REGULAR AUSLANDER-REITEN COMPONENTS CONTAINING DIRECTING MODULES

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Abstract. We describe completely all regular Auslander-Reiten components of artin algebras containing directing modules. We prove also that the Auslander-Reiten quiver of an artin algebra admits at most finitely many DT r-orbits containing directing modules.

1. Introduction

Throughout the paper $A$ will denote a fixed artin algebra over a commutative artin ring $R$ and $n$ will be the rank of the Grothendieck group $K_0(A)$ of $A$. By a module we always mean a finitely generated right module. We shall denote by $\text{mod } A$ the category of all (finitely generated right) $A$-modules, by $\text{rad}(\text{mod } A)$ the radical of $\text{mod } A$, and by $\text{rad}^\infty(\text{mod } A)$ the intersection of all powers $\text{rad}^i(\text{mod } A)$, $i \geq 0$, of $\text{rad}(\text{mod } A)$. From the existence of the Auslander-Reiten sequences in $\text{mod } A$ we know that $\text{rad}(\text{mod } A)$ is generated by the irreducible maps as a left and as a right ideal (see [1]). We denote by $D$ the standard duality $\text{Hom}_R(-, I)$, where $I$ is the injective envelope of $R/\text{rad } R$ in $\text{mod } R$. Further, we denote by $\Gamma_A$ the Auslander-Reiten quiver of $A$ and by $\tau_A$, $\tau^-_A$ the Auslander-Reiten operators $D\text{Tr}$, $\text{Tr}D$ on $\text{mod } A$, respectively. We will not distinguish between an indecomposable $A$-module, its isomorphism class, and the vertex in $\Gamma_A$ corresponding to it. A connected component $C$ of $\Gamma_A$ is said to be regular if $C$ contains neither a projective nor an injective module. Following [13], a connected component $C$ of $\Gamma_A$ is said to be generalized standard if $\text{rad}^\infty(X, Y) = 0$ for all modules $X$ and $Y$ from $C$. Finally, following [9], an indecomposable $A$-module $M$ is called directing if it does not belong to a cycle $M \rightarrow M_1 \rightarrow \cdots \rightarrow M_r \rightarrow M$ of nonzero nonisomorphisms between indecomposable $A$-modules.

Directing modules have played an important role in the representation theory of artin algebras: preprojective and preinjective components in general and...
connecting components of tilted algebras consist entirely of directing modules. It was proved in [11] that a hereditary algebra $H$ of type $\Delta$ has a regular tilting module if and only if the quiver $\Delta$ has at least three vertices and is neither of Euclidean nor of Dynkin type. Moreover, if $T$ is a regular tilting $H$-module and $B = \text{End}_H(T)$ is the associated tilted algebra, then the connecting component of $\Gamma_B$ is a regular component consisting entirely of directing modules and is of the form $Z\Delta^\text{op}$, where $\Delta^\text{op}$ is the opposite quiver of $\Delta$. In [14] the author and Smalø proved that such components exhaust all regular components consisting entirely of directing modules.

We shall prove here the following theorem on the structure of regular Auslander-Reiten components of artin algebras containing directing modules.

**Theorem.** Let $\mathcal{C}$ be a regular connected component of $\Gamma_A$ containing a directing module. Write $A = P \oplus Q$, where the simple direct summands of $P/\text{rad } P$ are exactly the simple composition factors of modules in $\mathcal{C}$. Denote by $t_Q(A)$ the ideal of $A$ generated by all images of all maps from $Q$ to $A$, and put $B = A/t_Q(A)$. Then

(i) $\mathcal{C}$ has only finitely many $\tau_A$-orbits and all modules from $\mathcal{C}$ are directing.

(ii) $\mathcal{C}$ is a generalized standard component of $\Gamma_A$.

(iii) $t_Q(A)$ is the annihilator of $\mathcal{C}$ in $A$.

(iv) $B$ is a tilted algebra of the form $\text{End}_H(T)$ with $H$ a (wild) hereditary artin algebra and $T$ a regular tilting $H$-module.

(v) $\mathcal{C}$ is the connecting component of $\Gamma_B$.

In the course of the proof, we obtain a condition for an indecomposable $A$-module $M$, in terms of its neighbourhood in $\Gamma_A$, to be nondirecting. In particular, we obtain the following consequence of this criterion and [4, 15].

