PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 120, Number 1, January 1994

REGULAR AUSLANDER-REITEN COMPONENTS CONTAINING DIRECTING MODULES

ANDRZEJ SKOWROŃSKI

(Communicated by Maurice Auslander)

ABSTRACT. We describe completely all regular Auslander-Reiten components of artin algebras containing directing modules. We prove also that the Auslander-Reiten quiver of an artin algebra admits at most finitely many DTr-orbits containing directing modules.

1. INTRODUCTION

Throughout the paper A will denote a fixed artin algebra over a commutative artin ring R and n will be the rank of the Grothendieck group $K_0(A)$ of A. By a module we always mean a finitely generated right module. We shall denote by $\operatorname{mod} A$ the category of all (finitely generated right) A-modules, by $\operatorname{rad}(\operatorname{mod} A)$ the radical of $\operatorname{mod} A$, and by $\operatorname{rad}^\infty(\operatorname{mod} A)$ the intersection of all powers $\operatorname{rad}^i(\operatorname{mod} A)$, $i \geq 0$, of $\operatorname{rad}(\operatorname{mod} A)$. From the existence of the Auslander-Reiten sequences in $\operatorname{mod} A$ we know that $\operatorname{rad}(\operatorname{mod} A)$ is generated by the irreducible maps as a left and as a right ideal (see [1]). We denote by D the standard duality $\operatorname{Hom}_R(-, I)$, where I is the injective envelope of $R/\operatorname{rad} R$ in $\operatorname{mod} R$. Further, we denote by Γ_A the Auslander-Reiten quiver of A and by τ_A, τ_A^- the Auslander-Reiten operators DTr, TrD on $\operatorname{mod} A$, respectively. We will not distinguish between an indecomposable A-module, its isomorphism class, and the vertex in Γ_A corresponding to it. A connected component $\mathscr{C}$ of Γ_A is said to be regular if $\mathscr{C}$ contains neither a projective nor an injective module. Following [13], a connected component $\mathscr{C}$ of Γ_A is said to be generalized standard if $\operatorname{rad}^\infty(X, Y) = 0$ for all modules X and Y from $\mathscr{C}$. Finally, following [9], an indecomposable A-module M is called directing if it does not belong to a cycle $M \to M_1 \to \cdots \to M_r \to M$ of nonzero nonisomorphisms between indecomposable A-modules.

Directing modules have played an important role in the representation theory of artin algebras: preprojective and preinjective components in general and

Received by the editors January 2, 1992 and, in revised form, April 8, 1992; the contents of this paper were presented at the CMS Annual Seminar/NATO Advanced Research Workshop on "Representations of Algebras and Related Topics" (Carleton University, Ottowa, August, 1992) and at the RIMS Symposium "Representation Theory of Finite Groups and Finite Dimensional Algebras" (Kyoto, October, 1991).

1991 *Mathematics Subject Classification*. Primary 16G70; Secondary 16G10.

Research partially supported by Polish Scientific Grant KBN No. 1222/2/91.

© 1993 American Mathematical Society
0002-9939/93 \$1.00 + \$.25 per page

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

connecting components of tilted algebras consist entirely of directing modules. It was proved in [11] that a hereditary algebra H of type Δ has a regular tilting module if and only if the quiver Δ has at least three vertices and is neither of Euclidean nor of Dynkin type. Moreover, if T is a regular tilting H-module and $B = \operatorname{End}_H(T)$ is the associated tilted algebra, then the connecting component of Γ_B is a regular component consisting entirely of directing modules and is of the form $Z\Delta^{\mathrm{op}}$, where Δ^{op} is the opposite quiver of Δ. In [14] the author and Smalø proved that such components exhaust all regular components consisting entirely of directing modules.

We shall prove here the following theorem on the structure of regular Auslander-Reiten components of artin algebras containing directing modules.

