \qquad t \in \sum_{d|l}\xi_d\,,$$

where ξ_d is a primitive dth root of unity. Also as $x^{-1}tx = t^r$; $(l\,, r) = 1$, and $t^x = \sum_{d|l}\xi_d^r$. By [7, VI.1.16], if e is an idempotent $\mathbb{Q}\langle t\rangle$, then $x^{-1}ex = e$, thus $x^{-1}\mathbb{Q}(\xi_d)x = \mathbb{Q}(\xi_d)$ and $\theta_x : \xi_l \mapsto \xi_l^r$ is a $\mathbb{Q}$-automorphism of $\mathbb{Q}(\xi_l)$.

Let $R = \bigoplus\sum_{d|l}\mathbb{Z}[\xi_d]$. Then by [7, II.2.9], $|U(R) : U(Z\langle t\rangle)| < \infty$. Now $U(\mathbb{Z}[\xi_l]) = \langle \pm\xi_l\rangle \times A$, where A is free abelian of finite rank. Thus for some positive integer s, $V = U(\mathbb{Z}[\xi_l])^{2ls} \subseteq U(\mathbb{Z}\langle t\rangle)$.

Clearly $\mathscr{G} = \langle V\,, x\rangle$ is finitely generated and torsion free. We claim that $\mathscr{G}$ is actually abelian.

In fact, if $\mathscr{G}$ is nilpotent, then $\mathscr{G}/\zeta(\mathscr{G})$ is torsion free as $\mathscr{G}$ is torsion free. Further, $\langle t\rangle$ is normal in G so x^n commutes with t for some n, i.e., $x^n \in \zeta(\mathscr{G})$. Hence $\mathscr{G}$ is abelian. This also covers the case if $\mathscr{G}$ is Engel as a finitely generated soluble Engel group is nilpotent.

In the other cases ((2), (3), and (4)) $\mathscr{G}'$ is torsion, but since $\mathscr{G}' \subseteq V$ and V is torsion free, it follows that $\mathscr{G}$ is abelian.

Finally we have that x commutes elementwise with V. Hence $\mathbb{Q}(V) \subseteq \overline{\mathbb{Q}(\xi_l)}$, where $\overline{\mathbb{Q}(\xi_l)}$ is the fixed field of the automorphism θ_x. Since $\operatorname{rank} U(\mathbb{Z}[\xi_l]) = \operatorname{rank}(V)$, by [7, II.2.10], $x^{-1}\xi_l x = \xi_l^{-1}$ and thus $x^{-1}tx = t^{-1}$.

3. Condition $(*)$ is sufficient

We first prove the following lemma.

Lemma. *Let G be a group satisfying $(*)$ and let $G/T(G)$ be right ordered. Then $U(\mathbb{Z}G) = \mathscr{H}G$ for some $\mathscr{H} \subseteq \zeta(U(\mathbb{Z}G))$.*

Proof. By [4], $U(\mathbb{Z}G) = U(\mathbb{Z}T(G))G$. If $T(G)$ is central, then let $\mathscr{H} = U(\mathbb{Z}T(G))$. If $T(G)$ is a Hamiltonian 2-group, then $U(\mathbb{Z}T(G)) = \pm T(G)$ [7, II.2.1 and II.2.2] and so $U(\mathbb{Z}G) = \pm G$. Here $\mathscr{H} = \{+1, -1\}$.

Further, for any $\alpha = \sum_{t\in T(G)} \alpha(t)t$ let $\alpha^* = \sum_{t\in T(G)} \alpha(t)t^{-1}$. If $T(G)$ is noncentral and abelian, then for any $\alpha \in U(\mathbb{Z}T(G))$, $x^{-1}\alpha x = \alpha$ or α^*. Therefore $\mathscr{H} = \{\alpha \in U(\mathbb{Z}T(G)) | \alpha = \alpha^*\}$ is a central subgroup of $U(\mathbb{Z}G)$.

To prove that $U(\mathbb{Z}G) = \mathscr{H}G$, we let $\alpha = \sum_{i=1}^m z_i x_i \in U(\mathbb{Z}T(G))$, with $\varepsilon(\alpha) = 1$, where ε is the augmentation map. Then

$$\begin{aligned}
\alpha &= 1 + \sum_{i=1}^{m} z_i(x_i - 1) \\
&= \left(\prod_{i=1}^{m} x_i^{z_i}\right) + \theta, \qquad \theta \in I_{\mathbb{Z}}^2(T(G)) \\
&= t + \theta, \qquad t = \prod_{i=1}^{m} x_i^{z_i} \in T(G).
\end{aligned}$$

Here $I_{\mathbb{Z}}(T(G))$ denotes the augmentation ideal of $\mathbb{Z}T(G)$. Thus $\alpha = (1+\delta)t$; $\delta = \theta t^{-1} \in I_{\mathbb{Z}}^2(T(G))$.