**Corollary 1.** Let $\mathcal{C}$ be a regular connected component of $\Gamma_A$ having infinitely many $\tau_A$-orbits. Then for any module $M$ in $\mathcal{C}$ there exists a cycle $M \rightarrow M_1 \rightarrow \cdots \rightarrow M_k \rightarrow M$ in mod $A$ with $M_1, \ldots, M_k$ from $\mathcal{C}$.

As another consequence of this criterion and the above theorem we obtain the following result.

**Corollary 2.** $\Gamma_A$ admits at most finitely many $\tau_A$-orbits containing directing modules.

Combining this corollary with a result in [5] we also obtain the following fact.

**Corollary 3.** Let $A$ be representation-infinite and $R$ be an algebraically closed field. Then $\Gamma_A$ admits infinitely many $\tau_A$-orbits without directing modules.

The results of this paper have been partially announced in the author’s survey article [12].

2. Preparatory lemmas

We shall use the following well-known fact from [1].

**Lemma 1.** Let $I$ be an ideal in $A$, $B = A/I$, and $M$ be a $B$-module. Then $\tau_B M$ is a submodule of $\tau_A M$.

The following lemma will play a crucial role in our investigations.
Lemma 2. Let \( M_1, \ldots, M_r \) be pairwise nonisomorphic indecomposable \( A \)-modules such that \( \operatorname{Hom}_A(M_i, \tau_A M_j) = 0 \) for all \( 1 \leq i, j \leq r \). Let \( M = M_1 \oplus \cdots \oplus M_r, I \) be the annihilator of \( M \) in \( A \), and \( B = A/I \). Then \( M \) is a partial tilting \( B \)-module. In particular, \( r \leq n \).

Proof. We have from our assumption that \( \operatorname{Hom}_A(M, \tau_A M) = 0 \), and then, by Lemma 1, \( \operatorname{Hom}_B(M, \tau_B M) = 0 \). Hence \( \operatorname{Ext}_B^1(M, M) \cong \operatorname{DHom}_B(M, \tau_B M) = 0 \). Clearly, \( M \) is a faithful \( B \)-module. This implies that \( \operatorname{pd}_B M \leq 1 \) (see [8, (1.5)]). Indeed, it is enough to show that \( \operatorname{Hom}_B(DB, \tau_B M) = 0 \) (see [9, (2.4)]). Since \( M \) is a faithful \( B \)-module, \( DM \) is a faithful \( B^{op} \)-module, and we have a monomorphism \( B^{op} \to (DM)^s \) for some \( s \geq 1 \). Then there is an epimorphism \( M^s \to DB \) in \( \operatorname{mod} B \), and consequently \( \operatorname{Hom}_B(DB, \tau_B M) = 0 \) because \( \operatorname{Hom}_B(M, \tau_B M) = 0 \). Therefore, \( M \) is a partial tilting \( B \)-module and, according to a result of Bongartz [3], \( M \) may be extended to a tilting \( B \)-module. Hence \( r \) is less than or equal to the rank of \( K_0(B) \), which is again less than or equal to \( n \). This finishes the proof.

By a path in \( \operatorname{mod} A \) we mean a sequence \( X_0 \to X_1 \to \cdots \to X_{s-1} \to X_s \) of nonzero nonisomorphisms between indecomposable \( A \)-modules. A path \( M_0 \to M_1 \to \cdots \to M_{r-1} \to M_r \) in \( \Gamma_A \) is called sectional if \( M_{i-2} \not\to \tau_A M_i \) for each \( i \), \( 2 \leq i \leq r \).

The following lemma gives some sufficient conditions for an indecomposable \( A \)-module \( M \) to be nondirecting.

Lemma 3. Let \( M \) be an indecomposable \( A \)-module and \( C \) the connected component of \( \Gamma_A \) containing \( M \). Assume that there is a sectional path \( M = M_0 \to M_1 \to \cdots \to M_{n-1} \to M_n \) in \( C \) such that the following conditions are satisfied:

(i) The modules \( M_1, \ldots, M_n \) are nonprojective and pairwise nonisomorphic.

(ii) For any \( i, 1 \leq i \leq n-1 \), and any sectional path in \( C \) of the form \( U_i \to U_{i-1} \to \cdots \to U_1 = \tau_A M_{i+1} \to U_0 = M_i \) such that \( t \leq n \), the modules \( U_1, \ldots, U_t \) are nonprojective and nonisomorphic to the modules \( M_0, M_1, \ldots, M_n \).