Theorem. *Let $\mathscr{C}$ be a regular connected component of Γ_A containing a directing module. Write $A = P \oplus Q$, where the simple direct summands of $P/\operatorname{rad} P$ are exactly the simple composition factors of modules in $\mathscr{C}$. Denote by $t_Q(A)$ the ideal of A generated by all images of all maps from Q to A, and put $B = A/t_Q(A)$. Then*

(i) *$\mathscr{C}$ has only finitely many τ_A-orbits and all modules from $\mathscr{C}$ are directing.*
(ii) *$\mathscr{C}$ is a generalized standard component of Γ_A.*
(iii) *$t_Q(A)$ is the annihilator of $\mathscr{C}$ in A.*
(iv) *B is a tilted algebra of the form $\operatorname{End}_H(T)$ with H a (wild) hereditary artin algebra and T a regular tilting H-module.*
(v) *$\mathscr{C}$ is the connecting component of Γ_B.*

In the course of the proof, we obtain a condition for an indecomposable A-module M, in terms of its neighbourhood in Γ_A, to be nondirecting. In particular, we obtain the following consequence of this criterion and [4, 15].

Corollary 1. *Let $\mathscr{C}$ be a regular connected component of Γ_A having infinitely many τ_A-orbits. Then for any module M in $\mathscr{C}$ there exists a cycle $M \to M_1 \to \cdots \to M_k \to M$ in $\operatorname{mod} A$ with $M_1, \ldots, M_k$ from $\mathscr{C}$.*

As another consequence of this criterion and the above theorem we obtain the following result.

Corollary 2. *Γ_A admits at most finitely many τ_A-orbits containing directing modules.*

Combining this corollary with a result in [5] we also obtain the following fact.

Corollary 3. *Let A be representation-infinite and R be an algebraically closed field. Then Γ_A admits infinitely many τ_A-orbits without directing modules.*

The results of this paper have been partially announced in the author's survey article [12].

2. Preparatory lemmas

We shall use the following well-known fact from [1].

Lemma 1. *Let I be an ideal in A, $B = A/I$, and M be a B-module. Then $\tau_B M$ is a submodule of $\tau_A M$.*

The following lemma will play a crucial role in our investigations.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Lemma 2. *Let* $M_1, \ldots, M_r$ *be pairwise nonisomorphic indecomposable* A*-modules such that* $\operatorname{Hom}_A(M_i, \tau_A M_j) = 0$ *for all* $1 \leq i, j \leq r$. *Let* $M = M_1 \oplus \cdots \oplus M_r$, I *be the annihilator of* M *in* A, *and* $B = A/I$. *Then* M *is a partial tilting* B*-module. In particular,* $r \leq n$.

Proof. We have from our assumption that $\operatorname{Hom}_A(M, \tau_A M) = 0$, and then, by Lemma 1, $\operatorname{Hom}_B(M, \tau_B M) = 0$. Hence $\operatorname{Ext}^1_B(M, M) \cong \mathrm{D}\overline{\operatorname{Hom}}_B(M, \tau_B M) = 0$. Clearly, M is a faithful B-module. This implies that $pd_B M \leq 1$ (see [8, (1.5)]). Indeed, it is enough to show that $\operatorname{Hom}_B(\mathrm{D}B, \tau_B M) = 0$ (see [9, (2.4)]). Since M is a faithful B-module, $\mathrm{D}M$ is a faithful B^{op}-module, and we have a monomorphism $B^{\mathrm{op}} \to (\mathrm{D}M)^s$ for some $s \geq 1$. Then there is an epimorphism $M^s \to \mathrm{D}B$ in $\operatorname{mod} B$, and consequently $\operatorname{Hom}_B(\mathrm{D}B, \tau_B M) = 0$ because $\operatorname{Hom}_B(M, \tau_B M) = 0$. Therefore, M is a partial tilting B-module and, according to a result of Bongartz [3], M may be extended to a tilting B-module. Hence r is less than or equal to the rank of $K_0(B)$, which is again less than or equal to n. This finishes the proof.

By a path in $\operatorname{mod} A$ we mean a sequence $X_0 \to X_1 \to \cdots \to X_{s-1} \to X_s$ of nonzero nonisomorphisms between indecomposable A-modules. A path $M_0 \to M_1 \to \cdots \to M_{r-1} \to M_r$ in Γ_A is called sectional if $M_{i-2} \not\cong \tau_A M_i$ for each i, $2 \leq i \leq r$.

