

ASSOCIATIVE AND JORDAN SHIFT ALGEBRAS

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ABSTRACT. Let R be the shift algebra, i.e., the associative algebra presented by generators u, v and the relation $uv = 1$. As N. Jacobson showed, R contains an infinite family of matrix units. In this paper, we describe the Jordan algebra R^+ and its unital special universal envelope by generators and relations. Moreover, we give a presentation for the Jordan triple system defined on R by $P_x y = xy^*x$ where $*$ is the involution on R with $u^* = v$. As a consequence, we prove the existence of an infinite rectangular grid in a Jordan triple system V containing tripotents c and d with $V_2(c) = V_2(d) \oplus (V_2(c) \cap V_1(d))$ and $V_2(c) \cap V_1(d) \neq 0$.

1. THE SHIFT ALGEBRA

Let k be a commutative ring and R the associative unital k -algebra presented by generators u, v and the relation $uv = 1$. We call R the *shift algebra* since it has a faithful representation on $k^{(\infty)} = \sum_{i=0}^{\infty} k \cdot \varepsilon_i$ by the shift operators

$$\begin{aligned}u: \varepsilon_i &\mapsto \varepsilon_{i-1}, \varepsilon_0 \mapsto 0, \\v: \varepsilon_i &\mapsto \varepsilon_{i+1}.\end{aligned}$$

In [1] Jacobson essentially proved the following theorem on the structure of R .

Theorem 1. (a) R is a free k -module with basis $1, u^n, v^n$ ($n \geq 1$), and $e_{ij} = v^i(1 - vu)u^j$ ($i, j \geq 0$).

(b) The e_{ij} satisfy the multiplication table for matrix units.

(c) The subspace E spanned by the e_{ij} is an ideal, namely, the ideal generated by $e_{00} = 1 - vu$, and $R/E \cong k[X, X^{-1}]$ (Laurent polynomials).

We remark that R has an involution $*$ with $u^* = v$, but the elements u and v of R obviously do not play symmetrical roles; in particular, there is no automorphism of R interchanging u and v and no involution fixing the generators. In this note we introduce a *symmetric shift algebra* S on two generators defined by left-right symmetric relations and, hence, carrying an involution π fixing the generators. The defining relations of S are expressible by Jordan products. This leads us to define a *Jordan shift algebra* J and a *Jordan shift triple system* T . We show that $J \cong R^+$ is special with unital special universal

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envelope (S, π) and that T is isomorphic to R with the Jordan triple product $P_x y = xy^*x$. As a consequence we obtain the existence of an infinite set of orthogonal idempotents in a unital Jordan algebra containing a noninvertible regular pair x, y (i.e., $U_x y = x, U_y x = y$) with vanishing Bergmann operators $W(x, y) (= \text{Id} - V_{x,y} + U_x U_y)$ and $W(y, x)$, and the existence of an infinite rectangular grid in a Jordan triple system V containing tripotents c and d with the property that $V_2(c) = V_2(d) \oplus (V_2(c) \cap V_1(d))$ and $V_2(c) \cap V_1(d) \neq 0$.

Standard references for notation and terminology are [2, 5].

2. THE SYMMETRIC SHIFT ALGEBRA

The symmetric shift algebra is the associative unital k -algebra S presented by generators s, t and relations

$$(1) \quad s = sts, \quad t = tst,$$

$$(2) \quad (1 - st)t^n(1 - ts) = (1 - ts)s^n(1 - st) = 0 \quad \text{for all } n \geq 0.$$

From the definition it is clear that S has an involution π and an automorphism ρ of period 2, commuting with each other, and acting on the generators by

$$s^\pi = s, \quad t^\pi = t, \quad s^\rho = t, \quad t^\rho = s.$$

Hence $*$ = $\pi \circ \rho = \rho \circ \pi$ is an involution with $s^* = t$. In the sequel, the locution “by symmetry” will mean “by an application of π, ρ , or $*$ ”.

Theorem 2. (a) S is a free k -module with basis $1, s^n, t^n$ ($n \geq 1$), and $g_{ij} = s^i(1 - st)t^j, h_{ij} = t^i(1 - ts)s^j$ ($i, j \geq 0$).

(b) The g_{ij} and h_{ij} satisfy the multiplication table for matrix units and annihilate each other.

(c) The subspaces G, H spanned by the g_{ij}, h_{ij} , respectively, are ideals, namely, the ideals generated by $g_{00} = 1 - st, h_{00} = 1 - ts$, respectively. They are interchanged by π and ρ , with $S/G \cong R \cong S/H$ and $S/(G \oplus H) \cong k[X, X^{-1}]$.

