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ON WEAK CONVERGENCE IN $H^1(\mathbf{R}^d)$

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ABSTRACT. We prove that the a.e. convergence of a sequence of functions bounded in $H^1(\mathbf{R}^d)$ to a function in $L^1(\mathbf{R}^d)$ implies weak convergence.

Let $H^1(\mathbf{R}^d)$ denote the Hardy space of functions on $\mathbf{R}^d$ whose Poisson maximal function lies in L^1, and let $\mathrm{BMO}(\mathbf{R}^d)$ denote the dual space [2] of functions of bounded mean oscillation. The space $\mathrm{VMO}(\mathbf{R}^d)$, the closure of the Schwartz space $\mathscr{S}$ in $\mathrm{BMO}(\mathbf{R}^d)$, is the predual of $H^1(\mathbf{R}^d)$. We answer a question of Lions and Meyer by proving the following result. The theorem is also valid for Martingale H^1, and the proof we give carries over directly to that setting.

Theorem. *Suppose $\{f_n\}$ is a sequence of $H^1(\mathbf{R}^d)$ functions such that $\|f_n\|_{H^1} \leq 1$ for all n and such that $f_n(x) \to f(x)$ for almost every $x \in \mathbf{R}^d$. Then $f \in H^1(\mathbf{R}^d)$, $\|f\|_{H^1} \leq 1$, and*

$$\int_{\mathbf{R}^d} f_n \varphi \, dx \to \int_{\mathbf{R}^d} f \varphi \, dx$$

for all $\varphi \in \mathrm{VMO}(\mathbf{R}^d)$.

Proof. We may suppose that $\|\varphi\|_{L^1}$, $\|\varphi\|_{L^\infty}$, $\|\partial\varphi/\partial x_j\|_{L^\infty} \leq 1$ and that support φ is compact. Fix $\delta > 0$ and pick $\eta > 0$ so that $\eta \exp\{\delta^{-1}\} \leq \delta^{1+d}$ and, whenever $|E| \leq C\eta \exp\{\delta^{-1}\}$, $\int_E |f|\, dx \leq \delta$. This can be done because by Fatou's lemma $\|f\|_{L^1} \leq 1$. Now pick n large enough so that

$$|E_n| = |\{x \in \operatorname{support} \varphi : |f_n(x) - f(x)| > \eta\}| \leq \eta .$$

Define

$$\tau(x) = \max\{0,\ 1 + \delta \log M\chi_{E_n}(x)\},$$

where $M(\cdot)$ denotes the Hardy-Littlewood maximal function. Then $0 \leq \tau \leq 1$, $\tau = 1$ a.e. dx on E_n, and by a result due to Coifman and Rochberg [1], $\|\tau\|_{\mathrm{BMO}} \leq C\delta$. By the weak $(1, 1)$ estimate for the maximal function [3],

$$|\operatorname{support} \tau| \leq C|E_n| \exp\{\delta^{-1}\} \leq C\eta \exp\{\delta^{-1}\}.$$

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Consequently,

$$\int_{\text{support }\tau} |f|\,dx \le \delta\,.$$

We now write

$$\begin{aligned}\left|\int_{\mathbf{R}^d}(f-f_n)\varphi\,dx\right| &\le \left|\int_{\mathbf{R}^d}(f-f_n)\varphi(1-\tau)\,dx\right| + \left|\int_{\mathbf{R}^d}(f-f_n)\varphi\tau\,dx\right| \\ &\le \eta\|\varphi\|_{L^1} + \int_{\text{support }\tau}|f\varphi|\,dx + \left|\int_{\mathbf{R}^d} f_n\varphi\tau\,dx\right| \\ &\le \delta+\delta+C\|f_n\|_{H^1}\|\varphi\tau\|_{\mathrm{BMO}}\,.\end{aligned}$$

The proof of the theorem will therefore be established as soon as we verify

$$\|\varphi\tau\|_{\mathrm{BMO}} \le C\delta\,.$$

Since $\|\varphi\tau\|_{L^1} \le C\eta\exp\{\delta^{-1}\} \le \delta^{1+d}$, it is sufficient to test the BMO norm over cubes of sidelength $\le \delta$. Let τ_Q and φ_Q denote the mean values of τ, φ over such a cube Q, and note that $|\varphi-\varphi_Q| \le d\delta$ on Q. Therefore,

$$\begin{aligned}\frac{1}{|Q|}\int_Q|\varphi\tau-\varphi_Q\tau_Q|\,dx &\le \frac{1}{|Q|}\int_Q|\varphi\tau-\varphi_Q\tau|\,dx + \frac{|\varphi_Q|}{|Q|}\int_Q|\tau-\tau_Q|\,dx \\ &\le d\delta + \|\tau\|_{\mathrm{BMO}} \le C\delta\,.\end{aligned}$$

Standard limiting arguments show $\|f\|_{H^1} \le 1$.

We remark that if f_n lie in the analytic Hardy space $H^1_a(\mathbf{R}^1)$, a proof can be given by using only H^∞ theory. By taking subsequences and looking at $\log|f_n|$, one can find $\varphi_m \in H^\infty$ such that $\|\varphi_m\|_{H^\infty} \le 1$, $\|f_n\varphi_m(1+|x|^2)\|_{L^\infty} \le m$, and $\varphi_m(x) \to 1$ a.e. dx. Taking limits in n, $\|f\varphi_m\|_{H^1} \le 1$, and then taking limits in m, $\|f\|_{H^1} \le 1$. Returning to the original sequence, one applies Jensen's inequality to $f_n - f$ to establish that $f_n(z) - f(z) \to 0$. This suffices to prove $f_n \to f$ weakly.

Remark. The theorem is not abstract. In other words, if one attempts to use only the facts $\|f_n\|_{L^1} \le 1$, $f_n(x) \to f(x)$ a.e. dx, and $\mathscr{S}$ is dense in VMO, one will not arrive at a correct proof.

References

1. R. R. Coifman and R. Rochberg, *Another characterization of BMO*, Proc. Amer. Math. Soc. **79** (1980), 249–254.

2. C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137–193.

3. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, NJ, 1970.

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