The following lemma gives some sufficient conditions for an indecomposable A-module M to be nondirecting.

Lemma 3. *Let* M *be an indecomposable* A*-module and* $\mathscr{C}$ *the connected component of* Γ_A *containing* M. *Assume that there is a sectional path* $M = M_0 \to M_1 \to \cdots \to M_{n-1} \to M_n$ *in* $\mathscr{C}$ *such that the following conditions are satisfied:*

(i) *The modules* $M_1, \ldots, M_n$ *are nonprojective and pairwise nonisomorphic.*

(ii) *For any* i, $1 \leq i \leq n-1$, *and any sectional path in* $\mathscr{C}$ *of the form* $U_t \to U_{t-1} \to \cdots \to U_1 = \tau_A M_{i+1} \to U_0 = M_i$ *such that* $t \leq n$, *the modules* $U_1, \ldots, U_t$ *are nonprojective and nonisomorphic to the modules* $M_0, M_1, \ldots, M_n$.

Then there is a cycle $M \to X_1 \to \cdots \to X_k \to M$ *in* $\operatorname{mod} A$ *with* $X_1, \ldots, X_k$ *from* $\mathscr{C}$.

Proof. Obviously we may assume that $M_0 \not\cong M_i$ for each i, $1 \leq i \leq n$. Then, by (i), the modules $M_0, \ldots, M_n$ are pairwise nonisomorphic, and, by Lemma 2, there are $0 \leq p, q \leq n$ such that $\operatorname{Hom}_A(M_p, \tau_A M_q) \neq 0$. In particular, we have in $\operatorname{mod} A$ a path $M_p \to \tau_A M_q \to M_{q-1}$, if $q \geq 1$, and a path $M_p \to \tau_A M_0 \to \tau_A M_1 \to M_0$, if $q = 0$. Let r be the least number with $1 \leq r \leq n$ and such that there is in $\operatorname{mod} A$ a path

$$(*) \qquad M_i = Z_s \xrightarrow{f_s} Z_{s-1} \to \cdots \to Z_2 \xrightarrow{f_2} Z_1 = \tau_A M_r \xrightarrow{f_1} Z_0 = M_{r-1}$$

with $Z_1, \ldots, Z_s$ from $\mathscr{C}$ and $0 \leq i \leq n$. We shall show that $r = 1$. In this case, we have in $\operatorname{mod} A$ a required cycle $M = M_0 \to \cdots \to M_i = Z_s \to \cdots \to Z_1 \to Z_0 = M$. Suppose that $r \geq 2$. We claim that there is a path $(*)$ with $s > n$, $f_1, \ldots, f_n$ irreducible, $Z_1, \ldots, Z_s$ from $\mathscr{C}$, and $Z_j \not\cong \tau_A Z_{j-2}$ for each j, $2 \leq j \leq n$. We may assume that, if f_i does not belong to $\operatorname{rad}^{\infty}(\operatorname{mod} A)$, then f_i is irreducible, because otherwise we replace it by a finite sequence of irreducible maps. Suppose now that some f_i belongs to

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

$\operatorname{rad}^{\infty}(\operatorname{mod} A)$ and let m be the least index with this property. Then there is in $\operatorname{mod} A$ an infinite sequence of irreducible maps

$$\cdots V_{j+1} \xrightarrow{g_{j+1}} V_j \xrightarrow{g_j} \cdots \to V_1 \xrightarrow{g_1} V_0 = Z_{m-1}$$

such that all V_j are indecomposable and $\operatorname{Hom}_A(Z_m, V_j) \neq 0$ for all $j \geq 0$. But then we have in $\operatorname{mod} A$ a path

$$M_i = Z_s \xrightarrow{f_s} \cdots \to Z_m \xrightarrow{g_n} V_{n-1} \to \cdots \to V_1 \xrightarrow{g_1} V_0$$
$$= Z_{m-1} \xrightarrow{f_{m-1}} \cdots \to Z_1 \xrightarrow{f_1} Z_0,$$