(d) S has a faithful representation on $k^{(\infty)} \oplus k^{(\infty)} = \sum_{i=0}^{\infty} k\varepsilon_i \oplus k\eta_i$ via $s(\varepsilon_i) = (1 - \delta_{i0})\varepsilon_{i-1}, s(\eta_i) = \eta_{i+1}, t(\varepsilon_i) = \varepsilon_{i+1}, t(\eta_i) = (1 - \delta_{i0})\eta_{i-1}$.

Proof. We have $g_{ij}^\pi = h_{ji} = g_{ji}^\rho$; hence, $G^\pi = G^\rho = H$. Let us set $g_{pq} = 0$ and $h_{pq} = 0$ if p or q is negative.

First we show

$$(3) \quad g_{ij}h_{kl} = 0,$$

$$(4) \quad t^i s^j = \begin{cases} t^{i-j} - \sum_{k=1}^j h_{i-k, j-k} & \text{if } i > j, \\ s^{j-i} - \sum_{k=1}^i h_{i-k, j-k} & \text{if } i \leq j, \end{cases}$$

$$(5) \quad s^n g_{ij} = g_{i+n, j}, \quad t^n g_{ij} = g_{i-n, j},$$

$$(6) \quad g_{ij} s^n = g_{i, j-n}, \quad g_{ij} t^n = g_{i, j+n},$$

$$(7) \quad g_{ij} g_{kl} = \delta_{jk} g_{il}.$$

Indeed, (3) follows from (2), while (4) and the first formula in (5) are clear from the definitions. For the second one, note in case $n > i$ that $t^n g_{ij} = t^n s^i g_{0j} = t^{n-i} g_{0j}$ (by (3) and (4)) = 0 by (1) and in case $n \leq i$ that $t^n g_{ij} = t^n s^i g_{0j} = s^{i-n} g_{0j} = g_{i-n, j}$ by (3) and (4).

By symmetry we have (6). The multiplication table (7) for the g_{ij} is now an easy consequence of the *shift formulas* (5) and (6):

$$\begin{aligned} g_{ij}g_{kl} &= g_{ij}s^k(1-st)t^l - g_{i,j-k}g_{0l} \\ &= s^i(1-st)t^jg_{kl} = g_{i0}g_{k-j,l}. \end{aligned}$$

Thus if $j \neq k$ then either $g_{i,j-k} = 0$ or $g_{k-j,l} = 0$, whereas for $j = k$ we get $g_{ij}g_{jl} = g_{i0}g_{0l} = g_{il}$ by $(1-st)^2 = 1-st$. The multiplication table for the h_{ij} is obtained by symmetry.

From (3)–(7) and the symmetrical formulas it follows that the linear span of the g_{ij}, h_{ij} and the powers of s and t is a subalgebra of S and thus is all of S and that G and H are ideals of S annihilating each other. By the universal property of S we have surjective homomorphisms $\varphi_1, \varphi_2: S \rightarrow R$ given by $\varphi_1(s) = u, \varphi_1(t) = v$ and $\varphi_2(s) = v, \varphi_2(t) = u$ respectively. The homomorphism $\varphi = \varphi_1 \oplus \varphi_2: S \rightarrow R \oplus R$ then maps $s^n \mapsto u^n \oplus v^n, t^n \mapsto v^n \oplus u^n, g_{ij} \mapsto 0 \oplus e_{ij}, h_{ij} \mapsto e_{ij} \oplus 0$. Hence (a) follows from Theorem 1(a). Since $\ker(\varphi_1) = G$ and $\ker(\varphi_2) = H$, we have $S/G \cong R \cong S/H$. Also $(G+H)/G \cong E \cong (G+H)/H$, whence $S/(G \oplus H) \cong R/E \cong k[X, X^{-1}]$. If ψ denotes the representation of R on $k^{(\infty)}$ mentioned in §1 then $(\psi \oplus \psi) \circ \varphi$ is a representation of S as claimed in (d). This completes the proof.

3. THE JORDAN SHIFT ALGEBRA

Observe that (1) and (2) make sense in a special Jordan algebra since $sts = U_s t$ and $(1-st)z(1-ts) = z - V_{s,t}z + U_s U_t z = W(s, t)z$. We therefore define the *Jordan shift algebra* as the unital quadratic Jordan algebra J over k presented by generators a, b and relations

$$\begin{aligned} (8) \quad & a = U_a b, \quad b = U_b a, \\ (9) \quad & W(a, b)b^n = W(b, a)a^n = 0 \quad \text{for all } n \geq 0. \end{aligned}$$

These relations are symmetric in a and b . Hence there is an automorphism τ of period two of J interchanging a and b .

