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COMBINATORIAL DIMENSION OF FRACTIONAL CARTESIAN PRODUCTS

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ABSTRACT. The combinatorial dimension of a fractional Cartesian product is the optimal value of an associated linear programming problem.

Fractional Cartesian products were introduced in a context of harmonic analysis to fill gaps between Cartesian products of spectral sets [1]. These fractional products subsequently gave rise to the idea of combinatorial dimension, a measurement of interdependencies that plays a natural role in harmonic and stochastic analysis (e.g., [2, 4, 6]). In this note, which is completely elementary and self-contained, we compute the combinatorial dimension of a general fractional Cartesian product by solving an associated linear programming problem.

Definition 1 [2, 5]. Let $E_1, \dots, E_n$ be infinite sets, and let $F \subset E_1 \times \cdots \times E_n$. For an integer $s > 0$, let

$$\Psi_F(s) = \max\{|F \cap (A_1 \times \cdots \times A_n)| \colon A_i \subset E_i,\ |A_i| \le s,\ i = 1, \dots, n\}. \tag{1}$$

Define the *combinatorial dimension* of F to be

$$\dim F = \limsup_{s\to\infty}(\ln \Psi_F(s)/\ln s). \tag{2}$$

Equivalently, for each real number $a > 0$, let

$$d_F(a) = \limsup_{s\to\infty}(\Psi_F(s)/s^a), \tag{3}$$

and then $\dim F = \inf\{a \colon d_F(a) < \infty\}$.

The parameter $\dim F$, for $F \subset E_1 \times \cdots \times E_n$, can be viewed as a measurement of interdependence between the canonical projections from $E_1 \times \cdots \times E_n$ onto E_j, $j = 1, \dots, n$, restricted to F. The idea of *combinatorial dimension* was motivated at the outset by problems in harmonic analysis [1]. Indeed, it turns out that $\dim F$ can precisely be linked to harmonic-analytic parameters that are naturally associated with spectral sets [2, 3].

Clearly, $\dim(E_1 \times \cdots \times E_n) = n$, and if $F \subset E_1 \times \cdots \times E_n$ is infinite then $1 \le \dim F \le n$. For arbitrary $\alpha \in (1, n)$, randomly produced examples of α-dimensional sets can be found in [5]. General fractional Cartesian products, archetypal instances of which appeared in [1, 3], provide explicit examples

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of sets with fractional combinatorial dimension; we know of no other general scheme which produces explicit examples of sets with arbitrary fractional dimension.

For a positive integer m, let $[m] = \{1, \dots, m\}$. Let m be some fixed positive integer, and let X be some fixed infinite set. For a subset $S \subset [m]$, let X^s be the set of all functions from S into X. In particular, we can identify $X^{[m]}$ and X^m. If $S_1 \subset S_2 \subset [m]$, then we let $t \mapsto t|S_1$ be the projection from X^{S_2} onto X^{S_1}; that is, if $t \in X^{S_2}$ then $t|S_1 \in X^{S_1}$ is such that, for each $i \in S_1$, $(t|S_1)(i) = t(i)$.

Definition 2. Let X be an infinite set. Let $\mathscr{U} = \{S_1, \dots, S_n\}$ be a cover of $[m]$; that is, $[m] = S_1 \cup \cdots \cup S_n$. The *fractional Cartesian product* of X based on $\mathscr{U}$ is

$$X^{(\mathscr{U})} = \{(t|S_1, \dots, t|S_n): t \in X^m\}. \tag{4}$$

Now let $\mathscr{U} = \{S_1, \dots, S_n\}$ be a cover of $[m]$, and consider the problem of computing the combinatorial dimension of $X^{(\mathscr{U})}$, where $X^{(\mathscr{U})}$ is viewed as a subset of the n-fold Cartesian product $X^{S_1} \times \cdots \times X^{S_n}$. In the case that $\mathscr{U}$ consists of k-element subsets of $[m]$ $(k \geq 1)$ so that $|\{j: i \in S_j\}| = I \geq 1$, for all $i \in [m]$, it is shown in [3, Corollary 2.6] that $\dim X^{(\mathscr{U})} = m/k$. In the general case, we show that the combinatorial dimension of an arbitrary fractional Cartesian product is the optimal value of the following linear programming problem:

$$\begin{gathered} \text{Maximize } x_1 + \cdots + x_m \\ \text{subject to the constraints that each } x_i \geq 0 \text{ and that} \\ \sum_{i \in S_j} x_i \leq 1 \quad \text{for each } j \in [n]. \end{gathered} \tag{5}$$

Since $\mathscr{U}$ is a cover of $[m]$, the feasible set is bounded, and thus the optimal value exists. Let $\alpha = \alpha(\mathscr{U})$ be this optimal value.