Then there is a cycle \( M \to X_1 \to \cdots \to X_k \to M \) in \( \operatorname{mod} A \) with \( X_1, \ldots, X_k \) from \( C \).

Proof. Obviously we may assume that \( M_0 \not\cong M_i \) for each \( i, 1 \leq i \leq n \). Then, by (i), the modules \( M_0, \ldots, M_n \) are pairwise nonisomorphic, and, by Lemma 2, there are \( 0 \leq p, q \leq n \) such that \( \operatorname{Hom}_A(M_p, \tau_A M_q) \not= 0 \). In particular, we have in \( \operatorname{mod} A \) a path \( M_p \to \tau_A M_q \to M_{q-1}, \) if \( q \geq 1 \), and a path \( M_p \to \tau_A M_0 \to \tau_A M_1 \to M_0, \) if \( q = 0 \). Let \( r \) be the least number with \( 1 \leq r \leq n \) and such that there is in \( \operatorname{mod} A \) a path

\[
(*) \quad M_i = Z_s \xrightarrow{f_i} Z_{s-1} \to \cdots \to Z_2 \xrightarrow{f_I} Z_1 = \tau_A M_r \xrightarrow{f_0} Z_0 = M_{r-1}
\]

with \( Z_1, \ldots, Z_s \) from \( C \) and \( 0 \leq i \leq n \). We shall show that \( r = 1 \). In this case, we have in \( \operatorname{mod} A \) a required cycle \( M = M_0 \to \cdots \to M_r = Z_r \to \cdots \to Z_1 \to Z_0 = M \). Suppose that \( r \geq 2 \). We claim that there is a path \( (*) \) with \( s > n, f_1, \ldots, f_n \) irreducible, \( Z_1, \ldots, Z_s \) from \( C \), and \( Z_j \not\cong \tau_A Z_{j-2} \) for each \( j, 2 \leq j \leq n \). We may assume that, if \( f_i \) does not belong to \( \operatorname{rad}^{\infty} (\operatorname{mod} A) \), then \( f_i \) is irreducible, because otherwise we replace it by a finite sequence of irreducible maps. Suppose now that some \( f_i \) belongs to
rad$$^\infty(\mod A)$$ and let \( m \) be the least index with this property. Then there is in \( \mod A \) an infinite sequence of irreducible maps

$$\ldots V_{j+1} \overset{g_{j+1}}{\to} V_j \overset{g_j}{\to} \ldots \to V_1 \overset{g_1}{\to} V_0 = Z_{m-1}$$

such that all \( V_j \) are indecomposable and \( \Hom_A(Z_m, V_j) \neq 0 \) for all \( j \geq 0 \). But then we have in \( \mod A \) a path

$$M_i = Z_s \overset{f_s}{\to} \ldots \to Z_m \overset{g_m}{\to} V_{n-1} \to \ldots \to V_1 \overset{g_1}{\to} V_0$$

where \( f_1, \ldots, f_{m-1}, g_1, \ldots, g_n \) are irreducible, and all modules forming this path belong to \( \mathcal{C} \). Finally, suppose that there is \( a \) such that \( a < n, \ a < s, \ f_1, \ldots, f_a \) are irreducible, \( Z_j \not\subseteq \tau_A Z_{j-2} \) for each \( 2 \leq j \leq a \), and \( Z_{a+1} \cong \tau_A Z_{a-1} \). Then from (i) and (ii), the modules \( Z_0, \ldots, Z_{a-1} \) are nonprojective, and consequently we have in \( \mod A \) a path

$$M_i = Z_s \to \ldots \to Z_{a+1} \cong \tau_A Z_{a-1} \to \cdots \to \tau_A Z_0 \to M_{r-2},$$