Theorem 3. *The assignment $a \mapsto u, b \mapsto v$ induces an isomorphism $\varphi: J \rightarrow R^+$ (the Jordan algebra associated to R), which is compatible with τ and $*$.*

Proof. Clearly, φ is compatible with τ and $*$ and maps a^n and b^n respectively to u^n and v^n . Define $c_{ij} \in J$ by

$$c_{ij} = \begin{cases} a^j \circ b^i - a^{j+1} \circ b^{i+1} & \text{if } i, j \geq 0, \\ 0 & \text{if } i < 0 \text{ or } j < 0. \end{cases}$$

Then

$$\begin{aligned} \varphi(c_{ij}) &= u^j v^i + v^i u^j - u^{j+1} v^{i+1} - v^{i+1} u^{j+1} \\ &= u^j(1-uv)v^i + v^i(1-vu)u^j = 0 + e_{ij}. \end{aligned}$$

Therefore the theorem will follow from Theorem 1(a) once we show that J is the linear span of the c_{ij} and the powers of a and b . Let $L = A + B + C$ be this linear span where A, B , and C denote respectively the linear span of the powers of a, b , and of the c_{ij} . Since L contains the generators of J , it will be enough to prove that L is a subalgebra of J or that L is invariant

under all Jordan multiplications by elements of J . By [2, Proposition 3.2.4] the multiplications of J are generated by the operators V_a, V_b, U_a, U_b , and $U_{a,b}$. By symmetry in a and b it therefore suffices to show that L contains the following subspaces:

$$(10) \quad \begin{array}{lll} \text{(i)} & V_a A + U_a A, & \text{(ii)} \quad V_a B, & \text{(iii)} \quad U_a B, \\ \text{(iv)} & U_{a,b} A, & \text{(v)} \quad V_a C + U_a C, & \text{(vi)} \quad U_{a,b} C. \end{array}$$

The following well-known identities, valid in any unital Jordan algebra, will be needed [2, Chapter III]:

$$\begin{aligned} (11) \quad & V(x^i)x^j = 2x^{i+j}, \quad U(x^i)x^j = x^{2i+j}, \\ (12) \quad & V(x^i, x^j) = V(x^{i+j}), \\ (13) \quad & U(x^i, x^j) = V(x^i)V(x^j) - V(x^{i+j}), \\ (14) \quad & U(x^{i+j}, x^i) = V(x^j)U(x^i) = U(x^i)V(x^j), \\ (15) \quad & V(x, y)U_x = U(U_x y, x), \quad V(x, y)V_x = V(U_x y) + U_x V_y. \end{aligned}$$

In J we now have

$$(16) \quad U_a b^{i+1} = a \circ b^i - b^{i-1}$$

for $i \geq 1$ and

$$(17) \quad \{a^j b^i a\} - \{a^{j+1} b^{i+1} a\} = c_{i-1, j},$$

$$(18) \quad U_a c_{ij} = c_{i-1, j+1},$$

$$(19) \quad V_a c_{ij} = c_{i-1, j} + c_{i, j+1},$$

for $i, j \geq 0$. Indeed, (16) follows from $U_a b^{i+1} = U_a U_b b^{i-1} = V(a, b)b^{i-1} - b^{i-1}$ (by (9)) = $V(b^{i-1}, b)a - b^{i-1} = b^i \circ a - b^{i-1}$ (by (12)). To prove (17) for $i \geq 1$ we use (13), (14), and (16):

$$\begin{aligned} \{a^j b^i a\} - \{a^{j+1} b^{i+1} a\} &= V(a^j)V_a b^i - V(a^{j+1})b^i - V(a^j)U_a b^{i+1} \\ &= V(a^j)(a \circ b^i - U_a b^{i+1}) - a^{j+1} \circ b^i \\ &= V(a^j)b^{i-1} - a^{j+1} \circ b^i = c_{i-1, j}, \end{aligned}$$

whereas for $i = 0$ we obtain $a^j \circ a - \{a^{j+1} b a\} = a^j \circ a - V(a^j)U_a b = a^j \circ (a - U_a b) = 0 = c_{-1, j}$ by (14) and (8). Formula (18) follows from (14) and (17):

$$\begin{aligned} U_a c_{ij} &= U_a V(a^j)b^i - U_a V(a^{j+1})b^{i+1} \\ &= \{a^{j+1} b^i a\} - \{a^{j+2} b^{i+1} a\} = c_{i-1, j+1}. \end{aligned}$$

Finally, we prove (19) using (13) and (17):

$$\begin{aligned} V_a c_{ij} &= V_a V(a^j)b^i - V_a V(a^{j+1})b^{i+1} \\ &= \{a^j b^i a\} + a^{j+1} \circ b^i - \{a^{j+1} b^{i+1} a\} - a^{j+2} \circ b^{i+1} \\ &= c_{i-1, j} + c_{i, j+1}. \end{aligned}$$