where $f_1, \ldots, f_{m-1}, g_1, \ldots, g_n$ are irreducible, and all modules forming this path belong to $\mathscr{C}$. Finally, suppose that there is a such that $a < n$, $a < s$, $f_1, \ldots, f_a$ are irreducible, $Z_j \not\cong \tau_A Z_{j-2}$ for each $2 \leq j \leq a$, and $Z_{a+1} \cong \tau_A Z_{a-1}$. Then from (i) and (ii), the modules $Z_0, \ldots, Z_{a-1}$ are nonprojective, and consequently we have in $\operatorname{mod} A$ a path

$$M_i = Z_s \to \cdots \to Z_{a+1} \cong \tau_A Z_{a-1} \to \cdots \to \tau_A Z_0 \to M_{r-2},$$

a contradiction to our choice of r. Similarly, if $(*)$ is a sectional path in Γ_A, then $s > n$ because $Z_s = M_i$ and from our assumption (ii). Therefore, we proved the existence of a path $(*)$ with $s > n$, $f_1, \ldots, f_n$ irreducible, $Z_j \not\cong \tau_A Z_{j-2}$ for each $2 \leq j \leq n$, and $Z_1, \ldots, Z_s$ from $\mathscr{C}$. Then, by (i) and (ii), the modules $Z_0, Z_1, \ldots, Z_n$ are nonprojective. We claim that they are also pairwise nonisomorphic. Indeed, if $Z_k \cong Z_h$ for some $n \geq k > h \geq 0$, then, by a result of Bautista and Smalø [2], we have $Z_{h+1} \cong \tau_A Z_{k-1}$, and hence again a path in $\operatorname{mod} A$ of the form

$$M_i = Z_s \to \cdots \to Z_{h+1} \cong \tau_A Z_{k-1} \to \cdots \to \tau_A Z_0 \to M_{r-2},$$

a contradiction. Finally, by Lemma 2, there are then b and c, $0 \leq b, c \leq n$, such that $\operatorname{Hom}_A(Z_b, \tau_A Z_c) \neq 0$, and we have in $\operatorname{mod} A$ a path of the form

$$M_i = Z_s \to \cdots \to Z_b \to \tau_A Z_c \to \cdots \to \tau_A Z_0 \to M_{r-2},$$

which again contradicts our choice of r. Therefore, $r = 1$. This finishes our proof.

For a connected component $\mathscr{C}$ of Γ_A, we denote by $\mathscr{C}_s$ the stable part of $\mathscr{C}$, obtained from $\mathscr{C}$ by removing the τ_A-orbits of projective and injective modules.

The following lemma shows possible applications of Lemma 3.

Lemma 4. *Let $\mathscr{C}$ be a connected component of Γ_A and $\mathscr{D}$ be a connected component of $\mathscr{C}_s$. Assume that $\mathscr{D}$ has infinitely many τ_A-orbits and no oriented cycles. Let M be a module in $\mathscr{D}$ such that the length of any walk in $\mathscr{C}$ from a nonstable module to M is at least $2n$. Then there is a sectional path $M = M_0 \to M_1 \to \cdots \to M_n$ in $\mathscr{C}$ such that $\mathscr{C}$ has no paths of the form $M_i = Z_s \to Z_{s-1} \to \cdots \to Z_1 = \tau_A M_j$ with $s \leq n$ and $0 \leq i, j \leq n$. In particular, there is a cycle $M \to X_1 \to \cdots \to X_k \to M$ in* $\operatorname{mod} A$ *with $X_1, \ldots, X_k$ from $\mathscr{C}$.*