a contradiction to our choice of \( r \). Similarly, if (\text{\textasteriskcentered\textdagger}) is a sectional path in \( \Gamma_A \), then \( s > n \) because \( Z_s = M_i \), and from our assumption (ii). Therefore, we proved the existence of a path (\text{\textasteriskcentered\textdagger}) with \( s > n, f_1, \ldots, f_n \) irreducible, \( Z_j \not\subseteq \tau_A Z_{j-2} \) for each \( 2 \leq j \leq n \), and \( Z_1, \ldots, Z_n \) from \( \mathcal{C} \). Then, by (i) and (ii), the modules \( Z_0, Z_1, \ldots, Z_n \) are nonprojective. We claim that they are also pairwise nonisomorphic. Indeed, if \( Z_k \cong Z_h \) for some \( n \geq k > h \geq 0 \), then, by a result of Bautista and Smalø [2], we have \( Z_{h+1} \cong \tau_A Z_{k-1} \), and hence again a path in \( \mod A \) of the form

$$M_i = Z_s \to \ldots \to Z_{h+1} \cong \tau_A Z_{k-1} \to \cdots \to \tau_A Z_0 \to M_{r-2},$$

a contradiction. Finally, by Lemma 2, there are then \( b \) and \( c \), \( 0 \leq b, c \leq n \), such that \( \Hom_A(Z_b, \tau_A Z_c) \neq 0 \), and we have in \( \mod A \) a path of the form

$$M_i = Z_s \to \ldots \to Z_b \to \tau_A Z_c \to \cdots \to \tau_A Z_0 \to M_{r-2},$$

which again contradicts our choice of \( r \). Therefore, \( r = 1 \). This finishes our proof.

For a connected component \( \mathcal{E} \) of \( \Gamma_A \), we denote by \( \mathcal{E} \) the stable part of \( \mathcal{E} \), obtained from \( \mathcal{E} \) by removing the \( \tau_A \)-orbits of projective and injective modules.

The following lemma shows possible applications of Lemma 3.

**Lemma 4.** Let \( \mathcal{E} \) be a connected component of \( \Gamma_A \) and \( \mathcal{D} \) be a connected component of \( \mathcal{E} \). Assume that \( \mathcal{D} \) has infinitely many \( \tau_A \)-orbits and no oriented cycles. Let \( M \) be a module in \( \mathcal{D} \) such that the length of any walk in \( \mathcal{E} \) from a nonstable module to \( M \) is at least \( 2n \). Then there is a sectional path \( M = M_0 \to M_1 \to \cdots \to M_n \) in \( \mathcal{E} \) such that \( \mathcal{E} \) has no paths of the form \( M_i = Z_s \to Z_{s-1} \to \cdots \to Z_1 = \tau_A M_j \) with \( s \leq n \) and \( 0 \leq i, j \leq n \). In particular, there is a cycle \( M \to X_1 \to \cdots \to X_k \to M \) in \( \mod A \) with \( X_1, \ldots, X_k \) from \( \mathcal{E} \).

**Proof.** Since \( \mathcal{D} \) is a stable translation quiver with infinitely many \( \tau_A \)-orbits and without oriented cycles, we infer from [4, 15] (see also [6]), that \( \mathcal{D} \cong Z\Delta \) for some infinite, locally finite, connected, valued quiver \( \Delta \) without oriented cycles.
Let $\mathcal{C}$ be a fixed $\tau_A$-orbit in $\mathcal{D}$. We claim that $\mathcal{D}$ contains a path from $M$ to a module in $\mathcal{C}$. Indeed, since $\mathcal{D}$ is connected, there is a walk $M = Y_0 \rightarrow Y_1 \rightarrow \ldots \rightarrow Y_r$ in $\mathcal{D}$ with $Y_i$ from $\mathcal{C}$, where $Y_i \rightarrow Y_{i+1}$ means either $Y_i \rightarrow Y_{i+1}$ or $Y_i \leftarrow Y_{i+1}$. We may assume $r \geq 1$. Suppose that there is a path in $\mathcal{D}$ from $M$ to $\tau_A^{-q}Y_i$ for some $q \geq 0$, $0 \leq i \leq r - 1$. Then either $\tau_A^{-q}Y_i \rightarrow \tau_A^{-q}Y_{i+1}$ or $\tau_A^{-q}Y_i \rightarrow \tau_A^{-q}Y_{i+1}$ is an arrow in $\mathcal{D}$, and consequently $\mathcal{D}$ contains either a path from $M$ to $\tau_A^{-q}Y_{i+1}$ or a path from $M$ to $\tau_A^{-q}Y_{i+1}$. Then the claim follows by induction on $i$, $0 \leq i \leq r$.