We can now show (10): (i) follows from (11); (ii) from $a \circ b^{n+1} = 2b^n - c_{n, 0}$ by definition of the c_{ij} ; (iii) from $U_a b = a$, (ii), and (16); (iv) from $U_{a,b} a^n = a^{n+1} \circ b = 2a^n - c_{0n}$ by (12); and (v) from (18) and (19). Finally, for (vi) it

suffices by the foregoing to prove $V_{a,b}C \subset L$ since $U_{a,b} = V_aV_b - V_{a,b}$. This follows from

$$(20) \quad \{abc_{ij}\} = (2 - \delta_{0j})c_{ij}.$$

To show (20) for $j > 0$, use (8), (15), and (18) to compute

$$\{abc_{ij}\} = V_{a,b}U_a c_{i+1,j-1} = U(U_a b, a)c_{i+1,j-1} = 2U_a c_{i+1,j-1} = 2c_{ij},$$

whereas

$$\begin{aligned} \{abc_{i0}\} &= 2\{abb^i\} - V_{a,b}V_a b^{i+1} \\ &= 2a \circ b^{i+1} - V(U_a b)b^{i+1} - U_a V_b b^{i+1} \quad (\text{by (12) and (15)}) \\ &= a \circ b^{i+1} - 2U_a b^{i+2} \quad (\text{by (8)}) \\ &= a \circ b^{i+1} - 2(a \circ b^{i+1} - b^i) \quad (\text{by (16)}) \\ &= c_{i0}. \end{aligned}$$

This completes the proof.

Note that while there is *associative asymmetry* in R as associative algebra (no automorphism switching u, v) there is *complete Jordan symmetry* between u and v in R^+ .

Corollary 1. (a) J is a free k -module with basis $1, a^n, b^n$ ($n \geq 1$), and $c_{ij} = a^j \circ b^i - a^{j+1} \circ b^{i+1}$ ($i, j \geq 0$), and $W_{a,b}J = 0 = W_{b,a}J$.

(b) For $i, j \geq 0$, the c_{ij} satisfy the Jordan multiplication table for matrix units and the general Jordan shift formulas

$$(21) \quad U(a^n)c_{ij} = c_{i-n,j+n}, \quad U(b^n)c_{ij} = c_{i+n,j-n},$$

$$(22) \quad V(a^n)c_{ij} = c_{i-n,j} + c_{i,j+n}, \quad V(b^n)c_{ij} = c_{i+n,j} + c_{i,j-n}.$$

(c) The subspace C spanned by the c_{ij} is an ideal of J and $J/C \cong k[X, X^{-1}]^+$ with $X = a + C$, $X^{-1} = b + C$.

Recall that $c_{pq} = 0$ if either p or q is negative. For the proof, one uses $\varphi(c_{ij}) = e_{ij}$, $\varphi(a) = u$, $\varphi(b) = v$, and computes in R . (By the same method, it is easy to establish the complete multiplication table for the above basis of J .) Alternatively, (21) can be obtained from (18) and $U(x^n) = U(x)^n$, while for (22) one can argue by induction: $n = 1$ is (19) and for $n > 1$ one uses $V(a^n) = V_a V(a^{n-1}) - U_a V(a^{n-2})$ by (13), (14).

Corollary 2. Let A be a unital Jordan algebra containing noninvertible elements x, y such that $U_x y = x$, $U_y x = y$, $W_{x,y} = W_{y,x} = 0$. Then the elements $z_{ij} = x^j \circ y^i - x^{j+1} \circ y^{i+1}$ ($i, j \geq 0$) are nonzero Jordan matrix units. In particular, the z_{ii} are nonzero orthogonal idempotents.

Proof. Let $\psi: J \rightarrow A$ be the unital homomorphism mapping $a \mapsto x$, $b \mapsto y$. Then $\psi(c_{ij}) = z_{ij}$. Assume $z_{mn} = 0$ for some m, n . By applying ψ to (21) and (22) one sees easily that $\psi(c_{ij}) = 0$ for all $i, j \geq 0$. Hence ψ factors via J/C . Since the canonical images of a and b in J/C are inverses by Corollary 1(c), so are x and y in A , a contradiction.

Corollary 3. The assignment $a \mapsto s$, $b \mapsto t$ induces a Jordan homomorphism $\sigma: J \rightarrow S^+$ which is an isomorphism between J and $H(S, \pi)$, the set of π -hermitian elements in S . The unital special universal envelope of J is isomorphic to (S, σ) .