We now show that $(1+\delta)^* = 1+\delta$, hence $1+\delta \in \mathscr{H}$. Let $\beta = 1+\delta$ and $\gamma = \beta^*\beta^{-1}$. Then $\gamma^*\gamma = 1$ and $\varepsilon(\gamma) = 1$ imply that $\gamma = x \in T(G)$. This gives the result that $x = \beta^*\beta^{-1} \in (1 + I_{\mathbb{Z}}^2(T(G))) \cap T(G) = T(G)' = 1$. Hence $\beta^* = \beta$ implies that $U(\mathbb{Z}T(G)) = \mathscr{H}T(G)$ and $U(\mathbb{Z}G) = \mathscr{H}G$. □

Now due to the previous lemma, the converse follows easily since $U(\mathbb{Z}G) = \mathscr{H}G$. For $\alpha_1, \dots, \alpha_n \in U(\mathbb{Z}G)$, there are $\beta_1, \dots, \beta_n$ in $\mathscr{H}$ and $x_1, \dots, x_n$ in G such that $\alpha_k = \beta_k x_k$, $k = 1, \dots, n$. Thus $(\alpha_1, \dots, \alpha_n) = (x_1, \dots, x_n)$. Also $\mathscr{H}\zeta(G) \subseteq \zeta(U(\mathbb{Z}G))$, thus $U(\mathbb{Z}G)/\mathscr{H}\zeta(G) \cong G/\zeta(G)$.

4. Further results

The discussions of §§2 and 3 show that if P is a group-theoretical property which is subgroup closed, then $U(\mathbb{Z}G) \in P$ implies that $G \in P$ and satisfies $(*)$ if it can be proved that $T(G)$ is either abelian or a Hamiltonian group and every subgroup of $T(G)$ is normal in G. Then the group $\mathcal{G}$ defined in §2 is abelian. To show the converse, the lemma of §3 should hold. Thus we have the following straightforward results, which we state as corollaries.

Corollary 1. *$U(\mathbb{Z}G)$ is locally nilpotent if and only if G is locally nilpotent and satisfies $(*)$.*

Corollary 2. *$U(\mathbb{Z}G)$ is locally FC if and only if G is locally FC and satisfies $(*)$.*

Corollary 3. *$U(\mathbb{Z}G)$ is hypercentral if and only if G is hypercentral and satisfies $(*)$.*

This is so because hypercentral groups are locally nilpotent.

Corollary 4. *$U(\mathbb{Z}G)$ satisfies the normalizer condition if and only if G satisfies the normalizer condition and $(*)$.*

Corollary 5. *Every subgroup of $U(\mathbb{Z}G)$ is subnormal if and only if every subgroup of G is subnormal and G satisfies $(*)$.*

Corollary 6. *If $U(\mathbb{Z}G)/\zeta(U(\mathbb{Z}G))$ is torsion, then $G/\zeta(G)$ is torsion and satisfies $(*)$. Conversely, if $G/\zeta(G)$ is torsion and satisfies $(*)$ and $G/T(G)$ is right ordered, then $U(\mathbb{Z}G)/\zeta(U(\mathbb{Z}G))$ is torsion.*

The following is an analogue of the theorem in [3] for integral group rings.

Theorem. *$U(\mathbb{Z}G)$ is soluble and n-Engel if and only if $U(\mathbb{Z}G)$ is nilpotent.*

Proof. Suppose that $U(\mathbb{Z}G)$ is soluble and n-Engel. Then G is soluble and n-Engel and satisfies $(*)$.

It is sufficient to show that G is nilpotent. Since $G/T(G)$ is torsion free soluble n-Engel, by [6, Corollary 7.36] $G/T(G)$ is nilpotent. Hence $\gamma_l(G) \subseteq T(G)$ for some l. Here $\gamma_l(G)$ is the lth term of the lower central chain of G.

If $T(G)$ is central, then G is nilpotent. If $T(G)$ is noncentral, then for every $t \in T(G)$ and $x \in G$, $(t, {}_n x) = 1$. Now as $x^{-1}tx = t$ or t^{-1}, we have $o(t) \leq 2^n$. Thus $T(G)$ is of exponent at most 2^n.

Again, since $x^{-1}tx = t$ or t^{-1} for $t \in T(G)$, $x \in G$, it follows that $(t, x_1, \dots, x_n) = 1$ for every $t \in T(G)$ and $x_i \in G$, $i = 1, \dots, n$. Thus $\gamma_{l+n}(G) = 1$, i.e., G is nilpotent. □

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