Theorem 3. $\dim X^{(U)} = \alpha(\mathscr{U})$.

Proof. First we prove the inequality $\alpha(\mathscr{U}) \leq \dim X^{(\mathscr{U})}$. Let $(x_1, \dots, x_m)$ be an optimal vector; that is, $x_1, \dots, x_m$ satisfy the constraints of the linear programming problem, and $\alpha = x_1 + \cdots + x_m$. Moreover, by letting $(x_1, \dots, x_m)$ be an extreme point of the feasible set, we can assume that each of $x_1, \dots, x_m$ is rational. Let s be an arbitrarily large integer such that s^{x_i} is an integer for each $i \in [m]$. Let $D_1, \dots, D_m \subset X$ be such that $|D_i| = s^{x_i}$ for each $i \in [m]$, and consider $D = D_1 \times \cdots \times D_m \subset X^m$. For each $j \in [n]$, let $A_j = \times\{D_i: i \in S_j\}$. We have $A_j \subset X^{S_j}$ and obtain from the constraints in (5)

$$|A_j| = \prod_{i \in S_j} |D_i| = \prod_{i \in S_j} s^{x_i} \leq s. \tag{6}$$

Also, it is clear that $t \mapsto (t|S_1, \dots, t|S_n)$ is a bijection from D onto $X^{(\mathscr{U})} \cap (A_1 \times \cdots \times A_n)$, so that

$$|X^{(\mathscr{U})} \cap (A_1 \times \cdots \times A_n)| = |D| = \prod_{i=1}^{n} s^{x_i} = s^{\alpha}. \tag{7}$$

Since s is an arbitrary integer, (6) and (7) imply that $\alpha \le \dim X^{(\mathscr{U})}$.

To prove $\alpha \ge \dim X^{(\mathscr{U})}$, we consider the following dual linear programming problem:

$$\text{Minimize } y_1 + \cdots + y_n \tag{8}$$
$$\text{subject to the constraints that each } y_j \ge 0 \text{ and that}$$
$$\sum_{i \in S_j} y_j \ge 1 \quad \text{for each } i \in [m].$$

By the fundamental duality theorem of linear programming, the value of the dual problem exists and equals $\alpha = \alpha(\mathscr{U})$. Let $(y_1, \ldots, y_n)$ be an optimal vector. Let $A_1, \ldots, A_n$ be arbitrary finite subsets of $X^{S_1}, \ldots, X^{S_n}$, respectively. We will show

$$|X^{(\mathscr{U})} \cap (A_1 \times \cdots \times A_n)| \le |A_1|^{y_1} \cdots |A_n|^{y_n}. \tag{9}$$

In the case $|A_1| \le s, \ldots, |A_n| \le s$, (9) of course implies

$$\Psi_{X^{(\mathscr{U})}}(s) \le s^{\sum_j y_j} = s^\alpha, \tag{10}$$

establishing $\alpha \ge \dim X^{(\mathscr{U})}$.

We use the notation: if $S = \{i_1, \ldots, i_k\}$ then $\sum_{t_i \in X : i \in S} f(t_1, \ldots, t_m)$ denotes the iterated sum

$$\sum_{t_{i_1} \in X} \cdots \sum_{t_{i_k} \in X} f(t_1, \ldots, t_m).$$

In particular, if $S = \varnothing$ then $\sum_{t_i \in X : i \in S} f(t_1, \ldots, t_m) = f(t_1, \ldots, t_m)$. Observe that

$$|X^{(\mathscr{U})} \cap (A_1 \times \cdots \times A_n)| = \sum_{t_i \in X : i \in [m]} \prod_{j=1}^{n} \mathbf{1}_{A_j}(t_i : i \in S_j) \tag{11}$$

($\mathbf{1}_A$ denotes the indicator function of A). To obtain (9), we apply recursively in (11) the following multilinear Hölder inequality: Let $k > 1$ be an integer and let $p_1 \in (0, \infty], \ldots, p_k \in (0, \infty]$ be such that $(1/p_1) + \cdots + (1/p_k) \ge 1$. Let $f_1, \ldots, f_k$ be real-valued functions defined on X. Then

$$\sum_{t \in X} |f_1(t) \cdots f_k(t)| \le \|f_1\|_{p_1} \cdots \|f_k\|_{p_k} \tag{12}$$