Proof. The formulas

$$\sigma(a^n) = s^n, \quad \sigma(b^n) = t^n, \quad \sigma(c_{ij}) = g_{ji} + h_{ij} = h_{ij} + h_{ij}^\pi$$

combined with Theorem 2(a) and Corollary 1(a) show that σ is an isomorphism of J with $H(S, \pi)$. (Note that, by Theorem 2(a), $H(S, \pi)$ is spanned by $1, s^n, t^n$, and $h_{ij} + h_{ij}^\pi$.) Let (S', σ') be the unital special universal envelope of J . Then $\sigma = \eta \circ \sigma'$ by the universal property of S' where $\eta: S' \rightarrow S$ maps $s' = \sigma'(a)$ to s and $t' = \sigma'(b)$ to t . On the other hand, an application of σ' to (8) and (9) shows that the elements s', t' of S' satisfy (1) and (2). Hence there is a homomorphism $\eta': S \rightarrow S'$ sending s to s' and t to t' , and clearly η' is the inverse of η .

4. THE SHIFT TRIPLE SYSTEM

We keep the notation of the previous section. The relations defining J can be given a Jordan triple formulation as follows. Turn the k -module underlying J into a Jordan triple system J^τ by setting $P_x y = U_x y^\tau$. Then $e = 1_J$ and $f = a$ are tripotents (as is $g = b = f^\tau$):

$$(23) \quad e^{(3)} = e, \quad f^{(3)} = f,$$

where $x^{(3)} = P_x x = U_x x^\tau$ denotes the third power in the triple sense. This is immediate from (6) and $a^\tau = b$. Also, $\tau = P_e$; whence, in particular,

$$(24) \quad P_e^2 f = f.$$

Finally, the operators $W(x, y)$ in J and $B(x, y) = \text{Id} - L(x, y) + P_x P_y$ in J^τ (where we set $L(x, y)z = P(x + z)y - P_x y - P_z y = V(x, y^\tau)z$) are related by $B(x, y) = W(x, y^\tau)$. Since $J = (J^\tau)_e$ is the e -homotope of J^τ as a Jordan algebra, the powers x^n in J can be recovered in J^τ as the powers $x^{(n,e)}$ in the e -homotope of J^τ , with the convention that $x^{(0,e)} = e$. Thus we obtain from (9) the relations

$$(25) \quad B(f, f)(P_e f)^{(n,e)} = 0 \quad \text{for all } n \geq 0.$$

It is now natural to define the *shift triple system* as the Jordan triple system T over k presented by generators e and f and relations (23)–(25). We set $g = P_e f$.

We denote by R^* the Jordan triple structure on R given by $P_x y = xy^*x$.

Theorem 4. *The assignments $e \mapsto 1_J, f \mapsto a$ and $e \mapsto 1_R, f \mapsto u$ respectively induce isomorphisms $T \cong J^\tau$ and $T \cong R^*$.*

Proof. Clearly we have a homomorphism $\alpha: T \rightarrow J^\tau$ mapping e to 1 and f to a . To obtain a homomorphism in the opposite direction, we first observe that $T = T_2(e)$. Indeed, $T_2(e)$ is a subtriple containing the generators e and f , by (24). Hence the e -homotope T_e of T is a Jordan algebra with unit element e , quadratic operators $U_x = P_x P_e$, and involutive automorphism P_e interchanging g and f . From (23)–(25) it follows easily that f and g satisfy (8) and (9) in T_e . This gives us a Jordan algebra homomorphism $\beta: J \rightarrow T_e$ with $\beta(1) = e, \beta(a) = f, \beta(b) = P_e f = g$. It follows that $\beta \circ \tau = P_e \circ \beta$; thus, $\beta: J^\tau \rightarrow T$ is a Jordan triple homomorphism which is the inverse of α . Under the isomorphism $J \cong R^+$ of Theorem 3 the involutions τ and $*$ correspond to each other. This proves the second statement.

We leave it to the reader to formulate Corollary 1 of Theorem 3 in Jordan triple terms. Before proving an analogue of Corollary 2, we introduce the following terminology. A *shift pair* in a Jordan triple system V is a pair (c, d) of tripotents such that $d \in V_2(c)$ and

$$(26) \quad V_2(c) \cap V_0(d) = 0 \neq V_2(c) \cap V_1(d).$$

Note that (26) refers to all of $V_2(c)$, not just the part generated by c and d . By [3,1.8] c and d have compatible Peirce decompositions. Hence (26) implies

$$(27) \quad V_2(c) = V_2(d) \oplus (V_2(c) \cap V_1(d)),$$

$$(28) \quad B(d, d)V_2(c) = 0,$$

since $B(d, d)$ is the projection onto $V_0(d)$. We note that, for compatible tripotents c, d satisfying (28), the second condition of (26) is equivalent to either one of