Proof. Since $\mathscr{D}$ is a stable translation quiver with infinitely many τ_A-orbits and without oriented cycles, we infer from [4, 15] (see also [6]), that $\mathscr{D} \cong \mathbb{Z}\Delta$ for some infinite, locally finite, connected, valued quiver Δ without oriented cycles.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Let $\mathscr{O}$ be a fixed τ_A-orbit in $\mathscr{D}$. We claim that $\mathscr{D}$ contains a path from M to a module in $\mathscr{O}$. Indeed, since $\mathscr{D}$ is connected, there is a walk $M = Y_0 — Y_1 — \cdots — Y_r$ in $\mathscr{D}$ with Y_r from $\mathscr{O}$, where $Y_i — Y_{i+1}$ means either $Y_i \to Y_{i+1}$ or $Y_i \leftarrow Y_{i+1}$. We may assume $r \geq 1$. Suppose that there is a path in $\mathscr{D}$ from M to $\tau_A^{-q} Y_i$ for some $q \geq 0$, $0 \leq i \leq r-1$. Then either $\tau_A^{-q} Y_i \to \tau_A^{-q} Y_{i+1}$ or $\tau_A^{-q} Y_i \to \tau_A^{-q-1} Y_{i+1}$ is an arrow in $\mathscr{D}$, and consequently $\mathscr{D}$ contains either a path from M to $\tau_A^{-q} Y_{i+1}$ or a path from M to $\tau_A^{-q-1} Y_{i+1}$. Then the claim follows by induction on i, $0 \leq i \leq r$.

Consider now the set $\mathscr{P}$ of all paths $M = N_0 \to N_1 \to \cdots \to N_k$ of finite length in $\mathscr{D}$ starting at M and such that $\tau_A N_k$ is not a successor of M in $\mathscr{D}$. Observe that, if $M = N_0 \to N_1 \to \cdots \to N_k$ is a path from $\mathscr{P}$, then $\mathscr{D}$ has no paths from N_i to $\tau_A N_j$, $0 \leq i, j \leq k$. Indeed, if $N_i = V_0 \to V_1 \to \cdots \to V_p = \tau_A N_j$ is a path in $\mathscr{D}$, then $\mathscr{D}$ admits a path

$$\begin{aligned} M = N_0 \to N_1 \to \cdots \to N_i = V_0 \to V_1 \to \cdots \to V_p \\ = \tau_A N_j \to \tau_A N_{j+1} \to \cdots \to \tau_A N_k, \end{aligned}$$

a contradiction since $\tau_A N_k$ is not a successor of M in $\mathscr{D}$. In particular, the paths $M = N_0 \to N_1 \to \cdots \to N_m$, $1 \leq m \leq k$, also belong to $\mathscr{P}$. Since $\mathscr{D}$ has no oriented cycles, each τ_A-orbit $\mathscr{O}$ of $\mathscr{D}$ contains a module which is the target of a path from $\mathscr{P}$. Moreover, since $\mathscr{D}$ is locally finite and admits infinitely many τ_A-orbits, $\mathscr{P}$ contains paths of arbitrary large length. Then, from the above remarks, $\mathscr{P}$ contains a path $M = M_0 \to M_1 \to \cdots \to M_n$ of length n. Suppose that $\mathscr{C}$ contains a path $M_i = Z_s \to Z_{s-1} \to \cdots \to Z_1 \to \tau_A M_j$ with $s \leq n$, $0 \leq i, j \leq n$. Then, from our assumption on the length of walks from nonstable modules in $\mathscr{C}$ to M, we deduce that this path lies in $\mathscr{D}$. But this is impossible by the above property of paths from $\mathscr{P}$. Therefore, $\mathscr{C}$ does not contain paths of the form $M_i = Z_s \to Z_{s-1} \to \cdots \to Z_1 \to \tau_A M_j$ with $s \leq n$, $0 \leq i, j \leq n$. Clearly, this implies that the path $M = M_0 \to M_1 \to \cdots \to M_n$ is sectional. Finally, we also infer that for any sectional path in $\mathscr{C}$ of the form $U_t \to U_{t-1} \to \cdots \to U_1 = \tau_A M_{i+1} \to U_0 = M_i$ with $1 \leq i \leq n-1$, $t \leq n$, the modules $U_1, \ldots, U_t$ are nonprojective and nonisomorphic to the modules $M_0, M_1, \ldots, M_n$. Then, by Lemma 3, there is a cycle $M \to X_1 \to \cdots \to X_k \to M$ in $\operatorname{mod} A$ with $X_1, \ldots, X_k$ from $\mathscr{C}$. This finishes the proof.