($\|\cdot\|_p$ denotes the usual l^p-norm). For $q = 0, \ldots, m$, define

$$F(q) = \sum_{t_i \in X : i \in [m] \sim [q]} \prod_{j=1}^{n} \left(\sum_{t_i \in X : i \in [q] \cap S_j} \mathbf{1}_{A_j}(t_i : i \in S_j) \right)^{y_j}. \tag{13}$$

Observe that $F(0) = |X^{(\mathscr{U})} \cap (A_1 \times \cdots \times A_n)|$ and that $F(m) = |A_1|^{y_1} \cdots |A_n|^{y_n}$, so (9) is just $F(0) \le F(m)$. We let $q = 0, \ldots, m-1$ and proceed to show $F(q) \le F(q+1)$, thus establishing (9). First we rewrite $F(q)$ as

$$\begin{aligned} F(q) = & \sum_{t_i \in X : i \in [m] \sim [q+1]} \prod_{\{j : q+1 \notin S_j\}} \left(\sum_{t_i \in X : i \in [q] \cap S_j} \mathbf{1}_{A_j}(t_i : i \in S_j) \right)^{y_j} \\ & \cdot \sum_{t_{q+1} \in X} \prod_{\{j : q+1 \in S_j\}} \left(\sum_{t_i \in X : i \in [q] \cap S_j} \mathbf{1}_{A_j}(t_i : i \in S_j) \right)^{y_j}. \end{aligned} \tag{14}$$

Given the constraints in the dual linear programming problem (8) ($\sum_{q+1\in S_j} y_j \geq 1$), we apply (12) in (14) to the summation over t_{q+1} with exponents $1/y_j$ and functions $(\sum_{t_j\in X: i\in[q]\cap S_j} \mathbf{1}_{A_j}(t_i: i\in S_j))^{y_j}$, where $q+1\in S_j$. We thus obtain

$$
\begin{aligned}
F(q) \leq & \sum_{t_i\in X: i\in[m]\sim[q+1]} \prod_{\{j: q+1\notin S_j\}} \left(\sum_{t_i\in X: i\in[q]\cap S_j} \mathbf{1}_{A_j}(t_i: i\in S_j)\right)^{y_j} \\
(15) \qquad & \cdot \prod_{\{j: q+1\in S_j\}} \left(\sum_{t_{q+1}\in X} \sum_{t_i\in X: i\in[q]\cap S_j} \mathbf{1}_{A_j}(t_i: i\in S_j)\right)^{y_j} \\
= & \sum_{t_i\in X: i\in[m]\sim[q+1]} \prod_{j=1}^{n} \left(\sum_{t_i: i\in[q+1]\cap S_j} \mathbf{1}_{A_j}(t_i: i\in S_j)\right)^{y_j} \\
= & F(q+1). \quad \square
\end{aligned}
$$

Remarks. 1. Since the optimal value of the linear programming problem in (5) is a rational number, we deduce that, for every positive integer m and every cover $\mathscr{U}$ of $[m]$, the combinatorial dimension of $X^{(\mathscr{U})}$ is rational.

2. $F \subset E_1 \times \cdots \times E_n$ was said in [5] to be a γ-Cartesian product if

$$(16) \qquad 0 < \liminf_{s\to\infty}(\Psi_F(s)/s^{\gamma}) \leq \limsup_{s\to\infty}(\Psi_F(s)/s^{\gamma}) < \infty.$$

Clearly, the combinatorial dimension of a γ-Cartesian product is γ. However, the statement in (16) is stronger than the statement $\dim F = \gamma$; this can be illustrated by the random constructions in [5]. In the framework of the present paper, following an elementary computation based on (6), (7), and (9), we observe that, for $\alpha = \alpha(\mathscr{U})$,

$$\lim_{s\to\infty} \Psi_{X^{(\mathscr{U})}}(s)/s^{\alpha} = 1.$$

3. Corollary 2.6 in [3] can be directly deduced from our main theorem. Suppose that a cover $\mathscr{U} = \{S_1, \ldots, S_n\}$ of $[m]$ consists of k-element subsets of $[m]$ $(k \geq 1)$ so that $|\{j: i\in S_j\}| = I \geq 1$ for all $i\in[m]$. Then, by a standard argument, the optimal value of the linear programming problem (5) is $\geq m/k$, and that of its dual (8) is $\leq n/I = m/k$. Therefore, by Theorem 3 and the fundamental duality theorem, $\dim X^{(\mathscr{U})} = m/k$.

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