$$(P_d^2 - \text{Id})V_2(c) \neq 0, \quad P_d^2c \neq c.$$

Indeed, if E_i are the projection operators onto the Peirce spaces of d , we have $V_2(c) \cap V_1(d) = E_1(V_2(c)) = (\text{Id} - E_0 - E_2)V_2(c) = (\text{Id} - B(d, d) - P_d^2)V_2(c) = (\text{Id} - P_d^2)V_2(c)$, whence the first equivalence. Moreover, if $P_d^2c = c$ then $c \in V_2(d)$ and so $V_2(c) \cap V_1(d) \subset V_2(d) \cap V_1(d) = 0$, showing $V_2(c) \cap V_1(d) \neq 0$; hence, $P_d^2c \neq c$. Finally, $P_d^2c \neq c$ obviously implies $(P_d^2 - \text{Id})V_2(c) \neq 0$.

Lemma 1. *Let (c, d) be a shift pair in the Jordan triple system V , and let $c = c_2 + c_1$ be the Peirce decomposition of c with respect to d . Then c_1 and c_2 are nonzero orthogonal tripotents. Furthermore, we have*

$$(29) \quad V_2(c_1) \subset V_2(c) \cap V_1(d),$$

and (d, c_2) is a shift pair.

Proof. By [3, 1.9] c_1 and c_2 are orthogonal tripotents. If $c_1 = 0$ then $c = c_2 \in V_2(d)$ implies $V_2(c) \subset V_2(d)$, contradicting (26) and (27). Likewise, $c_2 = 0$ yields $c = c_1 \in V_1(d)$; hence, $P_c d \in V_2(c) \cap V_0(d) = 0$ and thus $d = P_c^2 d = 0$, which again contradicts (26).

From (27) and $c_1 \in V_2(c) \cap V_1(d) \subset V_2(c)$ it follows by the Peirce relations that $V_2(c_1) \subset V_2(c)$ and

$$\begin{aligned} V_2(c_1) &= P(c_1)V_2(c) \subset P(c_1)V_2(d) \oplus P(c_1)(V_1(d) \cap V_2(c)) \\ &\subset (V_0(d) \cap V_2(c)) \oplus (V_1(d) \cap V_2(c)). \end{aligned}$$

Since $V_2(c) \cap V_0(d) = 0$ by (26), this proves (29).

Finally, we must show

$$V_2(d) \cap V_0(c_2) = 0 \neq V_2(d) \cap V_1(c_2).$$

Since $c = c_1 + c_2$ is an orthogonal decomposition, we have

$$(30) \quad V_0(c_2) \cap V_2(c) = V_2(c_1) \cap V_2(c),$$

$$(31) \quad V_1(c_2) \cap V_2(c) = V_1(c_1) \cap V_2(c).$$

Hence

$$\begin{aligned} V_2(d) \cap V_0(c_2) &= V_2(d) \cap V_0(c_2) \cap V_2(c) \\ &= V_2(d) \cap V_2(c_1) \cap V_2(c) \quad (\text{by (30)}) \\ &\subset V_2(d) \cap V_1(d) \quad (\text{by (29)}) \\ &= 0. \end{aligned}$$

Assume $V_2(d) \cap V_1(c_2) = 0$. Then $V_2(d) = V_2(c_2)$ since $V_2(d)$ is the sum of its intersections with the Peirce spaces of c_2 . Thus d and c_2 are associated and therefore have the same Peirce spaces; in particular, $V_1(d) = V_1(c_2)$. From (29) and (31) we get

$$V_2(c_1) \subset V_2(c) \cap V_1(c_2) = V_2(c) \cap V_1(c_1) \subset V_1(c_1),$$

which implies $c_1 = 0$, a contradiction.

Recall that two collinear tripotents c, d are called *rigidly collinear* if $V_2(c) \subset V_1(d)$ or, equivalently, $V_2(d) \subset V_1(c)$. Clearly, if c, d are rigidly collinear then so are $c, -d$. Also recall that a rectangular $(N_1 \times N_2)$ -grid (cf. [5, II.2.3; 4]) in a Jordan triple system is a family $\mathcal{R} = (r_{ij}), (i, j) \in N_1 \times N_2$, of nonzero tripotents such that the collinear pairs in \mathcal{R} are those which have a first (“row”) index or a second (“column”) index in common, and $(r_{ij}, r_{il}, r_{kl}, r_{kj})$ is a quadrangle whenever $i \neq k$ and $j \neq l$. If any two collinear tripotents in \mathcal{R} are rigidly collinear, we say \mathcal{R} is *rigid*.