We shall also need the following lemma.

Lemma 5. *Let $\mathscr{C}$ be a regular connected component of Γ_A having only finitely many τ_A-orbits. Assume that there is in* $\operatorname{mod} A$ *a path*

$$X = Z_0 \xrightarrow{f_1} Z_1 \to \cdots \to Z_{m-1} \xrightarrow{f_m} Z_m = Y$$

with X and Y from $\mathscr{C}$ and such that f_i belongs to $\operatorname{rad}^\infty(\operatorname{mod} A)$ *for some $1 \leq i \leq m$. Then all modules in $\mathscr{C}$ are nondirecting.*

Proof. Since $\mathscr{C}$ has only finitely many τ_A-orbits, we infer by [4] that $\mathscr{C}$ consists of nonperiodic modules. Then, by [15], we have $\mathscr{C} \cong Z\Delta$, for some finite valued quiver Δ without oriented cycles. Let M be a module in $\mathscr{C}$. We shall show that M is nondirecting in $\operatorname{mod} A$. Let i be the least index such that f_i belongs to $\operatorname{rad}^\infty(\operatorname{mod} A)$. Then $Z_0, \ldots, Z_{i-1}$ belong to $\mathscr{C}$, and there is in $\mathscr{C}$ an infinite path

$$Z_{i-1} = V_0 \to V_1 \to \cdots \to V_s \to V_{s+1} \to \cdots$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

such that $\operatorname{rad}^{\infty}(V_s, Z_i) \neq 0$ for all $s \geq 0$. Since $\mathscr{C}$ has only finitely many τ_A-orbits, there is a path in $\mathscr{C}$ of the form $M = U_0 \to U_1 \to \cdots \to U_k = V_p$ for some $p \geq 0$. Let j now be the maximal index such that f_j belongs to $\operatorname{rad}^{\infty}(\operatorname{mod} A)$. Then $Z_j, \ldots, Z_m$ belong to $\mathscr{C}$, and there is in $\mathscr{C}$ an infinite path

$$\cdots \to W_{r+1} \to W_r \to \cdots \to W_1 \to W_0 = Z_j$$

such that either $\operatorname{rad}^{\infty}(Z_{j-1}, W_r) \neq 0$ for all $r \geq 0$, if $i < j$, or $\operatorname{rad}^{\infty}(V_p, W_r) \neq 0$ for all $r \geq 0$, if $i = j$. Again, since $\mathscr{C}$ has only finitely many τ_A-orbits, there is in $\mathscr{C}$ a path of the form $W_q = L_0 \to L_1 \to \cdots \to L_t = M$ for some $q \geq 0$. Then we have in $\operatorname{mod} A$ either a cycle

$$\begin{aligned} M = U_0 \to \cdots \to U_k = V_p \to Z_i \to \cdots \to Z_{j-1} \to W_q \\ = L_0 \to L_1 \to \cdots \to L_t = M, \end{aligned}$$

if $i < j$, or a cycle

$$M = U_0 \to \cdots \to U_k = V_p \to W_q = L_0 \to L_1 \to \cdots \to L_t = M,$$

if $i = j$. Therefore, M is nondirecting in $\operatorname{mod} A$.

3. Proof of the theorem

(i) Let M be a directing module in the component $\mathscr{C}$. Then, by [4], $\mathscr{C}$ does not contain periodic modules, and therefore $\mathscr{C} \cong Z\Delta$ for some valued locally finite quiver Δ without oriented cycles, by [15] (and also [6]). Since $\mathscr{C}$ is regular and M is a directing module in $\mathscr{C}$, we deduce from Lemma 4 that Δ is finite. Moreover, since $\mathscr{C}$ has no oriented cycles, from Lemma 5 we infer also that all modules from $\mathscr{C}$ are directing A-modules.