Consider now the set $\mathcal{P}$ of all paths $M = N_0 \rightarrow N_1 \rightarrow \ldots \rightarrow N_k$ of finite length in $\mathcal{D}$ starting at $M$ such that $\tau_A N_k$ is not a successor of $M$ in $\mathcal{D}$. Observe that, if $M = N_0 \rightarrow N_1 \rightarrow \ldots \rightarrow N_k$ is a path from $\mathcal{P}$, then $\mathcal{D}$ has no paths from $N_i$ to $\tau_A N_j$, $0 \leq i, j \leq k$. Indeed, if $N_i = V_0 \rightarrow V_1 \rightarrow \ldots \rightarrow V_p = \tau_A N_j$ is a path in $\mathcal{D}$, then $\mathcal{D}$ admits a path

$$M = N_0 \rightarrow N_1 \rightarrow \ldots \rightarrow N_i = V_0 \rightarrow V_1 \rightarrow \ldots \rightarrow V_p = \tau_A N_j,$$

a contradiction since $\tau_A N_k$ is not a successor of $M$ in $\mathcal{D}$. In particular, the paths $M = N_0 \rightarrow N_1 \rightarrow \ldots \rightarrow N_k$, $1 \leq m \leq k$, also belong to $\mathcal{P}$. Since $\mathcal{D}$ has no oriented cycles, each $\tau_A$-orbit $\mathcal{C}$ of $\mathcal{D}$ contains a module which is the target of a path from $\mathcal{P}$. Moreover, since $\mathcal{D}$ is locally finite and admits infinitely many $\tau_A$-orbits, $\mathcal{P}$ contains paths of arbitrary large length. Then, from the above remarks, $\mathcal{P}$ contains a path $M = M_0 \rightarrow M_1 \rightarrow \ldots \rightarrow M_n$ of length $n$. Suppose that $\mathcal{C}$ contains a path $M_i = Z_3 \rightarrow Z_{s-1} \rightarrow \ldots \rightarrow Z_1 \rightarrow \tau_A M_j$ with $s \leq n$, $0 \leq i, j \leq n$. Then, from our assumption on the length of walks from nonstable modules in $\mathcal{C}$ to $M$, we deduce that this path lies in $\mathcal{D}$. But this is impossible by the above property of paths from $\mathcal{P}$. Therefore, $\mathcal{C}$ does not contain paths of the form $M_i = Z_3 \rightarrow Z_{s-1} \rightarrow \cdots \rightarrow Z_1 \rightarrow \tau_A M_j$ with $s \leq n$, $0 \leq i, j \leq n$. Clearly, this implies that the path $M = M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_n$ is sectional. Finally, we also infer that for any sectional path in $\mathcal{C}$ of the form $U_t \rightarrow U_{t-1} \rightarrow \cdots \rightarrow U_1 = \tau_A M_{t+1} \rightarrow U_0 = M_t$ with $1 \leq i \leq n - 1$, $t \leq n$, the modules $U_1, \ldots, U_t$ are nonprojective and nonisomorphic to the modules $M_0, M_1, \ldots, M_n$. Then, by Lemma 3, there is a cycle $M = X_1 \rightarrow \cdots \rightarrow X_k \rightarrow M$ in $\text{mod } A$ with $X_1, \ldots, X_k$ from $\mathcal{C}$. This finishes the proof.

We shall also need the following lemma.

**Lemma 5.** Let $\mathcal{C}$ be a regular connected component of $\Gamma_A$ having only finitely many $\tau_A$-orbits. Assume that there is in $\text{mod } A$ a path

$$X = Z_0 \rightarrow Z_1 \rightarrow \cdots \rightarrow Z_{m-1} \rightarrow Z_m = Y$$

with $X$ and $Y$ from $\mathcal{C}$ and such that $f_i$ belongs to $\text{rad}^\infty(\text{mod } A)$ for some $1 \leq i \leq m$. Then all modules in $\mathcal{C}$ are nondirecting.