Lemma 2. (a) *Let (e_1, e_2, e_3, e_4) be a quadrangle of tripotents in a Jordan triple system V . If e_1, e_2 are rigidly collinear then so are e_i, e_{i+1} for $i = 2, 3, 4$ (indices mod 4).*

(b) *Let e_1, e_2, e_3 be pairwise collinear tripotents. If e_1, e_2 are rigidly collinear so are e_1, e_3 and e_2, e_3 .*

(c) *If a rectangular grid $\mathcal{R} = (r_{ij})$ contains one rigidly collinear pair then \mathcal{R} is rigid.*

Proof. (a) It is enough to find automorphisms $\varphi_i, i = 2, 3, 4$, of $V_2(e_1 + e_2) = V_2(e_2 + e_4)$ mapping $\{e_i, e_{i+1}\}$ to $\{e_1, e_2\}$. Set $\varphi_2 = P(e_2 + e_4), \varphi_3 = P(e_1 + e_3)P(e_2 + e_4), \varphi_4 = P(e_1 + e_3)$.

(b) The automorphism $\theta = B(e_1 + e_3, e_1 + e_3)$ of V [4, 1.1] exchanges e_1 and e_3 and maps e_2 to $-e_2$, so e_2 and e_3 are rigidly collinear. By symmetry, the same holds for e_1 and e_3 .

(c) Suppose that e_1, e_2 is a rigidly collinear pair in \mathcal{R} . By collinearity, we either have $(e_1, e_2) = (r_{ij}, r_{il})$ or $(e_1, e_2) = (r_{ij}, r_{kj})$ for suitable indices i, j, k, l . It suffices to consider the first case. From the grid properties and (a) it follows that $A = \{r_{mj} : m \in N_1\}$ is a rigidly collinear family and from (b) that $B = \{r_{in} : n \in N_2\}$ is also a rigidly collinear family. If (c, d) is a collinear pair in \mathcal{R} which is not contained in $A \cup B$ then there exist $e, f \in A \cup B$ such that (c, d, e, f) is a quadrangle. Hence, again by (a), (c, d) is rigidly collinear.

We return to the notation introduced earlier and prove

Lemma 3. *If k is a field then E is the unique minimal ideal of $R, R^+,$ and R^* .*

Proof. In R^* we have $x^* = P_1x$; hence, every ideal of R^* is $*$ -invariant and thus an ideal of R^+ . Clearly an ideal of R is one of R^+ . Therefore, it suffices to show that $E \subset K$ for every nonzero ideal K of R^+ . Since E is simple as a Jordan algebra, it is even enough to prove $E \cap K \neq 0$. Suppose $x = \alpha_0 \cdot 1 + \sum_{l=1}^p (\alpha_l u^l + \alpha_{-l} v^l) + y \in K \setminus E$ where $y \in E$ so that $\alpha_j \neq 0$ for some j . In R^+ we have the formulas $U(u^n)e_{ij} = e_{i-n, j+n} = 0$ for $n > i$ and $u^n \circ e_{00} = e_{-n, 0} + e_{0, n} = e_{0, n}$ for $n > 0$ (cf. (21) and (22)) and $U(u^n)v^l = u^{2n-l}$

for $n \geq l$. Since y is a finite linear combination of the e_{ij} , it follows that $U(u^n)y = 0$ for sufficiently large n ; thus, $x' = U(u^n)x = \sum_{l=-p}^p \alpha_l u^{2n+l} \in K$. Now $x' \circ e_{00} = \sum_{l=-p}^p \alpha_l e_{0,2n+l}$ is a nonzero element of $E \cap K$.

Theorem 5. *Let V be a Jordan triple system containing a shift pair (c, d) . Then the elements*

$$r_{ij} = \{d^{(j,c)}, c, (P_c d)^{(i,c)}\} - \{d^{(j+1,c)}, c, (P_c d)^{(i+1,c)}\} \quad (i, j \geq 0)$$

form a rigid rectangular $(\mathbb{N} \times \mathbb{N})$ -grid in V . If k is a field, there exists a unique Jordan triple isomorphism ψ from T onto the subtriple generated by c and d such that $\psi(e) = c$ and $\psi(f) = d$.