(ii) This is a direct consequence of (i) and Lemma 5.

(iii) From (i) we know that $\mathscr{C} \cong Z\Delta$ for some finite valued quiver Δ without oriented cycles. Let U be the direct sum of modules corresponding to all vertices of a fixed Δ in $\mathscr{C}$. Then, by [8, (1.2)], $\operatorname{ann} U = \operatorname{ann} \mathscr{C}$ and clearly U is a faithful module over $A/\operatorname{ann}\mathscr{C}$. On the other hand, $t_Q(A)$ is contained in $\operatorname{ann}\mathscr{C}$ and U is a sincere B-module, where $B = A/t_Q(A)$. We claim that U is not the middle term of a short chain in $\operatorname{mod} B$; that is, for any indecomposable B-module Z, either $\operatorname{Hom}_B(Z, U) = 0$ or $\operatorname{Hom}_B(U, \tau_B Z) = 0$. Suppose that $\operatorname{Hom}_B(Z, U) \neq 0$ and $\operatorname{Hom}_B(U, \tau_B Z) \neq 0$ for some indecomposable B-module Z. Then there are indecomposable direct summands X and Y of U and a path in $\operatorname{mod} A$ of the form $X \xrightarrow{f} \tau_B Z \to W \to Z \xrightarrow{g} Y$. Observe that if Z belongs to $\mathscr{C}$ then $\tau_B Z = \tau_A Z$, because $\mathscr{C}$ consists entirely of B-modules. Now, since $\mathscr{C} \cong Z\Delta$, Δ has no oriented cycles, and X, Y lie on some fixed Δ, we infer that one of the maps f or g belongs to $\operatorname{rad}^{\infty}(\operatorname{mod} A)$. But then, by Lemma 5, all modules from $\mathscr{C}$ are nondirecting, a contradiction. Consequently, U is not the middle term of a short chain in $\operatorname{mod} B$ and then, by [7, (3.1)], U is a faithful B-module. This proves that $t_Q(A) = \operatorname{ann} U = \operatorname{ann}\mathscr{C}$.

(iv) and (v). We know that U is a faithful B-module. Moreover, we infer, by Lemma 5, that $\operatorname{Hom}_B(U, \tau_B U) = 0$ and $\operatorname{Hom}_B(\tau_B^{-} U, U) = 0$. Then applying [8, (1.5)] and its dual, we obtain that $pd_B U \leq 1$ and $id_B U \leq 1$. Clearly, $\operatorname{Ext}_B^1(U, U) \cong D\overline{\operatorname{Hom}}_B(U, \tau_B U) = 0$. Finally, if $\operatorname{Hom}_B(U, X) \neq 0$ for some indecomposable B-module X which is not a direct summand of U, then $\operatorname{Hom}_B(\tau_B^{-} U, X) \neq 0$. Consequently, by [8, (1.6)], U is a tilting and

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

cotilting B-module. Then, by (ii), $H = \mathrm{End}_B(U)$ is a hereditary algebra of type Δ^{op}. Hence $B = \mathrm{End}_H(T)$ for a tilting H-module T and $\mathscr{C}$ is a connecting component of Γ_B. Since $\mathscr{C}$ is regular, then, by [10, p. 42; 11, Theorem], T is a regular tilting H-module. Moreover, in this case B is not a concealed algebra (see [9]) and $\mathscr{C}$ is the unique connecting component of Γ_B.

4. Proofs of Corollaries 1, 2, and 3

Proof of Corollary 1. The claim is obvious in the case where $\mathscr{C}$ is a stable tube. If $\mathscr{C}$ is not a stable tube, then by [4, 15], $\mathscr{C}$ has no oriented cycles and our claim is a direct consequence of Lemma 4.