**Proof.** Since $\mathcal{C}$ has only finitely many $\tau_A$-orbits, we infer by [4] that $\mathcal{C}$ consists of nonperiodic modules. Then, by [15], we have $\mathcal{C} \cong Z\Delta$, for some finite valued quiver $\Delta$ without oriented cycles. Let $M$ be a module in $\mathcal{C}$. We shall show that $M$ is nondirecting in $\text{mod } A$. Let $i$ be the least index such that $f_i$ belongs to $\text{rad}^\infty(\text{mod } A)$. Then $Z_0, \ldots, Z_{i-1}$ belong to $\mathcal{C}$, and there is in $\mathcal{C}$ an infinite path

$$Z_{i-1} = V_0 \rightarrow V_1 \rightarrow \cdots \rightarrow V_s \rightarrow V_{s+1} \rightarrow \cdots$$
such that \( \text{rad}^\infty(V_s, Z_s) \neq 0 \) for all \( s \geq 0 \). Since \( \mathcal{C} \) has only finitely many \( \tau_A \)-orbits, there is a path in \( \mathcal{C} \) of the form \( M = U_0 \rightarrow U_1 \rightarrow \cdots \rightarrow U_k = V_p \) for some \( p \geq 0 \). Let \( j \) now be the maximal index such that \( f_j \) belongs to \( \text{rad}^\infty(\text{mod } A) \). Then \( Z_j, \ldots, Z_m \) belong to \( \mathcal{C} \), and there is in \( \mathcal{C} \) an infinite path

\[
\cdots \rightarrow W_{r+1} \rightarrow W_r \rightarrow \cdots \rightarrow W_1 \rightarrow W_0 = Z_j
\]

such that either \( \text{rad}^\infty(Z_{j-1}, W_r) \neq 0 \) for all \( r \geq 0 \), if \( i < j \), or \( \text{rad}^\infty(V_p, W_r) \neq 0 \) for all \( r \geq 0 \), if \( i = j \). Again, since \( \mathcal{C} \) has only finitely many \( \tau_A \)-orbits, there is in \( \mathcal{C} \) a path of the form \( W_q = L_0 \rightarrow L_1 \rightarrow \cdots \rightarrow L_t = M \) for some \( q \geq 0 \). Then we have in \( \text{mod } A \) either a cycle

\[
M = U_0 \rightarrow \cdots \rightarrow U_k = V_p \rightarrow W_q = L_0 \rightarrow L_1 \rightarrow \cdots \rightarrow L_t = M,
\]

if \( i < j \), or a cycle

\[
M = U_0 \rightarrow \cdots \rightarrow U_k = V_p \rightarrow W_q = L_0 \rightarrow L_1 \rightarrow \cdots \rightarrow L_t = M,
\]

if \( i = j \). Therefore, \( M \) is nondirecting in \( \text{mod } A \).

3. Proof of the theorem

(i) Let \( M \) be a directing module in the component \( \mathcal{C} \). Then, by [4], \( \mathcal{C} \) does not contain periodic modules, and therefore \( \mathcal{C} \cong Z\Delta \) for some valued locally finite quiver \( \Delta \) without oriented cycles, by [15] (and also [6]). Since \( \mathcal{C} \) is regular and \( M \) is a directing module in \( \mathcal{C} \), we deduce from Lemma 4 that \( \Delta \) is finite. Moreover, since \( \mathcal{C} \) has no oriented cycles, from Lemma 5 we infer also that all modules from \( \mathcal{C} \) are directing \( \tau_A \)-modules.

(ii) This is a direct consequence of (i) and Lemma 5.

(iii) From (i) we know that \( \mathcal{C} \cong Z\Delta \) for some finite valued quiver \( \Delta \) without oriented cycles. Let \( U \) be the direct sum of modules corresponding to all vertices of a fixed \( \Delta \) in \( \mathcal{C} \). Then, by [8, (1.2)], \( \text{ann } U = \text{ann } \mathcal{C} \) and clearly \( U \) is a faithful module over \( A/\text{ann } \mathcal{C} \). On the other hand, \( t_Q(A) \) is contained in \( \text{ann } \mathcal{C} \) and \( U \) is a sincere \( B \)-module, where \( B = A/\text{ann } Q(A) \). We claim that \( U \) is not the middle term of a short chain in \( \text{mod } B \); that is, for any indecomposable \( B \)-module \( Z \), either \( \text{Hom}_B(Z, U) = 0 \) or \( \text{Hom}_B(U, \tau_BZ) = 0 \). Suppose that \( \text{Hom}_B(Z, U) \neq 0 \) and \( \text{Hom}_B(U, \tau_BZ) \neq 0 \) for some indecomposable \( B \)-module \( Z \). Then there are indecomposable direct summands \( X \) and \( Y \) of \( U \) and a path in \( \text{mod } A \) of the form \( X \xrightarrow{f} \tau_BZ \rightarrow W \rightarrow Z \xrightarrow{g} Y \). Observe that if \( Z \) belongs to \( \mathcal{C} \) then \( \tau_BZ = \tau_AZ \), because \( \mathcal{C} \) consists entirely of \( B \)-modules. Now, since \( \mathcal{C} \cong Z\Delta \), \( \Delta \) has no oriented cycles, and \( X, Y \) lie on some fixed \( \Delta \), we infer that one of the maps \( f \) or \( g \) belongs to \( \text{rad}^\infty(\text{mod } A) \). But then, by Lemma 5, all modules from \( \mathcal{C} \) are nondirecting, a contradiction. Consequently, \( U \) is not the middle term of a short chain in \( \text{mod } B \) and then, by [7, (3.1)], \( U \) is a faithful \( B \)-module. This proves that \( t_Q(A) = \text{ann } U = \text{ann } \mathcal{C} \).