Proof. From $d \in V_2(c)$ and (28) it follows that c and d satisfy (23)–(25). After identifying T and R^* by Theorem 4, we have a Jordan triple homomorphism $\psi: R^* \rightarrow V$ mapping $1 \mapsto c$ and $u \mapsto d$. One checks easily that $\psi(e_{ij}) = r_{ij}$ using $(R^*)_1 = R^+$, so $x^{(n,1)} = x^n$ for every $x \in R$, $u^j \circ v^i = \{u^i, 1, v^j\} = \{u^{(j,1)}, 1, (P_1 u)^{(i,1)}\}$, and $e_{ij} = u^j \circ v^i - u^{j+1} \circ v^{i+1}$. The e_{ij} form a rectangular grid in R^* . Hence the r_{ij} satisfy all grid relations, and therefore $r_{00} \neq 0$ implies $r_{ij} \neq 0$ (consider the quadrangle $(r_{00}, r_{0j}, r_{ij}, r_{i0})$). To see that $r_{00} \neq 0$ we consider the Peirce decomposition of 1 in R^* , which is easily seen to be

$$1 = vu + e_{00} \in R_2^*(u) \oplus R_1^*(u),$$

using $P_u^2(1) = P_u(u1^*u) = P_u(u^2) = uv^2u = vu$ and $L(u, v)e_{00} = uve_{00} + e_{00}vu = e_{00}(1 + vu) = e_{00}(2 - e_{00}) = e_{00}$. By applying ψ we get the Peirce decomposition $c = c_2 + c_1$ of c with respect to d in V where $c_2 = \psi(vu)$ and $c_1 = \psi(e_{00}) = r_{00}$ which is not zero by Lemma 1.

Next we prove rigidity. By Lemma 2 it suffices to have r_{00} and r_{01} rigidly collinear. A computation shows that vu is a tripotent in R^* and that the Peirce decomposition of u with respect to vu is

$$u = vu^2 + e_{01} \in R_2^*(vu) \oplus R_1^*(vu).$$

Indeed, $P_{vu}^2 u = P_{vu}(v^2u) = vu^2$ and $L(vu, vu)e_{01} = vu v u e_{01} + e_{01} v u v u = v u e_{01} + e_{01} v u = v u (1 - v u) u + (1 - v u) u v u = 0 + (1 - v u) u = e_{01}$. An application of ψ yields the Peirce decomposition $d = d_2 + d_1$ with respect to $c_2 = \psi(vu)$ in V , where $d_1 = \psi(e_{01}) = r_{01}$. By Lemma 1, (d, c_2) is a shift pair. Hence (29), applied to d, c_2, d_1 in place of c, d, c_1 and (27), (31) imply

$$\begin{aligned} V_2(r_{01}) &= V_2(d_1) \subset V_2(d) \cap V_1(c_2) \subset V_2(c) \cap V_1(c_2) \\ &= V_2(c) \cap V_1(c_1) \subset V_1(c_1) = V_1(r_{00}). \end{aligned}$$

It remains to prove that ψ is injective if k is a field. The kernel of ψ is an ideal of R^* not containing E and hence is zero by Lemma 3. This completes the proof.

5. CONCLUDING REMARKS

(i) Suppose the base ring k is a field. Then [1] the shift algebra R is primitive and E is the socle of R and also the unique minimal ideal of R . Consequently, one shows as above that if A is any associative unital k -algebra with two generators x, y satisfying $xy = 1 \neq yx$ then $A \cong R$. Similarly, one can prove that the symmetric shift algebra S is semiprimitive but not primitive,

with G and H as the only minimal ideals, and socle $G \oplus H$. The details are left to the reader.

(ii) From Lemma 1 and the proof of Theorem 5 one can deduce the following inductive construction of the tripotents r_{ii} and $r_{i,i+1}$. Starting with a shift pair (c, d) , define two sequences e_n, f_n of tripotents by $e_0 = c$, $e_1 = d$, and

$$e_n = e_{n+2} + f_n \in V_2(e_{n+1}) \oplus V_1(e_{n+1}),$$

the Peirce decomposition of e_n with respect to e_{n+1} . Then f_0, f_1, f_2, \dots is the “infinite staircase” $r_{00}, r_{01}, r_{11}, r_{12}, \dots$ in \mathcal{R} , from which \mathcal{R} can be reconstructed by [5, II.2.6]. The details are omitted.

(iii) The exact sequence $0 \rightarrow E \rightarrow R \rightarrow k[X, X^{-1}] \rightarrow 0$ has the C^* -algebra analogue

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{T} \rightarrow \mathcal{C}(S^1) \rightarrow 0$$

where \mathcal{T} denotes the Toeplitz algebra, $\mathcal{C}(S^1)$ the continuous functions on the circle, and \mathcal{K} the compact operators on the Hardy space. More precisely, let A be a C^* -algebra generated as a C^* -algebra by u and v satisfying $uv = 1 \neq vu$ and $v = u^*$. Then one can show that A and \mathcal{T} are isomorphic as C^* -algebras. It might be interesting to explore this connection further, studying, on the one hand, more general shift algebras corresponding to the Toeplitz algebras of bounded symmetric domains [6] and, on the other hand, Toeplitz JB^* -triples.

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