Proof of Corollary 2. Observe first that A admits only finitely many ideals of the form $t_Q(A)$, where Q is a direct summand of A. Hence, by the theorem, Γ_A admits at most finitely many connected components containing directing modules. Hence, it is enough to show that any connected component of Γ_A admits at most finitely many τ_A-orbits containing directing modules. Suppose that $\mathscr{C}$ is a connected component of Γ_A which admits infinitely many τ_A-orbits containing directing modules. Then, since $\mathscr{C}$ is locally finite, there is a connected component $\mathscr{D}$ of the stable part $\mathscr{C}_s$ of $\mathscr{C}$ which admits infinitely many τ_A-orbits containing directing modules. Moreover, since the number of nonstable τ_A-orbits in $\mathscr{C}$ is finite, there exists a directing module M in $\mathscr{D}$ such that the length of any walk in $\mathscr{C}$ from a nonstable module to M is at least $2n$. But then, from Lemma 4, M is nondirecting, a contradiction. Therefore, each connected component of Γ_A admits at most finitely many τ_A-orbits containing directing modules, and hence Γ_A has the same property. This finishes our proof.

Proof of Corollary 3. It was proved in [5] that a finite-dimensional algebra A over an algebraically closed field k is representation-finite if and only if Γ_A has only finitely many τ_A-orbits. Then Corollary 3 is a direct consequence of Corollary 2.

The author was recently informed by Idun Reiten that, independently, L. Peng and J. Xiao have also proved Corollary 2.

Acknowledgment

These results were obtained while I was visiting the University of Tsukuba. I would like to thank H. Tachikawa for the invitation and all colleagues in Tsukuba for their hospitality during my stay. I would also like to thank the referee for remarks concerning the presentation of the results included in the paper.

References

1. M. Auslander and I. Reiten, *Representation theory of artin algebras.* III; IV, Comm. Algebra **3** (1975), 239–294; **5** (1977), 443–518.
2. R. Bautista and S. O. Smalø, *Nonexistent cycles*, Comm. Algebra **11** (1983), 1755–1767.
3. K. Bongartz, *Tilted algebras*, Representations of Algebras, Lecture Notes in Math., vol. 903, Springer, Berlin, 1981, pp. 26–38.
4. D. Happel, U. Preiser, and C. M. Ringel, *Vinberg's characterization of Dynkin diagrams using subadditive functions with applications to* DTr*-periodic modules*, Representation Theory. II, Lecture Notes in Math., vol. 832, Springer, Berlin, 1980, pp. 280–294.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

5. S. Liu, *The degrees of irreducible maps and shapes of Auslander-Reiten quivers*, J. London Math. Soc. (2) **45** (1992), 32–54.
6. ——, *Semi-stable components of an Auslander-Reiten quiver*, Preprint No. 12/91, University of Trondheim.
7. I. Reiten, S. O. Smalø, and A. Skowroński, *Short chains and short cycles of indecomposable modules*, Proc. Amer. Math. Soc. **117** (1993), 343–354.
8. ——, *Short chains and regular components*, Proc. Amer. Math. Soc. **117** (1993), 601–612.
9. C. M. Ringel, *Tame algebras and integral quadratic forms*, Lecture Notes in Math., vol. 1099, Springer, Berlin, 1984.
10. ——, *Representation theory of finite-dimensional algebras*, Representations of Algebras (Proceedings, Durham 1985) Cambridge Univ. Press, London and New York, 1986, pp. 7-80.
11. ——, *The regular components of the Auslander-Reiten quiver of a tilted algebra*, Chinese Ann. Math. Ser. B **1** (1988), 1–18.
12. A. Skowroński, *Cycles of indecomposable modules*, Representation Theory of Finite Groups and Finite Dimensional Algebras (Proc. Res. Inst. Math. Sci., Kyoto 1991) J. Math. Soc. Japan (to appear).
13. ——, *Generalized standard Auslander-Reiten components*, J. Math. Soc. Japan (to appear).
14. A. Skowroński and S. O Smalø, *Directing modules*, J. Algebra **147** (1992), 137–146.
15. Y. Zhang, *The structure of regular components*, Canad. J. Math. **43** (1991), 652–672.

Institute of Mathematics, Nicholas Copernicus University, Chopina 12/18, 87–100 Toruń, Poland
E-mail address: skowron@mat.torun.edu.pl

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use