(iv) and (v). We know that \( U \) is a faithful \( B \)-module. Moreover, we infer, by Lemma 5, that \( \text{Hom}_B(U, \tau_BU) = 0 \) and \( \text{Hom}_B(\tau_BU, U) = 0 \). Then applying [8, (1.5)] and its dual, we obtain that \( pd_BU \leq 1 \) and \( id_BU \leq 1 \). Clearly, \( \text{Ext}_B(U, U) = 0 \). Finally, if \( \text{Hom}_B(U, X) \neq 0 \) for some indecomposable \( B \)-module \( X \) which is not a direct summand of \( U \), then \( \text{Hom}_B(\tau_BU, X) \neq 0 \). Consequently, by [8, (1.6)], \( U \) is a tilting and
cotilting $B$-module. Then, by (ii), $H = \End_B(U)$ is a hereditary algebra of type $\Delta^\text{op}$. Hence $B = \End_H(T)$ for a tilting $H$-module $T$ and $\mathcal{E}$ is a connecting component of $\Gamma_B$. Since $\mathcal{E}$ is regular, then, by [10, p. 42; 11, Theorem], $T$ is a regular tilting $H$-module. Moreover, in this case $B$ is not a concealed algebra (see [9]) and $\mathcal{E}$ is the unique connecting component of $\Gamma_B$.

4. PROOFS OF COROLLARIES 1, 2, AND 3

Proof of Corollary 1. The claim is obvious in the case where $\mathcal{E}$ is a stable tube. If $\mathcal{E}$ is not a stable tube, then by [4, 15], $\mathcal{E}$ has no oriented cycles and our claim is a direct consequence of Lemma 4.

Proof of Corollary 2. Observe first that $A$ admits only finitely many ideals of the form $t_Q(A)$, where $Q$ is a direct summand of $A$. Hence, by the theorem, $\Gamma_A$ admits at most finitely many connected components containing directing modules. Hence, it is enough to show that any connected component of $\Gamma_A$ admits at most finitely many $\tau_A$-orbits containing directing modules. Suppose that $\mathcal{E}$ is a connected component of $\Gamma_A$ which admits infinitely many $\tau_A$-orbits containing directing modules. Then, since $\mathcal{E}$ is locally finite, there is a connected component $\mathcal{D}$ of the stable part $\mathcal{E}_s$ of $\mathcal{E}$ which admits infinitely many $\tau_A$-orbits containing directing modules. Moreover, since the number of nonstable $\tau_A$-orbits in $\mathcal{E}$ is finite, there exists a directing module $M$ in $\mathcal{D}$ such that the length of any walk in $\mathcal{E}$ from a nonstable module to $M$ is at least $2n$. But then, from Lemma 4, $M$ is nondirecting, a contradiction. Therefore, each connected component of $\Gamma_A$ admits at most finitely many $\tau_A$-orbits containing directing modules, and hence $\Gamma_A$ has the same property. This finishes our proof.

Proof of Corollary 3. It was proved in [5] that a finite-dimensional algebra $A$ over an algebraically closed field $k$ is representation-finite if and only if $\Gamma_A$ has only finitely many $\tau_A$-orbits. Then Corollary 3 is a direct consequence of Corollary 2.

The author was recently informed by Idun Reiten that, independently, L. Peng and J. Xiao have also proved Corollary 2.

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